# One-Dimensional Fluid - Structure - Interaction (1D-FSI) Steady Flow Stable Formulation

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***Abstract*** **— This paper concerns the numerical solution of one-dimensional fluid-structure-interaction (1D-FSI) formulation which has been formulated by providing a pressure-area constitutive relation to complement the mass and linear momentum equations. However, typical spurious oscillations were found for the cases of relatively high pressure difference when Bubnov-Galerkin formulation was employed. In minimizing the oscillation, SUPG stabilization scheme was then formulated and shown as able to stabilize the solutions. For validation purposes, an analytical solution for the limited case of straight vessel has been derived for a specific pressure-area constitutive relation. This study can be important for future works in 1D-FSI employing pressure-area constitutive relation.**

***Keywords — Streamline-Upwind-Petrov-Galerkin, Finite Element Method, Biomechanics***

## 1. INTRODUCTION

Fluid-structure-interaction (FSI) for one-dimensional flow can be formulated by providing a pressure-area constitutive relation to complement the mass and linear momentum equations. Such coupling would allow the interaction between volumetric flow rate, *Q*, cross-sectional area, *A*, and pressure, *p*, of the flow. The constitutive relation can be given in general form as;

$$P - P_0 = f(A) \tag{1}$$

where $P$ and $P_0$ are the local and reference pressure respectively, and $f(A)$ highlights the dependency of the pressure's magnitude and distribution on the cross-sectional area of the flow. Various detailed forms of $f(A)$ have been proposed in the literature [1-13] To note, pressure-area constitutive relation as in Eqn. (1) is also termed as tube law elsewhere [5-6]. Employment of Eqn. (1) thus the formulation has a wide range of applications especially in the field of biomechanics.

Due to the complexity of the governing equations, solutions are mostly obtained numerically. In recent works of Sochi [10, 12], Bubnov-Galerkin finite element method has been formulated where good verifications of results were reported. However, when we repeated the formulation and applied it to high pressure differences that is, in the range higher than reported, spurious oscillations were observed. These oscillations are typical phenomenon of Bubnov-Galerkin formulation hence its shortcoming. In minimizing the oscillation, we then formulated the well-known stabilization scheme, Streamline-Upwind-Petrov-Galerkin (SUPG) for the problem.

Realizing the importance of having stabilized solution in ensuring the attainment of reliable information, it is the interest of this paper, therefore, to report such a formulation for future reference especially in the study of one-dimensional fluid-structure-interaction (1D-FSI) flow employing pressure-area constitutive relation.

## 2. GOVERNING EQUATIONS

1D-FSI steady flow is governed by the conservation laws of mass and linear momentum as follows [10, 12]:

Mass equation

$$\frac{\partial Q}{\partial x} = 0 \tag{2}$$

Momentum equation

$$\frac{\partial}{\partial x}\left(\frac{\alpha Q^2}{A}\right) + \frac{A}{\rho}\frac{\partial p}{\partial x} + \kappa\frac{Q}{A} = 0 \tag{3}$$

where $\rho$ is the fluid density, $\alpha$ is the momentum correction factor and $x$ is the axial coordinate. $\kappa$ is defined as viscosity friction coefficient as follows

$$\kappa = \frac{2\pi\alpha\mu}{\rho(\alpha - 1)} \tag{4}$$

where $\mu$ is the fluid viscosity. With regards to the second term in Eqn. (3), we have

$$\begin{aligned}\frac{A}{\rho}\frac{\partial p}{\partial x} = \frac{A}{\rho}\frac{\partial p}{\partial A}\frac{\partial A}{\partial x} &= \frac{\partial}{\partial x}\int_x \frac{A}{\rho}\frac{\partial p}{\partial A}\frac{\partial A}{\partial x}\,\partial x \\ &= \frac{\partial}{\partial x}\int_A \frac{A}{\rho}\frac{\partial p}{\partial A}\,dA\end{aligned} \tag{5}$$

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*2.1 Constitutive Relation Equations*

Since there are three dependent variables, $A$, $Q$ and $p$, a third equation, that is, the pressure-area constitutive relation, must be provided. Despite the various constitutive relations available from literatures, only two relationships are considered in this study since the main purpose is to demonstrate the formulation of a specific numerical technique. Herein, the first constitutive equation is termed as $p-A$ Model 1 and is given by Eqn. (6). Despite its simplistic nature, the equation is chosen because it is the relationship used in Sochi [12] which results we are comparing against. For completeness, a more realistic constitutive equation (as it involves experimental-fit parameters) is thus considered and termed herein as $p-A$ Model 2 (Eqn. (9)).

