ONE MODULO $N$ GRACEFULNESS OF $n$-POLYGONAL SNAKES, $C_n^{(t)}$ AND $P_{a,b}$

V.Ramachandran$^1$ C.Sekar$^2$

$^1$ Department of Mathematics, P.S.R Engineering College (Affiliated to Anna University Chennai), Sevalpatti, Tamil Nadu, India.

$^2$ Department of Mathematics, Aditanar College of Arts and Science (Affiliated to MS University Tirunelveli), Tiruchendur, Tamil Nadu, India.

Abstract

A function $f$ is called a graceful labelling of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the set $\{0, 1, 2, \ldots, q\}$ such that, when each edge $xy$ is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. A graph $G$ is said to be one modulo $N$ graceful (where $N$ is a positive integer) if there is a function $\phi$ from the vertex set of $G$ to $\{0, 1, N, (N + 1), 2N, (2N + 1), \ldots, N(q - 1), N(q - 1) + 1\}$ in such a way that (i) $\phi$ is 1-1 (ii) $\phi$ induces a bijection $\phi^*$ from the edge set of $G$ to $\{1, N + 1, 2N + 1, \ldots, N(q - 1) + 1\}$ where $\phi^*(uv) = |\phi(u) - \phi(v)|$. In this paper we prove that the $n$-Polygonal snakes, $C_n^{(t)}$ and $P_{a,b}$ are one modulo $N$ graceful for all positive integers $N$.

Keywords: Graceful, modulo $N$ graceful, $n$-Polygonal snakes, $C_n^{(t)}$ and $P_{a,b}$.

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1 Introduction

S.W.Golomb [1] introduced graceful labelling. Odd gracefulness was introduced by R.B.Gnanajothi [2]. C.Sekar [6] introduced one modulo three graceful labelling. In this paper we introduce the concept of one modulo $N$ graceful where $N$ is any positive integer. In the case $N = 2$, the labelling is odd graceful and in the case $N = 1$ the labelling is graceful. We prove that the $n$-Polygonal snakes, $C_n^{(t)}$ and $P_{a,b}$ are one modulo $N$ graceful for all positive integers $N$.

2 Main Results

Definition 2.1. A graph $G$ with $q$ edges is said to be one modulo $N$ graceful (where $N$ is a positive integer) if there is a function $\phi$ from the vertex set of $G$ to $\{0, 1, N, (N + 1), 2N, (2N + 1), \ldots, N(q - 1), N(q - 1) + 1\}$ in such a way that (i) $\phi$ is 1-1 (ii) $\phi$ induces a bijection $\phi^*$ from the edge set of $G$ to $\{1, N + 1, 2N + 1, \ldots, N(q - 1) + 1\}$ where $\phi^*(uv) = |\phi(u) - \phi(v)|$.

Definition 2.2. Consider $k$ copies of path $P_n$. An $n$-Polygonal snake containing $k$ number of $n$-Polygons is obtained from a path $v_1, v_2, \ldots, v_k$ by identifying the pendant vertices of $i$th copy of the path $P_n$ with $v_{i-1}$ and $v_i$ for $i = 1, 2, \ldots, k$.

Definition 2.3. The one point union of $t$ cycles of length $n$ is denoted by $C_n^{(t)}$. This graph has $(t(n-1)+1)$ vertices and $tn$ edges.

Definition 2.4. Let $u$ and $v$ be two fixed vertices. We connect $u$ and $v$ by means of "b" internally disjoint paths of length "a" each. The resulting graph is denoted by $P_{a,b}$.

Theorem 2.5. $n$-Polygonal snakes for $n \equiv 0(\text{mod}4)$ are one modulo $N$ graceful for every positive integer $N$. 
Proof: Let \( n = 4r, r \geq 1 \). Let there be \( k \) polygons. This graph has \( 4rk - (k - 1) \) vertices and \( 4rk \) edges. For \( 1 \leq j \leq k \) let \( u_{i}^{(j)}, j = 1, 2, \ldots, 4r \) be the vertices of \( i \)th polygon. Identify \( u_{1}^{(4r)} \) with \( u_{1}^{(1)} \), \( u_{2}^{(4r)} \) with \( u_{3}^{(1)} \), and so on.

