Optimization of Structural Design using Geometric Programming Method

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Abstract: The main objective of structural engineers throughout design history has been to obtain structure under the prescribed design conditions which can not only withstand external loads safety but also achieve an economic solution. This paper focuses on the use of geometric programming solution method to optimum design of plane truss structures. This approach is illustrated on planer truss optimization model and the results are discussed.

Keywords: Structural Optimization, Geometric Programming.

1. INTRODUCTION

A Geometric Program (GP) is a type of mathematical optimization problem characterized by objective and constraint functions that have a special form. It has useful theoretical and computational properties. Although GP in standard form is apparently a non convex optimization problem, it can be readily turned into a convex optimization problem; hence a local optimum is also global optimum. Here the advantage is that it is usually much simpler to work with the dual than the primal one. Solving a nonlinear programming problem by GP method with degree of difficulty (DD) plays a significant role.

Since late 1960's Geometric Programming (GP) has been known and used in different field like Operations Research, Engineering designs etc. The general theory of geometric programming and its engineering application was initially developed by Duffin,Peterson and Zener [10] and Zener [4] in their published book. A serious limitation in the application of this theory has been that all the functions involved in the problem are to be posynomials.This shortcoming was overcome by Wild and Beightler [5] in 1967 when they generalized the theory to allow the use of negative coefficients in both objective and constraints, and also to permit reversed inequality constraints. Generalized GP refers to minimizing a generalized posynomial subject to upper bound inequality constraints on generalized posynomials. This method is a general form of geometric programming method in which signomal functions are present in objective function and in constraints.

The main objective of a structural engineering is to design structures which withstand external loads safely and at a minimum cost or weight [2,3 and7].The desire to improve a design without compromising the structural integrity has been a strong driving force behind the development of various optimum design methods.

Finally this GP method is identified through the numerical example of two-bar truss and the analysis results show that the geometric programming method can always converges to the global optimal solution.

2. Truss Structural Optimization

The mathematical form of optimization problem for truss structure can be expressed as follows:

Find $A^T = \{A_1, A_2, ..., A_n\}$ (2.1)

To minimize $F = W(A) = \rho \sum_{i=1}^{n} L_i A_i$ (2.2)

Subject to $g_j^L \le g_j(A) \le g_j^U \qquad j = 1,2,3,.....,m$ (2.3)

and $A_i^{\min} \le A_i \le A_i^{\max} \qquad i = 1,2,3,.....,n$ (2.4)

Where A_i = the design variable i (member i cross-sectional area, n= the number of design variables, $W(A)$ = the objective function (the structural weight), ρ = the material density, L_i = the member of length, m = the number of inequality constraints (g), $A_i^{\min}$ and

$A_i^{\max}$ are the lower and the upper bounds of the i^{th} variable respectively. The lower bounds posed by equation-3 on the constraints include truss member stresses and joint displacements.

3. Geometric Programming Method:

A geometric program (GP) is a type of mathematical optimization problem characterized by objective and constraint functions that have a special form. GP is a methodology for solving algebraic non-linear optimization problems. Also linear programming is a subset of a geometric programming .The theory of geometric programming was initially developed about three decades ago and culminated in the publication of the seminal text in this area by Duffin, Peterson, and Zener [10]

The general constrained Primal Geometric Programming problem is as follows:

$$Minimize \;\; g_0(x) = \sum_{t=1}^{T_0} c_{0t} \prod_{n=1}^{N} x_n^{a_{0tn}} \tag{3.1}$$

Subject to

$$g_m(x) = \sum_{t=1}^{T_m} c_{mt} \prod_{n=1}^{N} x_n^{a_{mtn}} \le 1; \quad m = 1,2,3,......., M \tag{3.2}$$

$$x_n > 0, \qquad n = 1,2,.........., N.$$

Here $c_{0t} > 0$ and a_{0tn} be any real number. The objective function contains T_0 terms and T_m terms in the inequality constraints. Here the coefficient of each term is positive.So it is a constrained posynomial geometric programming problem. Let $T = T_0 + T_1 + ... + T_m$ be the total number of terms in the primal program. The degree of difficulty (DD) is defined as DD = Total no. of terms – (Total no. of variables -1) $= T - (N+1)$.The dual problem (with the objective function $d(w)$,where $w \equiv \{ w(w_{mt}), \forall m = 0,1,2......, M; t = 1,2,.....T_m \}$ is the decision vector) of the geometric programming problem (1) for the general posynomial case is as follows:

$$Maximize \;\; d(w) = \prod_{t=1}^{T_0} \left(\frac{c_{0t}}{w_{0t}} \right)^{w_{0t}} \prod_{m=1}^{M} \prod_{t=1}^{T_m} \left(\frac{c_{mt} \sum w_{mt}}{w_{mt}} \right)^{w_{mt}} \tag{3.3}$$