$p-A$ Model 1

Pressure-area constitutive relation used in Sherwin *et al.* [8], Quarteroni and Formaggia [9], and Sochi [10-13] is termed as $p-A$ Model 1 herein and given as

$$f(A) = \frac{\beta}{A_0}(\sqrt{A} - \sqrt{A_0}\,) \tag{6}$$

where $\beta$ is known as vessel stiffness, given as

$$\beta = \frac{\sqrt{\pi} h_0 E}{1-\upsilon^2} \tag{7}$$

and $A_0$ and $h_0$ are the cross sectional area of the flow and vessel's wall thickness at reference pressure $p_0$, respectively whilst $E$ and $\upsilon$ are Young's elastic modulus and Poisson's ratio of the vessel's wall, respectively. Accordingly, Eq. (5) can be written in expanded form as

$$\frac{\partial}{\partial x}\int_A \frac{A}{\rho}\frac{\partial p}{\partial A} dA = \frac{\partial}{\partial x}\left(\frac{\beta}{3\rho A_o} A^{\frac{3}{2}}\right) \tag{8}$$

*$p-A$ Model 2*

Pressure-area constitutive relation used in Ku *et al.* [3] and Downing and Ku [4] is termed as $p-A$ Model 2 herein and given as

$$f(A) = K_p\left(\left(\frac{A}{A_o}\right)^{n_1} - \left(\frac{A}{A_o}\right)^{-n_2}\right) \tag{9}$$

where $n_1$ and $n_2$ are parameters obtained from a fit to experimental data of pressure versus diameter curve for a bovine carotid artery as detailed in Downing and Ku [4]. The vessel stiffness, $K_p$ is defined as

$$K_p = \frac{E{h_0}^3}{12(1-\upsilon^2)R^3} \tag{10}$$

where $R$ is the mean flow radius. One of the set of values of $n_1$ and $n_2$ proposed in Downing and Ku [4] is used in this study, which are 7 and 2.5, respectively. Accordingly, for $p-A$ Model 2, Eqn. (5) can be written in expanded form as

$$\frac{\partial}{\partial x}\int_A \frac{A}{\rho}\frac{\partial p}{\partial A} dA = \frac{\partial}{\partial x}\left(\frac{K_p A}{\rho}\left[\frac{n_1 e^{n_1 \ln(\frac{A}{A_o})}}{n_1+1} - \frac{n_2 e^{-n_2 \ln(\frac{A}{A_o})}}{n_2-1}\right]\right) \tag{11}$$

## 3. SPURIOUS OSCILLATIONS

In numerical analysis of fluid dynamics, spurious oscillations can occur in flows with high Peclet number (for advection-diffusion problems) and high Reynolds number (for general flows) when solved using either central finite difference method or Bubnov-Galerkin finite element method. Mathematical wise, both are known to be closely related thus inherit the same numerical difficulty [14, 15, 16].

As mentioned, when we repeated the Bubnov-Galerkin formulation detailed in Sochi [12], whilst we obtained similar non-oscillatory results for the reported range of pressure differences ($<1000$Pa), we started to observe spurious oscillation for higher pressure differences. These observations are depicted in Figure 1.

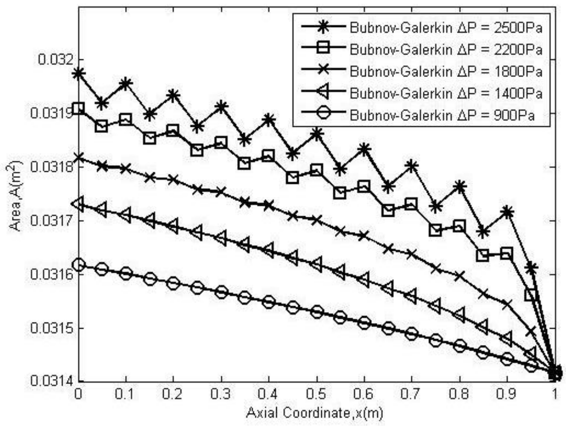


**Fig. 1** Oscillation due to the employment of Bubnov formulation for $p-A$ Model 1.The tube, fluid and flow parameters used are; $\rho = 1060\ \mathrm{kgm}^{-3}, \mu = 0.0035\ \mathrm{Pa\cdot s}$, $\alpha = 1.3333$, $L = 1$ m,$R = 0.1$ m and $\beta = 5\times10^4\ \mathrm{Pa\cdot m}$ (same data were used in Sochi [12])