Define

\[
\phi(u_{2i-1}) = \phi(u_{2i-1}^{(4r)}) = 4Nr(i - 1) \quad \text{for} \quad i = 1, 2, 3, 4, \ldots,
\]

\[
\phi(u_{2i}^{(1)}) = \phi(u_{2i}^{(4r)}) = 4Nr - (N - 1) - 2Nr + N - 4Nr(i - 1) \quad \text{for} \quad i = 1, 2, 3, 4, \ldots,
\]

\[
\phi(u_{2i-1}^{(j)}) = 4Nr - (N - 1) - \frac{N(j-2)}{2} - 4Nr(i - 1) \quad \text{for} \quad j = 2, 4, 6, 8, \ldots, 4r - 2 \quad \text{and} \quad i=1,2,3,4,\ldots
\]

\[
\phi(u_{2i}^{(j)}) = N + \frac{N(j-3)}{2} + 4Nr(i - 1) \quad \text{for} \quad j = 3, 5, 7, \ldots, 2r - 1 \quad \text{and} \quad i=1,2,3,4,\ldots
\]

\[
\phi(u_{2i-1}^{(j)}) = 2N + \frac{N(j-3)}{2} + 4Nr(i - 1) \quad \text{for} \quad j = 2r + 1, 2r + 3, \ldots, 4r - 1 \quad \text{and} \quad i=1,2,3,4,\ldots
\]

\[
\phi(u_{2i}^{(j)}) = N(2r + 1) + \frac{N(j-2)}{2} + 4Nr(i - 1) \quad \text{for} \quad j = 2, 4, 6, 8, \ldots, 4r - 2 \quad \text{and} \quad i=1,2,3,4,\ldots
\]

\[
\phi(u_{2i}^{(j)}) = 4Nr - (N - 1) - 2Nr - \frac{N(j-3)}{2} - 4Nr(i - 1) \quad \text{for} \quad j = 2r + 1, 2r + 3, \ldots, 4r - 1 \quad \text{and} \quad i=1,2,3,4,\ldots
\]

Clearly \( \phi \) is \( 1 \mod 4 \) and \( \phi \) defines a one modulo \( N \) graceful labelling of the \( n \)-Polygonal snakes for \( n \equiv 0(\mod 4) \).

Example 2.6. Graceful labelling of the 12-Polygonal snake. (No.of polygons = 5)

Example 2.7. Odd graceful labelling of the 8-Polygonal snake. (No.of polygons = 9)

Example 2.8. One modulo 4 graceful labelling of the 4-Polygonal snake. (No.of polygons = 4)
Theorem 2.9. \(n\)-Polygonal snakes for \(n \equiv 2(\text{mod} \ 4)\) are one modulo \(N\) graceful for every positive integer \(N\) if the number of polygons is even.

Proof: Let \(n = 4r + 2\) and let there be \(k = 2s\) polygon. This graph has \(4rk - (k - 1)\) vertices and \(4rk\) edges. For \(1 \leq j \leq k\) let \(u_i^{(j)}, j = 1, 2, \ldots 4r\) be the vertices of \(i\)th polygon. Identify \(u_1^{(4r)}\) with \(u_2^{(4r)}\), \(u_3^{(4r)}\) with \(u_1^{(4r)}\), and so on.

Define
\[
\phi(u_1^{(1)}) = \phi(u_2^{(4r+2)}) = N(4r + 3)(i - 1) \quad \text{for} \quad i = 1, 2, 3, 4, \ldots, s
\]
\[
\phi(u_2^{(2)}) = \phi(u_2^{(4r+2)}) = N(4r + 2)k - (N - 1) - 4Nr - N(4r + 1)(i - 1) \quad \text{for} \quad i = 1, 2, 3, 4, \ldots, s
\]
\[
\phi(u_3^{(j)}) = N + \frac{N(j-3)}{2} + N(4r + 3)(i - 1) \quad \text{for} \quad j = 3, 5, 7, 9, \ldots, 4r + 1 \text{ and } i = 1, 2, 3, 4, \ldots, s
\]
\[
\phi(u_4^{(j)}) = N(4r+2)k - (N - 1) - \frac{N(j-2)}{2} - N(4r + 1)(i - 1) \quad \text{for} \quad j = 3, 5, 7, \ldots, 4r + 1 \text{ and } i = 1, 2, 3, 4, \ldots, s
\]

Clearly \(\phi\) is \(1 - 1\) and \(\phi\) defines a one modulo \(N\) graceful labelling of the \(n\)-Polygonal snakes for \(n \equiv 0(\text{mod} 2)\).

