

Operations And Structural Properties of Fuzzy Signed Graphs

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Abstract - Fuzzy signed graphs have garnered significant attention in various fields such as social network analysis, decision-making processes, and system modelling due to their ability to represent complex relationships that involve both uncertainty and sign (positive or negative). This paper explores the operations on fuzzy signed graphs, focusing on the structural properties and mathematical foundations that underlie their manipulation. We propose novel operations, including union, intersection, Join within the context of fuzzy signed graphs, and provide detailed algorithms for each operation. Additionally, we analyze the impact of these operations on key graph-theoretic parameters such as connectivity, reachability, and the stability of the graph. Furthermore, we explore how these operations can be applied to real-world problems, such as modeling social influence, predicting interactions, and evaluating consensus in uncertain environments. The study highlights the flexibility of fuzzy signed graphs in capturing nuanced relationships and contributes to the development of more robust frameworks for handling uncertain and signed data in network-based analysis.

Keywords - Fuzzy signed graph, FSG union, FSG intersection, FSG join.

1.BASIC DEFINITIONS

1.1 Fuzzy Graph

Let $G(V, E, \mu)$ be a fuzzy graph corresponding to the Crisp graph G^* is a non-empty set V together with the pair of Functions $\mu: V \rightarrow [0,1]$ and $E: V \times V \rightarrow [0,1]$ such that for all $u, v \in V$, $E(u, v) \leq \min[\mu(u), \mu(v)]$.

1.2. Degree of vertex

The degree of a vertex v of the FG G is defined as $deg_G(v) = \sum_{u \neq v} E(u, v)$.

1.3. Signed graph:

Let $\Sigma(v, E, \sigma)$ be a signed graph is a graph in which each edge has a positive (or) negative sign

1.4. Degree of signed graph

The difference between the positive degree and negative at a vertex is denoted by $sdeg(v) = d^+(v) - d^-(v)$.

1.5 Fuzzy signed graph

Let μ be the SG it has ordered triple (v, δ, μ) where v is the set of all vertices V and δ is the set of all positive and negative edges and μ is the membership values of vertices and edges and also satisfying the condition $\mu(\delta^+) = \mu(u) \vee \mu(v)$, $\mu(\delta^-) = \mu(u) \wedge \mu(v)$.

1.6. Degree of fuzzy signed graph:

Let μ be the fuzzy signed graph the degree of vertex u is sum of the membership values of positive edges incident on u subtract sum of the membership value of negative edges incident on u .

2. Operations and Properties of Fuzzy Signed Graph

2.1 Union:

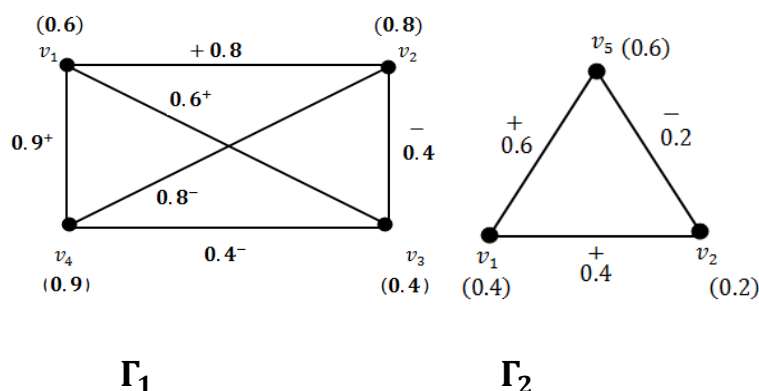
Let $\Gamma_1(V_1, \delta_1, \mu_1)$ and $\Gamma_2(V_2, \delta_2, \mu_2)$ are two fuzzy signed graph. Then the union of fuzzy signed graph is denoted by