In confirming the occurrence of the spurious oscillations, we then employed the same Bubnov-Galerkin formulation to a different pressure-area constitutive relation i.e. ($p-A$ Model 2) only to observe similar phenomenon, as depicted in Figure 2.

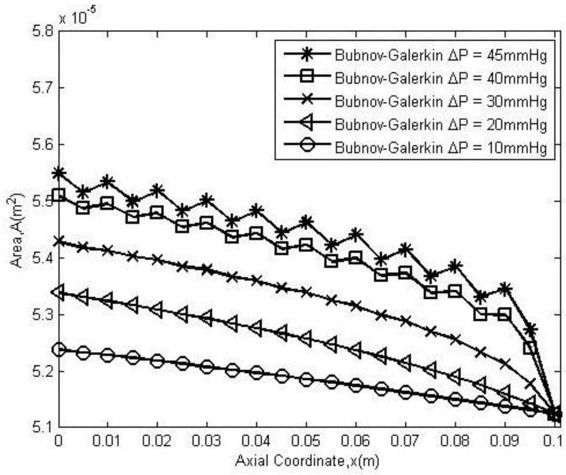


Fig. 2 Oscillation due to the employment of Bubnov formulation for $p-A$ Model 2. The tube, fluid and flow parameters used are; $\rho = 995$ kgm$^{-3}$, $\mu = 0.003\text{Pa}\cdot\text{s}$, $\alpha = 1.3333$, $L = 0.1$ m, $R = 0.003$ m and $K_p = 125$ Pa (same data were used in Downing and Ku [4])

Having confirmed the occurrence of the spurious oscillations in this 1D-FSI flow as the typical phenomenon of Bubnov-Galerkin formulation and also realized the importance for a stabilized solution, we then formulated SUPG stabilization scheme for this particular problem which derivation and results are reported herein.

## 4. ANALYTICAL SOLUTIONS

In this work, we verify our SUPG formulation against analytical solutions which are available for limited case of straight vessel. For $p-A$ Model 1, the analytical solution for the volumetric flow rate, $Q$ has been derived in Sochi [10, 12] which is given herein as

$$Q = \frac{-\kappa L + \sqrt{\kappa^2 L^2 - 4\alpha \ln\left(\frac{A_{in}}{A_{ou}}\right) \frac{\beta}{5\rho A_0}\left(A_{ou}^{\frac{5}{2}} - A_{in}^{\frac{5}{2}}\right)}}{2\alpha \ln\left(\frac{A_{in}}{A_{ou}}\right)} \qquad (12)$$

where $L$ is the length of vessel whilst $A_{in}$ and $A_{ou}$ are the flow cross sectional area at the inlet and outlet respectively.

### *4.1 Derivation of Analytical Solution for $p-A$ Model 2*

For $p-A$ Model 2, we have derived the analytical solution which derivation is detailed as follows. By inserting Eq. (11) into Eq. (3) and the fact that $Q$ is spatially constant (refer Eq. (2)), the momentum equation can be restated as

$$\frac{\partial}{\partial A}\left(\frac{\alpha Q^2}{A} + \frac{K_p n_1 A e^{n_1 \ln\frac{A}{A_0}}}{\rho(n_1+1)} - \frac{K_p n_2 A e^{-n_2 \ln\frac{A}{A_0}}}{\rho(-1+n_2)}\right)\frac{\partial A}{\partial x} + \kappa\frac{Q}{A} = 0 \qquad (13)$$

which can be further simplified as

$$\left(-\frac{\alpha Q^2}{A^2} + \frac{K_p}{\rho}\left(\left(\frac{A}{A_o}\right)^{n_1} n_1 + \left(\frac{A}{A_o}\right)^{-n_2} n_2\right)\right)\frac{\partial A}{\partial x} + \kappa\frac{Q}{A} = 0 \qquad (14)$$

