ONE MODULO $N$ GRACEFULNESS OF $n$-POLYGONAL SNAKES, $C_n^{(t)}$ AND $P_{a,b}$

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Abstract

A function $f$ is called a graceful labelling of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the set $\{0,1,2,\ldots,q\}$ such that, when each edge $xy$ is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. A graph $G$ is said to be one modulo $N$ graceful (where $N$ is a positive integer) if there is a function $\phi$ from the vertex set of $G$ to $\{0,1,N,(N + 1),2N, (2N + 1), \ldots , N(q - 1), N(q - 1) + 1\}$ in such a way that $(i) \phi$ is $1-1$ $(ii) \phi$ induces a bijection $\phi^*$ from the edge set of $G$ to $\{1,N+1,2N+1, \ldots , N(q - 1) + 1\}$ where $\phi^*(uv) = |\phi(u) - \phi(v)|$. In this paper we prove that the $n$-Polygonal snakes, $C_n^{(t)}$ and $P_{a,b}$ are one modulo $N$ graceful for all positive integers $N$.

Keywords : Graceful, modulo $N$ graceful, $n$-Polygonal snakes, $C_n^{(t)}$ and $P_{a,b}$.

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1 Introduction

S.W.Golomb [1] introduced graceful labelling. Odd gracefulness was introduced by R.B.Gnanajothi [2]. C.Sekar [6] introduced one modulo three graceful labelling. In this paper we introduce the concept of one modulo $N$ graceful where $N$ is any positive integer. In the case $N = 2$, the labelling is odd graceful and in the case $N = 1$ the labelling is graceful. We prove that the $n$-Polygonal snakes, $C_n^{(t)}$ and $P_{a,b}$ are one modulo $N$ graceful for all positive integers $N$.

2 Main Results

Definition 2.1. A graph $G$ with $q$ edges is said to be one modulo $N$ graceful (where $N$ is a positive integer) if there is a function $\phi$ from the vertex set of $G$ to $\{0,1,N,(N + 1),2N, (2N + 1), \ldots , N(q - 1), N(q - 1) + 1\}$ in such a way that $(i) \phi$ is $1-1$ $(ii) \phi$ induces a bijection $\phi^*$ from the edge set of $G$ to $\{1,N+1,2N+1, \ldots , N(q - 1) + 1\}$ where $\phi^*(uv) = |\phi(u) - \phi(v)|$.

Definition 2.2. Consider $k$ copies of path $P_{n}$. An $n$-Polygonal snake containing $k$ number of $n$-Polygons is obtained from a path $v_1,v_2,\ldots,v_k$ by identifying the pendant vertices of $i$th copy of the path $P_{n}$ with $v_{i-1}$ and $v_{i}$ for $i = 1,2,\ldots,k$.

Definition 2.3. The one point union of $t$ cycles of length $n$ is denoted by $C_n^{(t)}$. This graph has $(t(n-1)+1$ vertices and $tn$ edges.

Definition 2.4. Let $u$ and $v$ be two fixed vertices. We connect $u$ and $v$ by means of "$b$" internally disjoint paths of length "$a$" each. The resulting graph is denoted by $P_{a,b}$.

Theorem 2.5. $n$-Polygonal snakes for $n \equiv 0 (\text{mod} 4)$ are one modulo $N$ graceful for every positive integer $N$.
Proof: Let $n = 4r, r \geq 1$. Let there be $k$ polygons. This graph has $4rk - (k - 1)$ vertices and $4rk$ edges. For $1 \leq j \leq k$ let $u_i^{(j)}, j = 1, 2, \ldots, 4r$ be the vertices of $i$th polygon. Identify $u_1^{(4r)}$ with $u_2^{(1)}$, $u_2^{(4r)}$ with $u_3^{(1)}$, and so on.

Define

\[ \phi(u_1^{(2i-1)}) = \phi(u_2^{(4r)}) = 4N(i-1) \quad \text{for} \quad i = 1, 2, 3, 4, \ldots, \]

\[ \phi(u_2^{(2i)}) = \phi(u_1^{(4r)}) = 4Nr - (N - 1) - 2Nr - N - 4Nr(i-1) \quad \text{for} \quad i = 1, 2, 3, 4, \ldots, \]

\[ \phi(u_2^{(2i-1)}) = 4Nr - (N - 1) - \frac{N(j-2)}{2} - 4Nr(i-1) \quad \text{for} \quad j = 2, 4, 6, 8, \ldots, 4r - 2 \quad \text{and} \quad i=1,2,3,4, \ldots \]