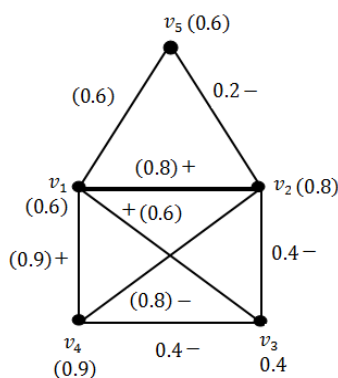
$$(\Gamma_1 \cup \Gamma_2)(V, \delta, \mu) = (\Gamma_1 \cup \Gamma_2)(V) = \Gamma_1(V_1) \cup \Gamma_2(V_2)$$

$$\mu(\Gamma_1 \cup \Gamma_2)(u) = \begin{cases} \mu_1(u) & u \in V_1 \\ \mu_2(u) & u \in V_2 \\ \mu_1(u) \vee \mu_2(u) & u \in V_1 \cap V_2 \end{cases}$$

$$\mu(\Gamma_1 \cup \Gamma_2)(uv) = \begin{cases} \mu_1(uv) & uv \in \delta_1 \\ \mu_2(uv) & uv \in \delta_2 \\ \mu_1(uv) \vee \mu_2(uv) & uv \in \delta_1 \cap \delta_2 (uv - \text{positive}) \\ \mu_1(uv) \wedge \mu_2(uv) & uv \in \delta_1 \cap \delta_2 (uv - \text{negative}) \end{cases}$$

and also must sign of the common edges are same.





$\Gamma_1 \cup \Gamma_2$

2.2 Definition

Let $\Gamma_1(v_1, \delta_1, \mu_1)$ and $\Gamma_2(v_2, \delta_2, \mu_2)$ be a two fuzzy signed graph then the union of two FSG is $\Gamma = (\Gamma_1 \cup \Gamma_2) = (v_1 \cup v_2, \delta_1 \cup \delta_2, \mu_1 \cup \mu_2)$. Then the degree of vertex x in $v_1 \cup v_2$ is denoted by $\deg_{(\Gamma_1 \cup \Gamma_2)}(u)$ is determined by the Common Edge is positive

$$\begin{aligned} \deg_{\Gamma_1 \cup \Gamma_2}(u) &= \sum \mu(\Gamma_1 \cup \Gamma_2)(u, v) \\ &= \sum_{uv \in \delta_1 - \delta_2} (\mu_1^+(uv) - \mu_1^-(uv)) + \sum_{uv \in \delta_2 - \delta_1} \mu_2^+(uv) - \mu_2^-(uv) \\ &\quad + \sum_{uv \in \delta_1 \cap \delta_2} \mu_1(uv) \vee \mu_2(uv) \end{aligned}$$

The Common Edge is negative

$$\begin{aligned} \deg_{(\Gamma_1 \cup \Gamma_2)}(u) &= \sum \mu(\Gamma_1 \cap \Gamma_2)(uv) \\ &= \sum_{uv \in \delta_1 - \delta_2} (\mu_1^+(uv) - \mu_1^-(uv)) + \sum_{uv \in \delta_2 - \delta_1} \mu_2^+(uv) - \mu_2^-(uv) \\ &\quad - \sum_{uv \in \delta_1 \cap \delta_2} \mu_1(uv) \wedge \mu_2(uv) \end{aligned}$$

2.3 Theorem:

Let $\Gamma_1(v_1, \delta_1, \mu_1)$ and $\Gamma_2(v_2, \delta_2, \mu_2)$ be a two FSG then

The common edge is positive

$$\deg_{\Gamma_1 \cup \Gamma_2}(u) = \deg_{\Gamma_1}(u) + \deg_{\Gamma_2}(u) - \sum_{uv \in \delta_1 \cap \delta_2} \mu_1(uv) \wedge \mu_2(uv)$$

The Common edge is negative

$$\deg_{\Gamma_1 \cup \Gamma_2}(u) = \deg_{\Gamma_1}(u) + \deg_{\Gamma_2}(u) + \sum_{uv \in \delta_1 \cap \delta_2} \mu_1(uv) \vee \mu_2(uv)$$