With some algebraic manipulations, Eq. (14) becomes

$$\frac{\partial x}{\partial A} = \frac{-\alpha\frac{Q^2}{A} + \frac{K_p A}{\rho}\left(\left(\frac{A}{A_o}\right)^{n_1} n_1 + \left(\frac{A}{A_o}\right)^{-n_2} n_2\right)}{-\kappa Q} \qquad (15)$$

By integrating Eqn. (15) with respect to $A$, we obtain

$$x = \frac{\alpha Q \ln\frac{A}{A_0}}{\kappa} + \frac{K_p n_2 A^2 e^{-n_2 \ln\frac{A}{A_0}}}{Q\kappa\rho(-2+n_2)} - \frac{K_p n_1 A^2 e^{n_1 \ln\frac{A}{A_0}}}{Q\kappa\rho(2+n_1)} + C \qquad (16)$$

where $C$ is the constant of integration which can be determined from the boundary condition where $A = A_{in}$ at $x = 0$, thus

$$C = -\frac{\alpha Q \ln\frac{A_{in}}{A_0}}{\kappa} - \frac{K_p n_2 A_{in}^2 e^{-n_2 \ln\frac{A_{in}}{A_0}}}{Q\kappa\rho(-2+n_2)} + \frac{K_p n_1 A_{in}^2 e^{n_1 \ln\frac{A_{in}}{A_0}}}{Q\kappa\rho(2+n_1)} \qquad (17)$$

Substituting Eqn. (17) into Eqn. (16), we obtain

$$x = \frac{\alpha Q \ln\frac{A}{A_0}}{\kappa} + \frac{K_p n_2 A^2 e^{-n_2 \ln\frac{A}{A_0}}}{Q\kappa\rho(-2+n_2)} - \frac{K_p n_1 A^2 e^{n_1 \ln\frac{A}{A_0}}}{Q\kappa\rho(2+n_1)} - \frac{\alpha Q \ln\frac{A_{in}}{A_0}}{\kappa} - \frac{K_p n_2 A_{in}^2 e^{-n_2 \ln\frac{A_{in}}{A_0}}}{Q\kappa\rho(-2+n_2)} + \frac{K_p n_1 A_{in}^2 e^{n_1 \ln\frac{A_{in}}{A_0}}}{Q\kappa\rho(2+n_1)} \qquad (18)$$

Now, to obtain a closed form solution for $Q$, we then employ the other boundary condition that is at the outlet which can be given as, $A = A_{ou}$ at $x = L$. This result in

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$$L = \frac{\alpha Q}{\kappa}\left(-\ln\frac{A_{in}}{A_0} + \ln\frac{A_{ou}}{A_0}\right) + \frac{1}{Q\kappa\rho}\left(-\frac{K_p n_2 A_{in}^2 e^{-n_2 \ln\frac{A_{in}}{A_0}}}{(-2+n_2)} + \frac{K_p n_2 A_{ou}^2 e^{-n_2 \ln\frac{A_{ou}}{A_0}}}{(-2+n_2)} - \frac{K_p n_1 A_{ou}^2 e^{n_1 \ln\frac{A_{ou}}{A_0}}}{(2+n_1)} + \frac{K_p n_1 A_{in}^2 e^{n_1 \ln\frac{A_{in}}{A_0}}}{(2+n_1)}\right) \tag{19}$$

Eqn. (19) can be rearranged to yield a quadratic polynomial in $Q$, given as

$$\frac{\alpha Q^2}{\kappa}\left(-\ln\frac{A_{ou}}{A_0} + \ln\frac{A_{in}}{A_0}\right) + LQ - \left(-\frac{K_p n_2 A_{in}^2 e^{-n_2 \ln\frac{A_{in}}{A_0}}}{\kappa\rho(-2+n_2)} + \frac{K_p n_2 A_{ou}^2 e^{-n_2 \ln\frac{A_{ou}}{A_0}}}{\kappa\rho(-2+n_2)} - \frac{K_p n_1 A_{ou}^2 e^{n_1 \ln\frac{A_{ou}}{A_0}}}{\kappa\rho(2+n_1)} + \frac{K_p n_1 A_{in}^2 e^{n_1 \ln\frac{A_{in}}{A_0}}}{\kappa\rho(2+n_1)}\right) = 0 \tag{20}$$