Example 2.10. One modulo 8 graceful labelling of the 10-Polygonal snake. (No.of polygons = 4)

Example 2.11. One modulo 5 graceful labelling of the 10-Polygonal snake. (No.of polygons = 6)
Example 2.12. Graceful labelling of the 6-Polygonal snake. (No. of polygons = 4)

![6-Polygonal snake diagram]

Theorem 2.13. Let $C_n^{(t)}$ denote the one point union of $t$ cycles of length $n$. $C_n^{(t)}$ is one modulo $N$ graceful when $n = 4, 8, t > 2$ and $n = 6, t$ even and $t \geq 4$ for every positive integer $N > 1$.

Proof: Case (i) $n = 4, t > 2$

For $1 \leq i \leq t$. Let $u_i^{(j)}, j = 1, 2, 3, 4$ be the vertices of the $i$th cycle with the one point identification of $u_i^{(1)}, u_i^{(2)}, \ldots, u_i^{(t)}$ at $u_0$.

Define

- $\phi(u_0) = 0$
- $\phi(u_i^{(j)}) = 4Nt - (N - 1) - \frac{N(j-2)}{2} - 2N(i - 1)$ for $j = 2, 4$ and $i = 1, 2, 3, 4, \ldots, t$
- $\phi(u_i^{(3)}) = 4Nt - 2N - 4N(i - 1)$ for $i = 1, 2, 3, 4, \ldots, t$

Clearly $\phi$ is $1 - 1$ and $\phi$ defines a one modulo $N$ graceful labelling of $C_n^{(t)}$ when $n = 4, t > 2$.

Example 2.14. One modulo 10 graceful labelling of $C_4^{(6)}$
Example 2.15. Odd graceful labelling of $C_4^{(4)}$

Case (ii) $n = 8, t > 2$

For $1 \leq i \leq t$. Let $u_{i}^{(j)}$, $j = 1, 2, \ldots, 8$ be the vertices of the $i$th cycle with the one point identification of $u_{1}^{(1)}, u_{2}^{(1)}, \ldots, u_{t}^{(1)}$ at $u_0$.

Define

$$
\phi(u_0) = 0
$$

$$
\phi(u_{i}^{(j)}) = \begin{cases} 
8Nt - (N - 1) - 2N(i - 1) - \frac{N(j - 2)}{2} & \text{if } i = 1, 2, 3, \ldots, t \text{ and } j = 2, 8 \\
6Nt - (N - 1) - 2N(i - 1) - \frac{N(j - 4)}{2} & \text{if } i = 1, 2, 3, \ldots, t \text{ and } j = 4, 6 \\
4Nt - N - 4N(i - 1) - \frac{2N(j - 3)}{4} & \text{if } i = 1, 2, 3, \ldots, t \text{ and } j = 3, 7 \\
6Nt - 2N - 4N(i - 1) & \text{if } i = 1, 2, 3, \ldots, t \text{ and } j = 5
\end{cases}
$$

Clearly $\phi$ is $1 - 1$ and $\phi$ defines a one modulo $N$ graceful labelling of $C_n^{(t)}$ when $n = 8, t > 2$.

Example 2.16. Odd graceful labelling of $C_8^{(4)}$
Example 2.17. One modulo 7 graceful labelling of $C_8^{(3)}$

Case (iii) $n = 6$, $t$ is even $t \geq 4$ let $t = 2s$

For $1 \leq i \leq 2s$. Let $u_i^{(j)}$, $j = 1, 2, \ldots, 6$ be the vertices of the $i$th cycle with the one point identification of $u_1^{(1)}, u_2^{(1)}, \ldots, u_t^{(1)}$ at $u_0$.