Subject to

$$\sum_{t=1}^{T_0} w_{0t} = 1, \qquad \text{(Normality condition)}$$

$$\sum_{m=0}^{M}\sum_{t=1}^{T_m} a_{mtn} w_{mt} = 0 \quad for\, n = 1,2,......,N. \qquad \text{(Othogonality conditions)}$$

$$w_{mt} > 0 \quad \forall m = 0,1,........,M;\ t = 1,2,........T_m.$$

For a primal problem with M variables, $T_0 + T_1 + ... + T_m$ terms and N constraints, the dual problem consists of $T_0 + T_1 + ... + T_m$ variables and M+ 1 constraint. The relation between these problems, the optimality has been shown [...] to satisfy

$$c_{0t}\prod_{n=1}^{N} x_n^{a_{0m}} = d^*(w^*)\times w_{0t}^* \quad t = 1,2,3,...,T_m \tag{3.4}$$

$$c_{mt}\prod_{n=1}^{N} x_n^{a_{mtn}} = \frac{w_{mt}^*}{\sum_{t=1}^{T_m} w_{mt}^*} \quad m = 1,2,3,....,M;\ t = 1,2,3,...,T_m \tag{3.5}$$

Taking logarithms in (3.4) and (3.5) and putting $t_n = \log x_n$ for $n = 1,2,..........,N.$ we shall get a system of linear equations of $t_n\,(n = 1,2,..........,N.)$.We can easily find primal variables from the system of linear equations.

Case I: For $T \geq N+1$,the dual program presents a system of linear equations for the dual variables where the number of linear equations is either less than or equal to the number of dual variables. A solution vector exists for the dual variable (Beightler and Philips [20]).

Case II: For $T < N+1$,the dual program presents a system of linear equations for the dual variables where the number of linear equation is greater than the number of dual variables. In this case, generally, no solution vector exists for the dual variables. However, one can get an approximate solution vector for this system using either the least squares or the linear programming method.

4. Numerical Example:

A numerical problem as follows:

The primal problem is

$$\left.\begin{aligned} &Minimize\ g_0(x) = 2x_1 + 5x_2 + 2x_3 + x_4 + 0.5x_1^{-1}x_2^{-1}x_3^{-1} \\ &Subject\ to\quad g_1(x) \equiv x_1^2 x_4^{-2} + x_2^2 x_4^{-2} \leq 1; \\ &\qquad\qquad g_2(x) \equiv 100x_1^{-1}x_2^{-1}x_3^{-1} \leq 1; \\ &\qquad\qquad\qquad x_1, x_2, x_3, x_4 > 0; \end{aligned}\right\} \tag{4.1}$$

This is a posynomial constraints geometric programming problem. This problem is having degree difficulty = 8-(4+1) =3. The problem is solved via dual geometric programming.

The corresponding dual of geometric programming (DGP) problem is:

$$\max d(w) =$$

$$\left[\left(\frac{2}{w_{01}}\right)^{w_{01}}\left(\frac{5}{w_{02}}\right)^{w_{02}}\left(\frac{2}{w_{03}}\right)^{w_{03}}\left(\frac{1}{w_{04}}\right)^{w_{04}}\left(\frac{0.5}{w_{05}}\right)^{w_{05}}\left(\frac{1}{w_{11}}\right)^{w_{11}}\left(\frac{1}{w_{12}}\right)^{w_{12}}(w_{11}+w_{12})^{w_{11}+w_{12}}(100)^{w_{21}}\right] \quad (4.2)$$

Subject to

$$w_{01}+w_{02}+w_{03}+w_{04}+w_{05}=1$$

For the primal variable x_1

$$w_{01}-w_{05}+2w_{11}-w_{21}=0; \quad (4.4)$$

For the primal variable x_2

$$w_{02}-w_{05}+2w_{12}-w_{21}=0; \quad (4.5)$$

For the primal variable x_3

$$w_{03}-w_{05}-w_{21}=0; \quad (4.6)$$

For the primal variable x_4

$$w_{04}-2w_{11}-2w_{12}=0; \quad (3.7)$$

$$w_{01}, w_{02}, w_{03}, w_{04}, w_{05}, w_{11}, w_{12}, w_{21} > 0$$

The dual variables and the corresponding maximum value of dual objective are given in the following table.