We can solve Eqn. (20) for the roots of $Q$ by applying the quadratic formula, thus obtain

$$Q = \frac{-L \pm \sqrt{L^2 - \frac{4\alpha}{\kappa}\left(-\ln\frac{A_{ou}}{A_0} + \ln\frac{A_{in}}{A_0}\right)\left(-\frac{K_p n_2}{\kappa\rho(-2+n_2)}\left[A_{ou}^2\left(\left(\frac{A_{ou}}{A_0}\right)^{-n_2} - A_{in}^2\left(\frac{A_{in}}{A_0}\right)^{-n_2}\right)\right] + \frac{K_p n_1}{\kappa\rho(2+n_1)}\left[A_{ou}^2\left(\left(\frac{A_{ou}}{A_0}\right)^{n_1} - A_{in}^2\left(\frac{A_{in}}{A_0}\right)^{n_1}\right)\right]\right)}}{\frac{2\alpha}{\kappa}\left(-\ln\frac{A_{ou}}{A_0} + \ln\frac{A_{in}}{A_0}\right)} \tag{21}$$

If we limit the solution to a specific condition of $A_{in} > A_{out}$, the two roots must be real. Also, in ensuring the flow to be consistent in direction with the pressure gradient, the root with the plus sign should be chosen so that positive flow rate is obtained. This is because, since $A_{in} > A_{out}$ it can be shown that the denominator is always positive and the square root is always greater than $L$. So, the flow rate can be given as

$$Q = \frac{-L + \sqrt{L^2 - \frac{4\alpha}{\kappa}\left(-\ln\frac{A_{ou}}{A_0} + \ln\frac{A_{in}}{A_0}\right)\left(-\frac{K_p n_2}{\kappa\rho(-2+n_2)}\left[A_{ou}^2\left(\left(\frac{A_{ou}}{A_0}\right)^{-n_2} - A_{in}^2\left(\frac{A_{in}}{A_0}\right)^{-n_2}\right)\right] + \frac{K_p n_1}{\kappa\rho(2+n_1)}\left[A_{ou}^2\left(\left(\frac{A_{ou}}{A_0}\right)^{n_1} - A_{in}^2\left(\frac{A_{in}}{A_0}\right)^{n_1}\right)\right]\right)}}{\frac{2\alpha}{\kappa}\left(-\ln\frac{A_{ou}}{A_0} + \ln\frac{A_{in}}{A_0}\right)} \tag{22}$$

Eqn. (22) is the equivalent of the Poiseuille equation for rigid tubes but instead of pressure difference, it is expressed in terms of the specified inlet and outlet areas i.e. $A_{in}$, $A_{ou}$ which actually represent the specification of pressure at boundaries through the constitutive equation given by Eqn. (9) (e.g. using Eqn. (9), $A_{in}$ or $A_{ou}$ is solved for the desired pressure at the boundary) .

## 5. FINITE ELEMENT FORMULATION

The weak form of the formulation (after conducting integration by parts to Eq. (2) and (3)) can be written as

$$\int_{x}\left(-\mathbf{F}\cdot\left(\frac{\partial \mathbf{w}}{\partial x}\right) + \mathbf{B}\cdot\mathbf{w}\right)dx + [\mathbf{F}\cdot\mathbf{w}]|_{x=0}^{x=L} = 0 \tag{23}$$

where $\mathbf{w}$ is the linear weighting functions whilst $\mathbf{F}$and $\mathbf{B}$ give the vector representation of Eq. (2) and (3), given herein as

$$\mathrm{F} = \begin{bmatrix} Q \\ \frac{\alpha Q^2}{A} + \int \frac{A}{\rho}\frac{\partial p}{\partial A} dA \end{bmatrix} \tag{24}$$

$$\mathrm{B} = \begin{bmatrix} 0 \\ \kappa \frac{Q}{A} \end{bmatrix} \tag{25}$$

The weak formulation of Eq. (37) is then coupled with the suitable boundary conditions through the introduction of compatibility conditions. The non-reflecting boundary conditions are used to project the differential equations in the direction of outgoing characteristic variables at the inlet and outlet in producing the compatibility conditions as proposed in Sochi [10, 12] and Thompson [17] and as detailed next. Based on the method of characteristics and by assuming that $A > 0$, the eigenvalues and left eigenvectors of a matrix $\mathbf{H}$ defined as