\[ \phi(u_2^{(2i-1)}) = 4Nr - (N - 1) - 2Nr - N - \frac{N(j-3)}{2} - 4Nr(i-1) \quad \text{for} \quad j = 2r + 1, 2r + 3, \ldots, 4r - 1 \quad \text{and} \quad i=1,2,3,4, \ldots \]

\[ \phi(u_2^{(2i)}) = N(2r+1) + \frac{N(j-2)}{2} + 4Nr(i-1) \quad \text{for} \quad j = 2, 4, 6, 8, \ldots, 4r - 2 \quad \text{and} \quad i=1,2,3,4, \ldots \]

\[ \phi(u_2^{(2i-1)}) = 4Nr - (N - 1) - 2Nr - N - \frac{N(j-3)}{2} - 4Nr(i-1) \quad \text{for} \quad j = 2r + 1, 2r + 3, \ldots, 4r - 1 \quad \text{and} \quad i=1,2,3,4, \ldots \]

Clearly $\phi$ is 1-1 and $\phi$ defines a one modulo $N$ graceful labelling of the $n$-Polygonal snakes for $n \equiv 0 \pmod{4}$.

**Example 2.6.** Graceful labelling of the 12-Polygonal snake. (No. of polygons = 5)

**Example 2.7.** Odd graceful labelling of the 8-Polygonal snake. (No. of polygons = 9)

**Example 2.8.** One modulo 4 graceful labelling of the 4-Polygonal snake. (No. of polygons = 4)
**Theorem 2.9.** $n$-Polygonal snakes for $n \equiv 2 \pmod{4}$ are one modulo $N$ graceful for every positive integer $N$ if the number of polygons is even.

**Proof:** Let $n = 4r + 2$ and let there be $k = 2s$ polygon. This graph has $4rk - (k - 1)$ vertices and $4rk$ edges. For $1 \leq j \leq k$ let $u^{(j)}_{i}$, $j = 1, 2, \ldots, 4r$ be the vertices of $i$th polygon. Identify $u^{(4r)}_{1}$ with $u^{(1)}_{2}$, $u^{(4r)}_{2}$ with $u^{(1)}_{3}$, and so on.

Define
\[
\phi(u^{(1)}_{2i-1}) = \phi(u^{(4r+2)}_{2i-1}) = N(4r + 3)(i - 1) \quad \text{for } i = 1, 2, 3, 4, \ldots, s
\]
\[
\phi(u^{(1)}_{2i}) = \phi(u^{(4r+2)}_{2i}) = N(4r + 2)k - (N - 1) - 4Nk - N(4r + 1)(i - 1) \quad \text{for } i = 1, 2, 3, 4, \ldots, s
\]
\[
\phi(u^{(j)}_{2i-1}) = N + \frac{N(j-3)}{2} + N(4r + 3)(i - 1) \quad \text{for } j = 3, 5, 7, 9, \ldots, 4r + 1 \text{ and } i = 1, 2, 3, 4, \ldots, s
\]
\[
\phi(u^{(j)}_{2i-1}) = N(4r+2)k - (N - 1) - \frac{N(j-2)}{2} - N(4r+1)(i - 1) \quad \text{for } j = 3, 5, 7, \ldots, 4r + 1 \text{ and } i = 1, 2, 3, 4, \ldots, s
\]
\[
\phi(u^{(j)}_{2i}) = N(4r+2)k - (N - 1) - 2Nk - \frac{N(j-3)}{2} - N(4r+1)(i - 1) \quad \text{for } j = 3, 5, 7, \ldots, 4r + 1 \text{ and } i = 1, 2, 3, 4, \ldots, s
\]
\[
\phi(u^{(j)}_{2i}) = N(2r+1) + \frac{N(j-2)}{2} + N(4r + 3)(i - 1) \quad \text{for } j = 2, 4, 6, \ldots, 2r \text{ and } i = 1, 2, 3, 4, \ldots, s
\]
\[
\phi(u^{(j)}_{2i}) = N(2r+1) + 2N + \frac{N(j-2)}{2} + N(4r + 3)(i - 1) \quad \text{for } j = 2r + 2, 2r + 4, \ldots, 4r \text{ and } i = 1, 2, 3, 4, \ldots, s
\]

Clearly $\phi$ is $1 \pmod{1}$ and $\phi$ defines a one modulo $N$ graceful labelling of the $n$-Polygonal snakes for $n \equiv 0 \pmod{4}$.