Proof

The common edge is positive from the definition

$$\begin{aligned} \deg_{\Gamma_1 \cup \Gamma_2}(u) &= \sum_{uv \in \delta_1 - \delta_2} (\mu_1^+(uv) - \mu_1^-(uv)) + \sum_{uv \in \delta_1 - \delta_2} \mu_2^+(uv) - \mu_2^-(uv) \\ &\quad + \sum_{uv \in \delta_1 \cap \delta_2} \mu_1(uv) \vee \mu_2(uv) \\ &= \sum_{uv \in \delta_1 - \delta_2} (\mu_1^+(uv) - \mu_1^-(uv)) + \sum_{uv \in \delta_1 - \delta_2} (\mu_2^+(uv) - \mu_2^-(uv)) \\ &\quad + \sum_{uv \in \delta_1 \cap \delta_2} \mu_1(uv) + \sum_{uv \in \delta_1 \cap \delta_2} \mu_2(uv) \\ &\quad - \sum_{uv \in \delta_1 \cap \delta_2} \mu_1(uv) \wedge \mu_2(uv) \\ &= \deg_{\Gamma_1}(u) + \deg_{\Gamma_2}(u) - \sum_{uv \in \delta_1 \cap \delta_2} \mu_1(uv) \wedge \mu_2(uv) \end{aligned}$$

($\therefore$ Common edge is positive so negative edge membership value 0)

The common edge is negative.

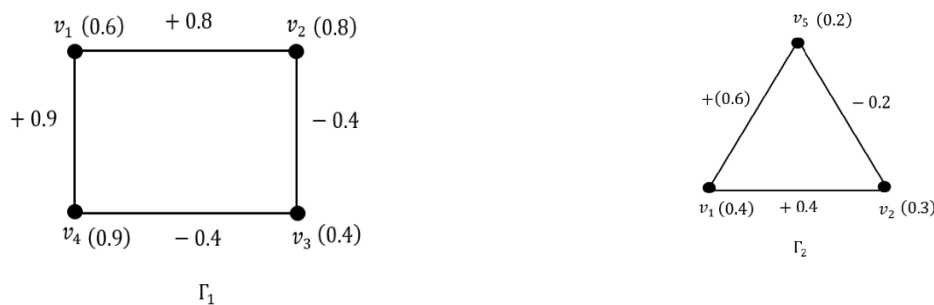
From the definition

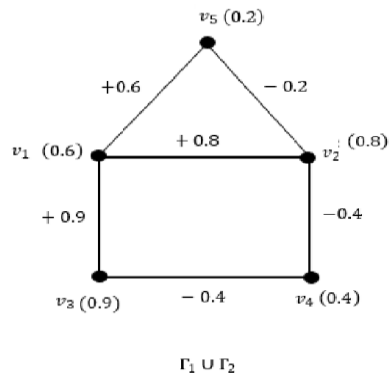
$$\begin{aligned} \deg_{\Gamma_1 \cup \Gamma_2}(u) &= \sum_{uv \in \delta_1 - \delta_2} (\mu_1^+(uv) - \mu_1^-(uv)) + \sum_{uv \in \delta_2 - \delta_1} \mu_2^+(uv) - \mu_2^-(uv) \\ &\quad - \sum_{uv \in \delta_1 \cap \delta_2} \mu_1(uv) \wedge \mu_2(uv) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{uv \in \delta_1 - \delta_2} (\mu_1^+(uv) - \mu_1^-(uv)) + \sum_{uv \in \delta_2 - \delta_1} \mu_2^+(uv) - \mu_2^-(uv) \\
 &\quad - \left(\sum_{uv \in \delta_1 \cap \delta_2} \mu_1(uv) + \sum_{uv \in \delta_1 \cap \delta_2} \mu_2(uv) \right. \\
 &\quad \left. - \sum_{uv \in \delta_1 \cap \delta_2} \mu_1(uv) \vee \mu_2(uv) \right) \\
 &= \sum_{uv \in \delta_1 - \delta_2} (\mu_1^+(uv) - \mu_1^-(uv)) + \sum_{uv \in \delta_2 - \delta_1} \mu_2^+(uv) - \mu_2^-(uv) \\
 &\quad - \sum_{uv \in \delta_1 \cap \delta_2} \mu_1(uv) - \sum_{uv \in \delta_1 \cap \delta_2} \mu_2(uv) \\
 &\quad + \sum_{uv \in \delta_1 \cap \delta_2} \mu_1(uv) \vee \mu_2(uv) \\
 &= \deg_{\Gamma_1}(u) + \deg_{\Gamma_2}(u) - \sum_{uv \in \delta_1 \cap \delta_2} \mu_1(uv) \vee \mu_2(uv)
 \end{aligned}$$