$$\mathbf{H} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} \tag{26}$$

can be obtained as follows. The eigenvalues can be obtained by solving

$$\det(\mathbf{H} - \boldsymbol{\lambda}\mathbf{I}) = 0 \tag{27}$$

where $\boldsymbol{\lambda}$ is the eigenvalues, $\mathbf{I}$ is identity matrix and $\mathbf{H}$ is defined in Eq. (26) as the matrix of partial derivative of $\mathbf{F}$ with respect to $\mathbf{U}$. The matrix $\mathbf{H}$ has two eigenvalues represented as $\lambda_{1,2}$ which can be obtained by the quadratic formula. Left eigenvectors are then obtained by solving the following system

$$\mathbf{LH} = \mathbf{\Lambda L} \tag{28}$$

where $\mathbf{L}$ is the left eigenvectors of $\mathbf{H}$ and $\mathbf{\Lambda}$ is given as

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{29}$$

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Once the left eigenvectors, **L** is obtained, the desired compatibility conditions is then given by

$$\mathrm{L}_{1,2}\left(\mathrm{H}\frac{\partial \mathrm{U}}{\partial x}+\mathrm{B}\right)=0 \qquad (30)$$

where $\mathbf{L}_{1,2}$ are the left-eigenvectors. The imposition of boundary condition is accomplished by replacing the continuity equation at the boundary node with Eq. (30). The expanded expressions of the eigenvalues, left eigenvectors and compatibility conditions for each of the constitutive model are given next.

$p-A$ Model 1

The eigenvalues of **H** for $p-A$ Model 1, obtained by solving Eq. (27) can be given in expanded form as

$$\lambda_{1,2}=\alpha\frac{Q}{A}\pm\sqrt{\frac{Q^2}{A^2}(\alpha^2-\alpha)+\left(\frac{\beta\sqrt{A}}{2\rho A_0}\right)} \qquad (31)$$

Inserting Eq. (31) into Eq. (29) and by solving Eq. (28), the left eigenvalues of **H** for $p-A$ Model 1, can be given as

$$\mathbf{L}_{1,2}=\left[-\alpha\frac{Q}{A}\pm\sqrt{\frac{Q^2}{A^2}(\alpha^2-\alpha)+\left(\frac{\beta\sqrt{A}}{2\rho A_0}\right)}\quad 1\right] \qquad (32)$$

Inserting Eq. (32) into Eq. (30), we then obtain the compatibility conditions for $p-A$ Model 1 which can be given as

$$\left(-\alpha\frac{Q}{A}\pm\sqrt{\frac{Q^2}{A^2}(\alpha^2-\alpha)+\left(\frac{\beta\sqrt{A}}{2\rho A_0}\right)}\right)\frac{\partial Q}{\partial x}+\left(-\alpha\frac{Q^2}{A^2}+\left(\frac{\beta\sqrt{A}}{2\rho A_0}\right)\right)\frac{\partial A}{\partial x}+\left(2\alpha\frac{\partial Q}{\partial x}+\kappa\right)\frac{Q}{A}=0 \qquad (33)$$

$p-A$ Model 2

The eigenvalues of **H** for $p-A$ Model 2, obtained by solving Eq. (27) can be given in expanded form as

$$\lambda_{1,2}=\alpha\frac{Q}{A}\pm\sqrt{\frac{Q^2}{A^2}(\alpha^2-\alpha)+\frac{K_p}{\rho}\left[\left(\frac{A}{A_0}\right)^{-n_2}n_2+\left(\frac{A}{A_0}\right)^{n_1}n_1\right]} \qquad (34)$$

Inserting Eq. (34) into Eq. (29) and by solving Eq. (28), the left eigenvalues of **H** for $p-A$ Model 2, can be given as

$$\mathbf{L}_{1,2}=\left[-\alpha\frac{Q}{A}\pm\sqrt{\frac{Q^2}{A^2}(\alpha^2-\alpha)+\frac{K_p}{\rho}\left[\left(\frac{A}{A_0}\right)^{-n_2}n_2+\left(\frac{A}{A_0}\right)^{n_1}n_1\right]}\quad 1\right] \qquad (35)$$