**Example 2.10.** One modulo 8 graceful labelling of the 10-Polygonal snake. (No.of polygons = 4)

**Example 2.11.** One modulo 5 graceful labelling of the 10-Polygonal snake. (No.of polygons = 6)
Example 2.12. Graceful labelling of the $6$-Polygonal snake. (No. of polygons $= 4$)

Theorem 2.13. Let $C^{(t)}_n$ denote the one point union of $t$ cycles of length $n$. $C^{(t)}_n$ is one modulo $N$ graceful when $n = 4, 8, t > 2$ and $n = 6, t$ even and $t \geq 4$ for every positive integer $N > 1$.

Proof: Case (i) $n = 4, t > 2$
For $1 \leq i \leq t$. Let $u^{(j)}_i, j = 1, 2, 3, 4$ be the vertices of the $i$th cycle with the one point identification of $u^{(1)}_i, u^{(1)}_2, \ldots, u^{(1)}_t$ at $u_0$.
Define

$\phi(u_0) = 0$

$\phi(u^{(j)}_i) = 4Nt - (N - 1) - \frac{N(j-2)}{2} - 2N(i-1)$ for $j = 2, 4$ and $i=1,2,3,4,\ldots,t$

$\phi(u^{(3)}_i) = 4Nt - 2N - 4N(i-1)$ for $i = 1, 2, 3, 4, \ldots, t$

Clearly $\phi$ is $1 - 1$ and $\phi$ defines a one modulo $N$ graceful labelling of $C^{(t)}_n$ when $n = 4, t > 2$.

Example 2.14. One modulo 10 graceful labelling of $C^{(6)}_4$
Example 2.15. Odd graceful labelling of $C_4^{(4)}$

Case (ii) $n = 8, t > 2$

For $1 \leq i \leq t$. Let $u_i^{(j)}, j = 1, 2, \ldots, 8$ be the vertices of the $i$th cycle with the one point identification of $u_1^{(1)}, u_2^{(1)}, \ldots, u_t^{(1)}$ at $u_0$.

Define

$$
\phi(u_0) = 0
$$

$$
\phi(u_i^{(j)}) = \begin{cases} 
8Nt - (N - 1) - 2N(i - 1) - \frac{N(j - 2)}{2} & \text{if } i = 1, 2, 3, \ldots, t \text{ and } j = 2, 8 \\
6Nt - (N - 1) - 2N(i - 1) - \frac{N(j - 4)}{2} & \text{if } i = 1, 2, 3, \ldots, t \text{ and } j = 4, 6 \\
4Nt - N - 4N(i - 1) - \frac{2N(j - 3)}{4} & \text{if } i = 1, 2, 3, \ldots, t \text{ and } j = 3, 7 \\
6Nt - 2N - 4N(i - 1) & \text{if } i = 1, 2, 3, \ldots, t \text{ and } j = 5
\end{cases}
$$

Clearly $\phi$ is $1 - 1$ and $\phi$ defines a one modulo $N$ graceful labelling of $C_n^{(t)}$ when $n = 8, t > 2$.

Example 2.16. Odd graceful labelling of $C_8^{(4)}$
Example 2.17. One modulo 7 graceful labelling of $C_8^{(3)}$

**Case (iii)** $n = 6$, $t$ is even $t \geq 4$ let $t = 2s$

For $1 \leq i \leq 2s$. Let $u_i^{(j)}$, $j = 1, 2, \ldots, 6$ be the vertices of the $i$th cycle with the one point identification of $u_1^{(1)}, u_2^{(1)}, \ldots, u_t^{(1)}$ at $u_0$.

Define

$$\phi(u_0) = 0$$

$$\phi(u_i^{(j)}) = \begin{cases} 
6Nt - (N - 1) - 2N(i - 1) - \frac{N(j-2)}{4} & \text{if } i = 1, 2, 3, \ldots, 2s \text{ and } j = 2, 6 \\
4Nt - N - 4N(i - 1) - \frac{2N(j-3)}{2} & \text{if } i = 1, 2, 3, \ldots, 2s \text{ and } j = 3, 5 \\
4Nt - (N - 1) - \frac{4N(j-1)}{2} & \text{if } i = 1, 3, 5, \ldots, 2s - 1 \text{ and } j = 4 \\
4Nt - (4N - 1) - \frac{4N(j-2)}{2} & \text{if } i = 2, 4, 6, \ldots, 2s \text{ and } j = 4 
\end{cases}$$

Clearly $\phi$ is $1 - 1$ and $\phi$ defines a one modulo $N$ graceful labelling of $C_n^{(t)}$ when $n = 6, t$ is even and $t \geq 4$. 
Example 2.18. One modulo 4 graceful labelling of $C_6^{(6)}$

Example 2.19. One modulo 5 graceful labelling of $C_6^{(4)}$

Theorem 2.20. $P_{a,b}$ for all $a \geq 2$ and for all odd $b$ is one modulo $N$ graceful for every positive integer $N$. Here $P_m$ is a path of length $m - 1$.