($\therefore$ common edge is negative so positive edge membership value 0)

2.4 Example

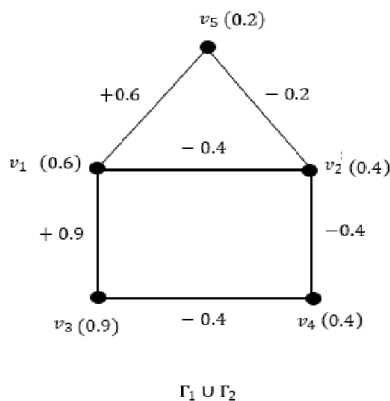
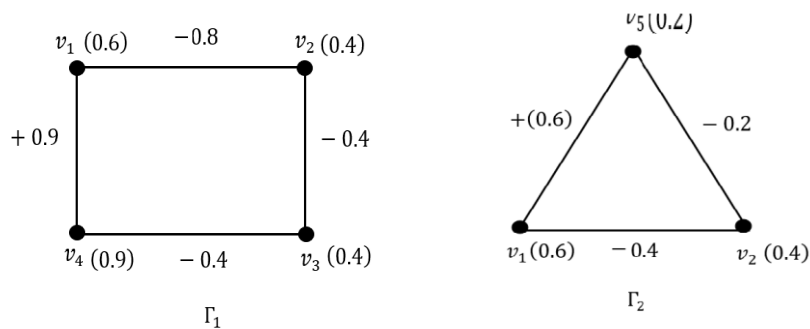




$$\deg(v_1) = 2.3$$

From definition:

$$\begin{aligned} \deg_{\Gamma_1} + \deg_{\Gamma_2} - \sum_{uv \in \delta_1 \cap \delta_2} \mu_1(uv) \wedge \mu_2(uv) \\ = 1.7 + 1.0 - 0.4 \\ = 2.3 \end{aligned}$$



$$\deg_{\Gamma_1 \cup \Gamma_2}(v_1) = 1.1$$

From definition:

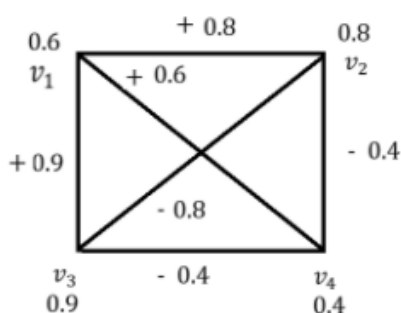
$$\begin{aligned}
 °_{\Gamma_1} + deg_{\Gamma_2} + \sum_{uv \in \delta_1 \cap \delta_2} (\mu_1(uv) \vee \mu_2(uv)) \\
 &= 0.1 + 0.2 + 0.8 \\
 &= 1.1
 \end{aligned}$$

2.5 Intersection of two fuzzy signed graph:

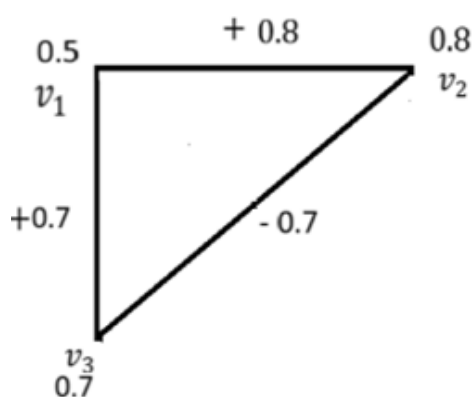
Let $\Gamma_1(v_1, \delta_1, \mu_1)$ and $\Gamma_2(v_2, \delta_2, \mu_2)$ is fuzzy two signed then the intersection of fuzzy signed graph is $(\Gamma_1 \cap \Gamma_2)(v, \delta, \mu)$ is

$$\mu(\Gamma_1 \cap \Gamma_2)(u) = \mu_1(u) \wedge \mu_2(u) \quad u \in v_1 \cap v_2$$