Define

$$
\phi(u_0) = 0
$$

$$
\phi(u_i^{(j)}) = \begin{cases} 
6Nt - (N-1) - 2N(i-1) - \frac{N(j-2)}{4} & \text{if } i = 1, 2, 3, \ldots, 2s \text{ and } j = 2, 6 \\
4Nt - N - 4N(i-1) - 2N(j-3) & \text{if } i = 1, 2, 3, \ldots, 2s \text{ and } j = 3, 5 \\
4Nt - (N-1) - \frac{4N(j-1)}{2} & \text{if } i = 1, 3, 5, \ldots, 2s - 1 \text{ and } j = 4 \\
4Nt - (4N-1) - \frac{4N(j-2)}{2} & \text{if } i = 2, 4, 6, \ldots, 2s \text{ and } j = 4 
\end{cases}
$$

Clearly $\phi$ is $1-1$ and $\phi$ defines a one modulo $N$ graceful labelling of $C_n^{(t)}$ when $n = 6, t$ is even and $t \geq 4.$
Example 2.18. One modulo 4 graceful labelling of $C_6^{(6)}$

Example 2.19. One modulo 5 graceful labelling of $C_6^{(4)}$

Theorem 2.20. $P_{a,b}$ for all $a \geq 2$ and for all odd $b$ is one modulo $N$ graceful for every positive integer $N$. Here $P_m$ is a path of length $m-1$.

Proof: Let $b = 2r + 1, r \geq 1$

Define

$$X(t) = \begin{cases} 1 & \text{if } t \leq r \\ 0 & \text{if } t > r \end{cases}$$

Define

$$\phi(u) = 0$$

$$\phi(v) = \begin{cases} \frac{Na(2r+1)}{2} & \text{if } a \text{ is even} \\ \frac{Na(b-1)}{2} + 1 & \text{if } a \text{ is odd} \end{cases}$$
For \( j = 1, 3, 5, \ldots \)
\[
\phi(v^{(i)}_j) = N(a(2r + 1) - 1) + 1 - N(i - 1) - (2r + 1)(j - 1) \quad \text{if } i = 1, 2, 3, \ldots, 2r + 1
\]

For \( j = 2, 4, 6, \ldots \)
\[
\phi(v^{(i)}_j) = X(i)\{2N(r + 1) + N(i - 1) + (2r + 1)(j - 2)\} + (1 - X(i))\{N + N(i - 1) + (2r + 1)(j - 2)\} \quad \text{if } i = 1, 2, 3, \ldots, 2r + 1
\]

Clearly \( \phi \) is 1 − 1 and \( \phi \) defines a one modulo \( N \) graceful labelling of \( P_{a,b} \) for all \( a \geq 2 \) and for all odd \( b \).

**Example 2.21.** One modulo 3 graceful labelling of \( P_{6,5} \)

**Example 2.22.** One modulo 4 graceful labelling of \( P_{5,7} \)

**Example 2.23.** Graceful labelling of \( P_{7,7} \)
Theorem 2.24. $P_{4,b}$ for all $b \geq 2$ is one modulo $N$ graceful for every positive integer $N$.

**Proof:**

Define

- $\phi(u) = 0$
- $\phi(v) = 2Nb$
- $\phi(v_{ij}^{(i)}) = N(ab - 1) + 1 - b(j - 1) - N(i - 1)$ for $i = 1, 2, \ldots, b$ and $j = 1, 3$
- $\phi(v_{ij}^{(2)}) = 2Nb - N - 2N(i - 1)$ for $i = 1, 2, \ldots, b$

Clearly $\phi$ is $1 - 1$ and $\phi$ defines a one modulo $N$ graceful labelling of $P_{4,b}$ for all $b \geq 2$

**Example 2.25.** One modulo 3 graceful labelling of $P_{4,9}$

**Example 2.26.** One modulo 5 graceful labelling of $P_{4,6}$
Example 2.27. Graceful labelling of $P_{4,4}$

\[
\begin{array}{c}
0 & 1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 & 9 \\
10 & 11 & 12 & 13 & 14 \\
15 & 16 & 17 & 18 & 19
\end{array}
\]

Theorem 2.28. $P_{2,b}$ for all $b \geq 2$ is one modulo $N$ graceful for every positive integer $N$.