**Proof:** Let $b = 2r + 1, r \geq 1$

Define 

$$X(t) = \begin{cases} 
1 & \text{if } t \leq r \\
0 & \text{if } t > r 
\end{cases}$$

Define 

$$\phi(u) = 0$$

$$\phi(v) = \begin{cases} 
\frac{Na(2r+1)}{2} & \text{if } a \text{ is even} \\
\frac{N(a^b-1)}{2} + 1 & \text{if } a \text{ is odd}
\end{cases}$$
For \( j = 1, 3, 5, \ldots \)
\[
\phi(v_j^{(i)}) = N(a(2r + 1) - 1) + 1 - N(i - 1) - (2r + 1)(j - 1) \quad \text{if } i = 1, 2, 3, \ldots, 2r + 1
\]

For \( j = 2, 4, 6, \ldots \)
\[
\phi(v_j^{(i)}) = X(i)\{2N(r + 1) + N(i - 1) + (2r + 1)(j - 2)\} + (1 - X(i))\{N + N(i - 1) + (2r + 1)(j - 2)\} \quad \text{if } i = 1, 2, 3, \ldots, 2r + 1
\]

Clearly \( \phi \) is 1-1 and \( \phi \) defines a one modulo \( N \) graceful labelling of \( P_{a,b} \) for all \( a \geq 2 \) and for all odd \( b \).

**Example 2.21.** One modulo 3 graceful labelling of \( P_{6,5} \)

**Example 2.22.** One modulo 4 graceful labelling of \( P_{5,7} \)

**Example 2.23.** Graceful labelling of \( P_{7,7} \)
Theorem 2.24. $P_{4,b}$ for all $b \geq 2$ is one modulo $N$ graceful for every positive integer $N$.

Proof:
Define
\begin{align*}
\phi(u) &= 0 \\
\phi(v) &= 2Nb \\
\phi(v_{ij}^{(1)}) &= N(ab - 1) + 1 - b(j - 1) - N(i - 1) \quad \text{for } i = 1, 2, \ldots, b \text{ and } j = 1, 3 \\
\phi(v_{ij}^{(2)}) &= 2Nb - N - 2N(i - 1) \quad \text{for } i = 1, 2, \ldots, b
\end{align*}
Clearly $\phi$ is 1-1 and $\phi$ defines a one modulo $N$ graceful labelling of $P_{4,b}$ for all $b \geq 2$

Example 2.25. One modulo 3 graceful labelling of $P_{4,9}$

Example 2.26. One modulo 5 graceful labelling of $P_{4,6}$
Example 2.27. Graceful labelling of $P_{4,4}$

Theorem 2.28. $P_{2,b}$ for all $b \geq 2$ is one modulo $N$ graceful for every positive integer $N$.

Proof:

Define

$\phi(u) = 0$

$\phi(v) = Nb$

$\phi(v^{(i)}) = 2Nb - (N - 1) - N(i - 1)$ for $i = 1, 2, \ldots, b$

Clearly $\phi$ is 1-1 and $\phi$ defines a one modulo $N$ graceful labelling of $P_{2,b}$ for all $b \geq 2$. 
Example 2.29. One modulo 6 graceful labelling of $P_{2,8}$

Example 2.30. Graceful labelling of $P_{2,6}$
Example 2.31. One modulo 8 graceful labelling of $P_{2,10}$

\[ \begin{align*}
\phi(u) &= 0 \\
\phi(v) &= \begin{cases} 
N[6 + 16\left\lfloor \frac{x^2-1}{2} \right\rfloor] + 1 & \text{if } r \text{ is odd} \\
N[14 + 16\left\lfloor \frac{x^2-1}{2} \right\rfloor] + 1 & \text{if } r \text{ is even}
\end{cases}
\end{align*} \]