Table-1: Dual Solution

w_{01}	w_{02}	w_{03}	w_{04}	w_{05}	w_{11}	w_{12}	w_{21}	$g_0(x)$
0.23111	0.30484	0.33332	0.13064	0.00011	0.05119	0.01422	0.33318	43.998

The dual primal relations are

$$2x_1 = w_{01}d(w);$$

$$5x_2 = w_{02}d(w);$$

$$2x_3 = w_{03}d(w);$$

$$x_4 = w_{04}d(w);$$

$$0.5x_1^{-1}x_2^{-1}x_3^{-1} = w_{05}d(w);$$

$$x_1^2x_4^{-2} = \frac{w_{11}}{w_{11}+w_{12}};$$

$$x_2^2x_4^{-2} = \frac{w_{12}}{w_{11}+w_{12}};$$

$$100x_1^{-1}x_2^{-1}x_3^{-1} = \frac{w_{21}}{w_{21}};$$

The primal variables and the corresponding minimum value of primal objective are given in the following table:

Table-2: Primal Solution

x_1^*	x_2^*	x_3^*	x_4^*	$g_0^*(x)$
5.08405	2.68255	7.33232	5.74837	43.998

5. APPLICATION

A two-bar truss shown in Fig.1 is designed to support the loading condition Consider the following data Nodal load (P) =100 KN ; Volume density (γ)=7.7 KN/m^3 ; Length (l)=2000 mm ;Width(x_B)=1000 mm; Allowable tensile stress($[\sigma_t]$)=150 MPa ;Allowable compressive stress($[\sigma_c]$)=100 MPa ;Cross-sectional area of bar 1(A_1)= 0 $mm^2 \le A_1 \le 1000\,mm^2$;Cross-sectional area of bar 2(A_2)=0 $mm^2 \le A_2 \le 1000\,mm^2$;Y coordinate of node B(y_B)=500 $mm \le y_B \le 1500\,mm$;The structure is subject to constraints in geometry, area, stress [9]. The maximum tensile stress is restricted to 150MPa, while the maximum compressive stress is restricted to 100MPa. The three design variables are A_1, A_2 and y_B. Obviously, this is minimization problem.

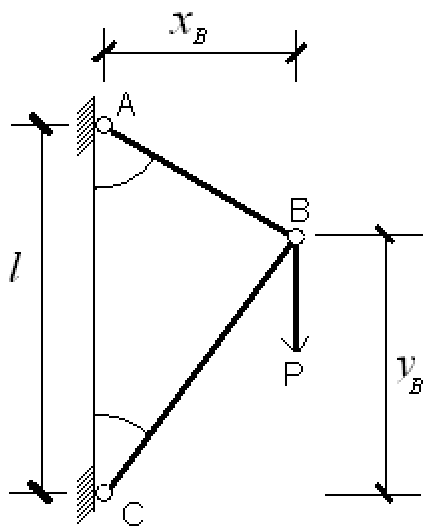

Figure-1: Design of the two-bar planar truss

The Optimization model of the two-bar truss is as follows:

$$\left.\begin{aligned}
&\min W = \gamma\left(A_1\sqrt{x_B^2 + (l - y_B)^2} + A_2\sqrt{x_B^2 + y_B^2}\right) \\
&\textit{subject to.}\quad \frac{P\sqrt{x_B^2 + (l - y_B)^2}}{lA_1} \le [\sigma_t]; \\
&\qquad\qquad \frac{P\sqrt{x_B^2 + y_B^2}}{lA_2} \le [\sigma_c]; \\
&\qquad\qquad 0.5 \le y_B \le 1.5 \qquad A_1 > 0; A_2 > 0;
\end{aligned}\right\} \tag{5.1}$$