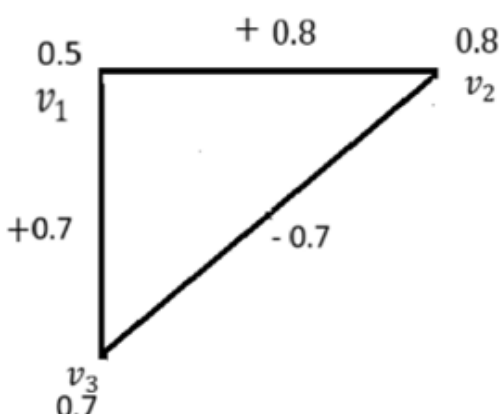
$$\mu(\Gamma_1 \cap \Gamma_2)(uv) = \mu_1(uv) \wedge \mu_2(uv) \quad uv \in \delta_1 \cap \delta_2$$



Γ_1



Γ_2



$\Gamma_1 \cap \Gamma_2$

2.6 Join of fuzzy signed graph

Let $\Gamma_1(v_1, \delta_1, \mu_1)$ and $\Gamma_2(v_2, \delta_2, \mu_2)$ be two fuzzy signed graphs. The join between them is defined as $(\Gamma_1 + \Gamma_2)(v, \delta, \mu)$ where one set of vertices, edges are given by

$$(\Gamma_1 + \Gamma_2)(u) = V_1 \cup V_2$$

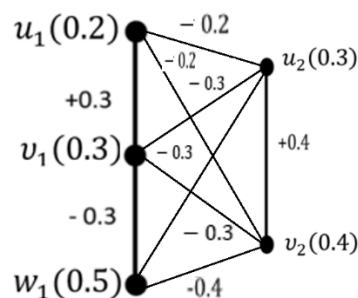
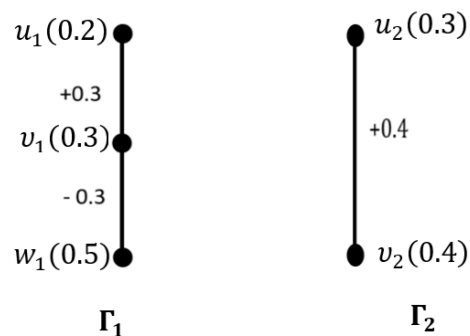
$$\mu(\Gamma_1 + \Gamma_2)(u) = \begin{cases} \mu_1(u) & u \in V_1 \\ \mu_2(u) & u \in V_2 \\ \mu_1(u) \vee \mu_2(u) & u \in V_1 \cap V_2 \end{cases}$$

$$(\Gamma_1 + \Gamma_2)(\delta v) = \delta_1 \cup \delta_2 \cup \delta^*$$

Where δ^* is the set of all edges joining vertices of V_1 with vertices of V_2 such that

$$\mu(\Gamma_1 + \Gamma_2)(uv) = \begin{cases} \mu_1(uv) & uv \in \delta_1 \\ \mu_2(uv) & uv \in \delta_2 \\ \mu_1(u) \wedge \mu_2(u) & uv \in \delta^* \end{cases}$$

The sign of the new edges is negative because we have to take the minimum value of the corresponding vertices



$$\Gamma_1 + \Gamma_2$$

2.7 Degree of a vertex in Join:

Here $V_1 \cap V_2 = \phi$, hence $\delta_1 \cap \delta_2 = \phi$

$$\begin{aligned} (\mu_1 \cup \mu_2)uv &= \mu_1(uv) \quad uv \in \delta_1 \\ &= \mu_2(uv) \quad uv \in \delta_2 \end{aligned}$$

By definition

$$\deg_{(\Gamma_1 + \Gamma_2)}(u) = \sum_{uv \in \delta_1 \cup \delta_2} (\mu_1 \cup \mu_2)(uv) - \sum_{uv \in \delta^*} \mu_1(u) \wedge \mu_2(v)$$

For any $u \in V_1$

$$\begin{aligned} \deg_{(\Gamma_1 + \Gamma_2)}(u) &= \sum_{uv \in \delta_1} \mu_1(uv) - \sum_{uv \in \delta^*} \mu_1(u) \wedge \mu_2(v) \\ &= \sum_{uv \in \delta_1} (\mu_1^+(uv) - \mu_1^-(uv)) - \sum_{uv \in \delta^*} \mu_1(u) \wedge \mu_2(v) \\ &= \deg_{\Gamma_1}(u) - \sum_{v \in V_2} \mu_1(u) \wedge \mu_2(v) \end{aligned}$$