For $i = 2, 3, 4, \ldots, 4r$

\[ \phi(v_j^{(i)}) = \begin{cases} 
N(4r-1)4r - (N-1) - N(i-2) - \frac{N}{2}(4r-1)(j-1) & \text{if } j = 1, 3, 5, \ldots, 2r - 1 \\
N(4r-1)4r - (2N-1) - N(i-2) - \frac{N}{2}(4r-1)(j-1) & \text{if } j = 2r + 1, 2r + 3, 5, \ldots, 4r - 3
\end{cases} \]

For $j = 2, 3, 4, \ldots, 2r$

\[ \phi(v_j^{(i)}) = 4Nr + N(i-2) + \frac{N}{2}(4r-1)(j-2) \] if $i = 2, 4, \ldots, 4r - 2$

For $i = 2r + 1, 2r + 2, \ldots, 4r$

\[ \phi(v_j^{(i)}) = 2Nr + N(i-2r - 1) + \frac{N}{2}(4r-1)(j-2) \] if $i = 2, 4, \ldots, 4r - 2$

\[ \phi(v_1^{(i)}) = \begin{cases} 
2Nr(4r-1) - (N-1) + (j-1) & \text{if } j = 1, 3, 5, \ldots, 4r - 3 \\
2Nr(4r-1) - N - (j-2) & \text{if } j = 2, 4, 6, \ldots, 4r - 2
\end{cases} \]

Clearly $\phi$ is $1 - 1$ and $\phi$ defines a one modulo $N$ graceful labelling of $P_{2r-1,4r}$ for all $r \geq 2$ and for all odd $b$.

Example 2.33. One modulo 3 graceful labelling of $P_{7,8}$
Example 2.34. One modulo 5 graceful labelling of $P_{3,4}$

Example 2.35. Graceful labelling of $P_{3,4}$

Theorem 2.36. $P_{a,b}$ for all even $a \geq 4$ is one modulo $N$ graceful and for all even $b \geq 4$ for every positive integer $N$.

Proof: Case (i) Let $a = 4r$, $r \geq 1$
Let $b = 2m$,
Define
For all even $a \geq 4$ and for all even $b \geq 4$.

**Example 2.37.** One modulo 5 graceful labelling of $P_{8,6}$

**Example 2.38.** One modulo 7 graceful labelling of $P_{4,4}$

**Example 2.39.** Graceful labelling of $P_{12,4}$
Case (ii) Let $a = 4r + 2$, $r \geq 1$

Let $b = 2m$,

Define

$$x(t) = \begin{cases} 1 & \text{if } t \leq m \\ 0 & \text{if } t > m \end{cases}$$

$$y(j) = \begin{cases} 1 & \text{if } j \equiv 1 \pmod{2} \\ 0 & \text{if } j \equiv 1 \pmod{2} \end{cases}$$

Define

$$\phi(u) = Nr$$

$$\phi(v) = N(4r + 2)m - Nr$$

$$\phi(v^{(i)}) = y(j)\{2N(4r + 2)m - (N - 1) - \frac{N(2r+1-j)}{2}\} + y(j+1)\{\frac{N(2r-j)}{2}\}$$

if $j = 1, 2, 3, \ldots, 2r + 1$.

$$\phi(v^{(i)}) = y(j)\{N(4r + 2)m + 1 + \frac{N(j-2r-3)}{2}\} + y(j+1)\{N(4r+2)m - \frac{N(j-2r-2)}{2}\}$$

if $j = 2r + 2, 2r + 3, \ldots, 4r + 1$.

For $i = 2, 3, \ldots, 2m$

$$\phi(v^{(i)}) = 2Nm(4r + 2) - (N - 1) - N(r + 1) - N(i - 2) - \frac{N(j-1)(2m-1)}{2}$$

if $j = 1, 3, \ldots, 4r + 1$.

For $i = 2, 3, \ldots, 2r$

$$\phi(v^{(i)}) = \{x(i)\{N(r + 2m + 1) + N(i - 2) + \frac{N(j-2)(2m-1)}{2}\} + (1 - x(i))\{N(r + m + 1) + N(i - m - 1) + \frac{N(j-2)(2m-1)}{2}\}\}$$

if $j = 2r + 2, 2r + 4, \ldots, 4r$.

Clearly $\phi$ is $1-1$ and $\phi$ defines a one modulo $N$ graceful labelling of $P_{a,b}$ for all even $a \geq 4$ and for all even $b \geq 4$.

**Example 2.40.** One modulo 4 graceful labelling of $P_{6,4}$

**Example 2.41.** One modulo 5 graceful labelling of $P_{10,4}$
Example 2.42. Graceful labelling of $P_{6,6}$

References