Proof:
Define
\[\phi(u) = 0\]
\[\phi(v) = Nb\]
\[\phi(v_i) = 2Nb - (N - 1) - N(i - 1)\] for $i = 1, 2, \ldots, b$
Clearly $\phi$ is $1-1$ and $\phi$ defines a one modulo $N$ graceful labelling of $P_{2,b}$ for all $b \geq 2$.
Example 2.29. One modulo 6 graceful labelling of $P_{2,8}$

Example 2.30. Graceful labelling of $P_{2,6}$
Example 2.31. One modulo 8 graceful labelling of $P_{2,10}$

\begin{center}
\begin{tikzpicture}
\node at (0,0) {0};
\node at (0,2) {80};
\node at (2,2) {153};
\node at (2,4) {145};
\node at (2,6) {137};
\node at (2,8) {129};
\node at (2,10) {121};
\node at (2,12) {113};
\node at (2,14) {105};
\node at (2,16) {97};
\node at (2,18) {89};
\node at (2,20) {81};
\node at (4,20) {81};
\node at (4,18) {89};
\node at (4,16) {97};
\node at (4,14) {105};
\node at (4,12) {113};
\node at (4,10) {121};
\node at (4,8) {129};
\node at (4,6) {137};
\node at (4,4) {145};
\node at (4,2) {153};
\node at (6,2) {0};
\end{tikzpicture}
\end{center}

Theorem 2.32. $P_{4r-1,4r}$ for all $r \geq 1$ is one modulo $N$ graceful for every positive integer $N$.

Proof:

Define

\[ \phi(u) = 0 \]

\[ \phi(v) = \begin{cases} 
N[6 + 16\{ \frac{x^2-1}{2} \}] + 1 & \text{if } r \text{ is odd} \\
N[14 + 16\{ \frac{x^2-1}{2} \}] + 1 & \text{if } r \text{ is even}
\end{cases} \]

For $i = 2, 3, 4, \ldots, 4r$

\[ \phi(v_j^{(i)}) = \begin{cases} 
N(4r-1)4r - (N-1) - N(i-2) - \frac{N}{2}(4r-1)(j-1) & \text{if } j = 1, 3, 5, \ldots, 2r-1 \\
N(4r-1)4r - (2N-1) - N(i-2) - \frac{N}{2}(4r-1)(j-1) & \text{if } j = 2r+1, 2r+3, 5, \ldots, 4r-3
\end{cases} \]

For $j = 2, 3, 4, \ldots, 2r$

\[ \phi(v_j^{(i)}) = 4Nr + N(i-2) + \frac{N}{2}(4r-1)(j-2) \]

For $i = 2r+1, 2r+2, \ldots, 4r$

\[ \phi(v_j^{(i)}) = 2Nr + N(i-2r-1) + \frac{N}{2}(4r-1)(j-2) \]

\[ \phi(v_j^{(i)}) = \begin{cases} 
2Nr(4r-1) - (N-1) + (j-1) & \text{if } j = 1, 3, 5, \ldots, 4r-3 \\
2Nr(4r-1) - N - (j-2) & \text{if } j = 2, 4, 6, \ldots, 4r-2
\end{cases} \]

Clearly $\phi$ is $1-1$ and $\phi$ defines a one modulo $N$ graceful labelling of $P_{4r-1,4r}$ for all $a \geq 2$ and for all odd $b$.

Example 2.33. One modulo 3 graceful labelling of $P_{7,8}$
Example 2.34. One modulo 5 graceful labelling of $P_{3,4}$

Example 2.35. Graceful labelling of $P_{3,4}$

Theorem 2.36. $P_{a,b}$ for all even $a \geq 4$ is one modulo $N$ graceful and for all even $b \geq 4$ for every positive integer $N$.

Proof: Case (i) Let $a = 4r$, $r \geq 1$

Let $b = 2m$,

Define
\[
x(t) = \begin{cases} 
1 & \text{if } t \leq m \\
0 & \text{if } t > m 
\end{cases}
\]
\[
y(j) = \begin{cases} 
1 & \text{if } j \equiv 1 \pmod{2} \\
0 & \text{if } j \equiv 1 \pmod{2} 
\end{cases}
\]

Define
\[
\phi(u) = N(r - 1)
\]
\[
\phi(v) = 4Nrm - N(r + 1)
\]
\[
\phi(v^{(i)}_j) = \begin{cases} 
y(j)\{8Nrm - (N - 1) - \frac{N(2r-1-j)}{2}\} + y(j + 1)\{Nr - N - \frac{Nj}{2}\} & \text{if } j = 1, 2, 3, \ldots, 2r - 1 \\
y(j)\{4Nrm - (N - 1) - \frac{N(j-2r-1)}{2}\} + y(j + 1)\{4Nrm - N - \frac{N(j-2r)}{2}\} & \text{if } j = 2r, 2r + 1, \ldots, 4r - 1
\end{cases}
\]