Now this optimization model is not in standard form of geometric programming model. First we transfer it into the standard geometric programming problem with suitable substitution $A_1 = x_1$, $A_2 = x_2$, $\sqrt{1+(2-y_B)^2} \le x_3$, $\sqrt{1+y_B^2} \le x_4$, $y_B = x_5$, $1+4x_3^{-2}x_5 = x_6$, Then the new form of posynomial Geometric Programming (GP) Problem is;

$$\left.\begin{aligned}
&Minimize\ \ W = 7.7x_1x_3 + 7.7x_2x_4 \\
&subject\ to\ \ \frac{1}{3}x_1^{-1}x_3 \le 1 \\
&\frac{1}{2}x_2^{-1}x_4 \le 1 \\
&x_4^{-2} + x_4^{-2}x_5^2 \le 1 \\
&5x_3^{-2}x_6^{-1} + x_3^{-2}x_5^2x_6^{-1} \le 1 \\
&x_6^{-1} + 4x_3^{-2}x_5x_6^{-1} = 1 \\
&0.5 \le x_5 \le 1.5 \\
&x_1, x_2, x_3, x_4, x_5, x_6 > 0
\end{aligned}\right\} \qquad (5.2)$$

When the constraint $0.5 \le x_5 \le 1.5$ of (5.2) is excluded, then (5.2) is a constrained posynomial geometric programming problem with degree of difficulty = 10-(6+1) =3. The problem is solved via dual programming.

The corresponding dual of geometric programming (DGP) problem is:

$$Maximize\ d(w) = \left(\frac{7.7}{w_{01}}\right)^{w_{01}}\left(\frac{7.7}{w_{02}}\right)^{w_{02}}\left(\frac{1}{3w_{11}}\right)^{w_{11}}\left(\frac{1}{2w_{21}}\right)^{w_{21}}\left(\frac{1}{w_{31}}\right)^{w_{31}}\left(\frac{1}{w_{32}}\right)^{w_{32}}\left(w_{31}+w_{32}\right)^{w_{31}+w_{32}}$$

$$\times\left(\frac{5}{w_{41}}\right)^{w_{41}}\left(\frac{1}{w_{42}}\right)^{w_{42}}\left(w_{41}+w_{42}\right)^{w_{41}+w_{42}}\left(\frac{1}{w_{51}}\right)^{w_{51}}\left(\frac{4}{w_{52}}\right)^{w_{52}}\left(w_{51}+w_{52}\right)^{w_{51}+w_{52}}$$

Subject to $w_{01} + w_{02} = 1$

For primal variable x_1

$$w_{01} - w_{11} = 0$$

For primal variable x_2

$$w_{02} - w_{21} = 0$$

For primal variable x_3

$$w_{01} + w_{11} - 2w_{41} - 2w_{42} - 2w_{52} = 0$$

For primal variable x_4

$$w_{02} + w_{21} - 2w_{31} - 2w_{32} = 0$$

For primal variable x_5

$$2w_{32}+2w_{42}+w_{52}=0$$

For primal variable x_6

$$w_{41}+w_{42}+w_{51}+w_{52}=0$$

The dual primal relations are

$$7.7x_1x_3=w_{01}d(w)$$

$$7.7x_2x_4=w_{02}d(w)$$

$$\frac{1}{3}x_1^{-1}x_3=\frac{w_{11}}{w_{11}}$$

$$\frac{1}{2}x_2^{-1}x_4=\frac{w_{21}}{w_{21}}$$

$$x_4^{-2}=\frac{w_{31}}{w_{31}+w_{32}}$$

$$x_4^{-2}x_5^2=\frac{w_{32}}{w_{31}+w_{32}}$$

$$5x_3^{-2}x_6^{-1}=\frac{w_{41}}{w_{41}+w_{42}}$$

$$x_3^{-2}x_5^2x_6^{-1}=\frac{w_{42}}{w_{41}+w_{42}}$$

$$x_3^{-2}x_5^2x_6^{-1}=\frac{w_{42}}{w_{41}+w_{42}}$$

$$x_6^{-1}=\frac{w_{51}}{w_{51}+w_{52}}$$

$$4x_3^{-2}x_5x_6^{-1}=\frac{w_{52}}{w_{51}+w_{52}}$$

Solving above equations we get optimal solution of primal variables $x_1^*=0.52068, x_2^*=0.640312, x_3^*=1.56205, x_4^*=1.280625, x_5^*=0.80, x_6^*=2.31147$ and $W=125.7667$. It is noted that $x_5=0.8\in[0.5,1.5]$

We get the optimal values of Cross-sectional area of bar '1' $A_1^* = x_1^* = 520.68\ mm^2$, Cross-sectional area of bar '2' $A_2^* = x_2^* = 640.31\ mm^2$, Y coordinate of node B $y_B^* = x_5^* = 0.80\,m$ and $W^* = 125.7667N$.