For any $u \in V_2$

$$\deg_{\Gamma_1 + \Gamma_2}(u) = \deg_{\Gamma_2}(u) - \sum_{v \in V_2} \mu_1(u) \wedge \mu_2(v)$$

2.8. Theorem

Let $\Gamma_1(v_1, \delta_1, \mu_1)$ and $\Gamma_2(v_2, \delta_2, \mu_2)$ be two fuzzy graphs

- i. if $\mu_1(u) > \mu_2(u)$ then $d_{\Gamma_1 + \Gamma_2}(u) = d_{\Gamma_1}(u) - O(\Gamma_2)$ if $u \in V_1$
 $= d_{\Gamma_2}(u) - \rho_1(u)$ if $u \in V_2$
- ii. if $\mu_2(u) \geq \mu_1(u)$ then $d_{(\Gamma_1 + \Gamma_2)}(u) = d_{\Gamma_1}(u) - \rho_2\gamma_1(u)$ if $u \in V_1$
 $= d_{\Gamma_2}(u) - O(\Gamma_1)$ if $u \in V_2$

Proof:

i. We have $\mu_1(u) \geq \mu_2(u)$.

For any $u \in V_1$

$$\begin{aligned} \deg_{(\Gamma_1 \cup \Gamma_2)}(u) &= \deg_{\Gamma_1}(u) - \sum_{v \in V_2} \mu_1(u) \wedge \mu_2(v) \\ &= \deg_{\Gamma_1}(u) + \mu_2(v) \\ &= d_{\Gamma_1}(u) - O(\Gamma_2) \end{aligned}$$

For any $u \in V_2$

$$\begin{aligned} \deg_{(\Gamma_1 \cup \Gamma_2)}(u) &= \deg_{\Gamma_2}(u) - \sum_{v \in V_1} \mu_1(u) \wedge \mu_2(v) \\ &= \deg_{\Gamma_2}(u) - \sum_{v \in V_1} \mu_2(v) \\ &= \deg_{\Gamma_2}(u) - \mu_2(v) \sum_{v \in V_1} 1 \\ &= d_{\Gamma_1}(u) - \rho_1 \mu_2(v) \end{aligned}$$

(ii) proof is similar to the proof (i).

2.9 Theorem

Let $\Gamma_1(v_1, \delta_1, \mu_1)$ and $\Gamma_2(v_2, \delta_2, \mu_2)$ be two fuzzy graphs such that $\mu_1 \wedge \mu_2$ is constant function then $\deg_{\Gamma_1 + \Gamma_2}(u) = \deg_{\Gamma_1}(u) - C\rho_2$ if $u \in V_1$

$$= \deg_{\Gamma_2}(u) - C\rho_1 \text{ if } u \in V_2$$

where $\mu_1 \wedge \mu_2 = C$ is constant for every $u \in V_1$ and $u \in V_2$.

Proof:

Let $\mu_1(u) \wedge \mu_2(u) = C$ for every $u \in V_1$ and $u \in V_2$

where C is constant.

For any $u \in V_1$

$$\begin{aligned}
 \deg_{\Gamma_1 \cup \Gamma_2}(u) &= \deg_{\Gamma_1}(u) - \sum_{v \in V_2} \mu_1(u) \wedge \mu_2(v) \\
 &= \deg_{\Gamma_2}(u) - \sum_{v \in V_2} C \\
 &= d_{\Gamma_1}(u) - \rho_2 C
 \end{aligned}$$

For any $u \in V_2$

$$\begin{aligned}
 \deg_{\Gamma_1 \cup \Gamma_2}(u) &= \deg_{\Gamma_2}(u) - \sum_{v \in V_1} \mu_1(u) \wedge \mu_2(v) \\
 &= \deg_{\Gamma_2}(u) - \sum_{v \in V_1} C \\
 &= d_{\Gamma_2}(u) - \rho_1 C
 \end{aligned}$$

CONCLUSION:

In this paper we have to found the operations of fuzzy signed graphs such as Union, Intersection, Join and the degree of vertices of these operations under some conditions and illustrated them through Examples.

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