For \(i = 2, 3, \ldots, 2m\)
\[
\phi(v^{(i)}_j) = \begin{cases} 
8Nrm - (N - 1) - Nr - N(i - 2) - \frac{N(j-2r-1)}{2} & \text{if } j = 1, 3, \ldots, 2r - 1 \\
8Nrm - (2N - 1) - Nr - N(i - 2) - \frac{N(j-1)(2m-1)}{2} & \text{if } j = 2r + 1, 2r + 3, \ldots, 4r - 1
\end{cases}
\]
\[
\phi(v^{(i)}_j) = x(i)\{N(2m + r - 1) + N(i - 2) + \frac{N(j-2r)(2m-1)}{2}\} + (1 - x(i))\{N(m + r - 1) + N(i - m - 1) + \frac{N(j-2r)(2m-1)}{2}\}
\]

Clearly \(\phi\) is \(1 - 1\) and \(\phi\) defines a one modulo \(N\) graceful labelling of \(P_{a,b}\) for all even \(a \geq 4\) and for all even \(b \geq 4\).

**Example 2.37.** One modulo 5 graceful labelling of \(P_{8,6}\)

\[
\begin{array}{cccccccccc}
231 & 0 & 236 & 115 & 116 & 110 & 121 \\
226 & 35 & 201 & 60 & 171 & 85 & 146 \\
221 & 40 & 196 & 65 & 166 & 90 & 141 \\
216 & 20 & 191 & 45 & 161 & 70 & 136 \\
211 & 25 & 186 & 50 & 156 & 75 & 131 \\
206 & 30 & 181 & 55 & 151 & 80 & 126 \\
\end{array}
\]

**Example 2.38.** One modulo 7 graceful labelling of \(P_{4,4}\)

\[
\begin{array}{cccc}
106 & 49 & 50 \\
99 & 28 & 71 \\
92 & 14 & 64 \\
85 & 21 & 57 \\
\end{array}
\]

**Example 2.39.** Graceful labelling of \(P_{12,4}\)
Case (ii) Let \( a = 4r + 2, \ r \geq 1 \)

Let \( b = 2m \),

Define

\[
x(t) = \begin{cases} 
1 & \text{if } t \leq m \\
0 & \text{if } t > m 
\end{cases}
\]

\[
y(j) = \begin{cases} 
1 & \text{if } j \equiv 1 \pmod{2} \\
0 & \text{if } j \equiv 1 \pmod{2} 
\end{cases}
\]

Define

\[
\phi(u) = Nr \\
\phi(v) = N(4r + 2)m - Nr \\
\phi(v_j^{(1)}) = y(j)\{2N(4r + 2)m - (N - 1) - \frac{N(2r+1-j)}{2}\} + y(j+1)\{\frac{N(2r-j)}{2}\} \quad \text{if } j = 1, 2, 3, \ldots, 2r + 1.
\]

\[
\phi(v_j^{(i)}) = y(j)\{N(4r+2)m+1+\frac{N(j-2r-3)}{2}\}+y(j+1)\{N(4r+2)m-\frac{N(j-2r-2)}{2}\} \quad \text{if } j = 2r+2, 2r+3, \ldots, 4r + 1.
\]

For \( i = 2, 3, \ldots, 2m \)

\[
\phi(v_j^{(i)}) = 2Nm(4r + 2) - (N - 1) - N(r + 1) - N(i - 2) - \frac{N(j-1)(2m-1)}{2} \quad \text{if } j = 1, 3, \ldots, 4r + 1.
\]

\[
\phi(v_j^{(i)}) = x(i)\{N(r + 2m + 1) + N(i - 2) + \frac{N(j-2)(2m-1)}{2}\} + (1 - x(i))\{N(r + m + 1) + N(i - m - 1) + \frac{N(j-2)(2m-1)}{2}\} \quad \text{if } j = 2r + 2r + 4, \ldots, 4r.
\]

Clearly \( \phi \) is \( 1 - 1 \) and \( \phi \) defines a one modulo \( N \) graceful labelling of \( P_{a,b} \) for all even \( a \geq 4 \) and for all even \( b \geq 4 \).

**Example 2.40. One modulo 4 graceful labelling of \( P_{6,4} \)**

**Example 2.41. One modulo 5 graceful labelling of \( P_{10,4} \)**
Example 2.42. Graceful labelling of $P_{6,6}$

References