This parametric model of the two bar planer truss is built in First order method in software ANSYS 10.0.The solving results are as follows:

Cross-sectional area of bar 1(A_1^*) $= 497.9\,mm^2$, Cross-sectional area of bar 2(A_2^*) = $671.5\,mm^2$, Y coordinate of node B (y_B^*) $= 0.89\ m$ and $W^* = 126.46N$.

This parametric model of the two bar planer truss is built in the MATLAB genetic algorithm toolbox .The solving result are as follows:

Cross-sectional area of bar 1(A_1^*) $= 520\ mm^2$, Cross-sectional area of bar 2(A_2^*) = $680\,mm^2$, Y coordinate of node B (y_B^*) $= 0.73\ m$ and $W^* = 128.1N$.

A comparison of the results between geometric programming problem (GP) method and other algorithms mentioned before is presented in table 3.

Table-3: Comparison of the results for the two-bar planer truss problem

Algorithm	**Design variable** $A_1(mm^2)$	**Design variable** $A_2(mm^2)$	**Y coordinate of node B** $Y_B(m)$	**Weight** $W(N)$
geometric programming (GP)	520.68	640.31	0.80	125.7667
MATLAB genetic algorithm toolbox (MGA)	520	680	0.73	128.1
First order method in ANSYS (FOMA)	497	671	0.89	126.46

It can be seen from the table-3. that the first–order method in ANSYS gives better results than that of the genetic algorithm native to MATLAB, but Geometric Programming (GP) method yields better result than that of the first–order method in

ANSYS and the genetic algorithm native to MATLAB.The chart of the comparison of results obtained by different algorithms is shown in Figure-2 .

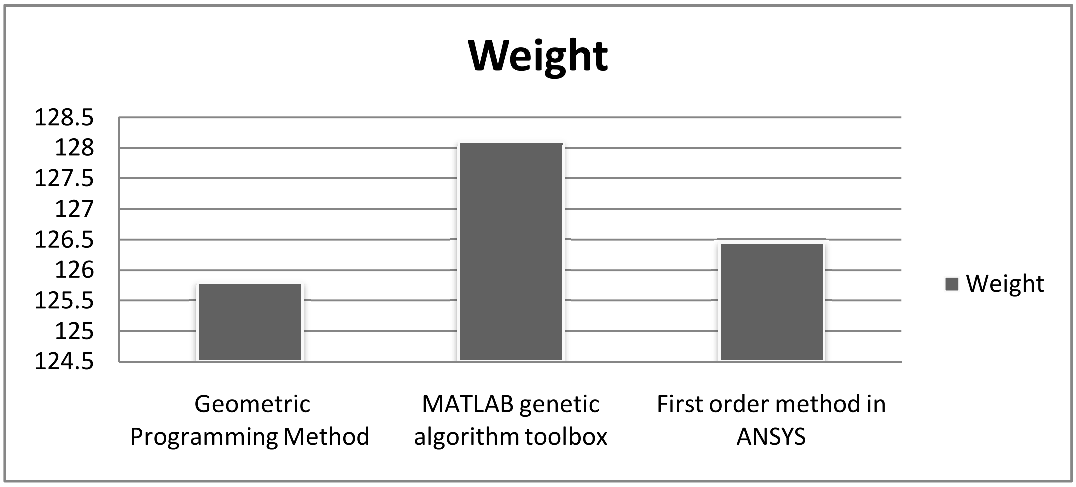

Figure-2: Comparison of the results under different methods

Conclusion: The successful results that are obtained in this study by GP solving method will contribute to further studies whenever the reliability of the structure is specified with respect to several criteria such as deflection, buckling and natural frequency of vibration.

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