Inserting Eq. (35) into Eq. (30), we then obtain the compatibility conditions for $p-A$ Model 2 which can be as

$$\left(-\alpha\frac{Q}{A}\pm\sqrt{\frac{Q^2}{A^2}(\alpha^2-\alpha)+\frac{K_p}{\rho}\left[\left(\frac{A}{A_0}\right)^{-n_2}n_2+\left(\frac{A}{A_0}\right)^{n_1}n_1\right]}\right)\frac{\partial Q}{\partial x}+\left(-\alpha\frac{Q^2}{A^2}+\frac{K_p}{\rho}\left[\left(\frac{A}{A_0}\right)^{-n_2}n_2+\left(\frac{A}{A_0}\right)^{n_1}n_1\right]\right)\frac{\partial A}{\partial x}+\left(2\alpha\frac{\partial Q}{\partial x}+\kappa\right)\frac{Q}{A}=0 \qquad (36)$$

*5.1 SUPG Formulation*

In employing the SUPG stabilization technique, the stabilization term is added to Eq. (23) to give

$$\int_{\mathrm{x}}\left(-\mathbf{F}\cdot\left(\frac{\partial\mathbf{w}}{\partial\mathrm{x}}\right)+\mathbf{B}\cdot\mathbf{w}\right)d\mathrm{x}+\int_x(\mathbf{P}(\mathbf{w})\,\boldsymbol{\tau}\,\mathbf{R}(\mathbf{U}))\,dx+[\mathbf{F}\cdot\mathbf{w}]|_{x=0}^{x=L}=0 \qquad (37)$$

where $\mathbf{P}(\mathbf{w})$ is the operator applied to the test function, $\mathbf{R}(\mathbf{U})$ is the residual of the governing equation, and $\boldsymbol{\tau}$ is the stabilization parameter, all as detailed in Donea and Huerta [18] and Soulaïmani and Fortin [19] and given herein as

$$\mathrm{P(w)}=\mathrm{H}\frac{\partial\mathrm{w}}{\partial x} \qquad (38)$$

$$\mathrm{R(U)}=\frac{\partial\mathrm{F}}{\partial x}+\mathrm{B} \qquad (39)$$

$$\tau=(bb)^{-\frac{1}{2}} \qquad (40)$$

$$b=\frac{\partial\xi}{\partial x}\mathrm{H} \qquad (41)$$

where $x=x(\xi)$ is the actual coordinates whilst $\xi$ refers to normalized local coordinates. To note, the weighting function **w** in Eq. (37) is the same as the shape functions, $N_i$. The expanded expression of Eq. (37) for each constitutive model is detailed next for the discretization that uses linear shape functions (i.e. $N_i$ where $i=1{,}2$).

$p-A$ Model 1

With the descriptions given by Eq. (38) to (41), expanded expression of Eq. (37) can be given for $p-A$ Model 1 as

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$$
\begin{aligned}
&-\int_x \begin{bmatrix} 0 & \dfrac{\partial N_j^T}{\partial x}N_i \\ \dfrac{\partial N_j^T}{\partial x}\dfrac{\beta}{3\rho A_o}\sqrt{N_mA_m}\,N_i & \dfrac{\partial N_j^T}{\partial x}\,\alpha\dfrac{N_kQ_k}{N_mA_m}N_i \end{bmatrix} dx \begin{Bmatrix} A \\ Q \end{Bmatrix} + \int_x \begin{bmatrix} 0 & 0 \\ 0 & N_j^T\dfrac{\kappa}{N_mA_m}N_i \end{bmatrix} dx \begin{Bmatrix} A \\ Q \end{Bmatrix} \\
&+\int_x \begin{bmatrix} 0 & \dfrac{\partial N_j^T}{\partial x}\left(-\alpha\dfrac{{N_kQ_k}^2}{{N_mA_m}^2}+\dfrac{\beta}{2\rho A_o}\sqrt{N_mA_m}\right) \\ \dfrac{\partial N_j^T}{\partial x} & \dfrac{\partial N_j^T}{\partial x}\left(2\alpha\dfrac{N_kQ_k}{N_mA_m}\right) \end{bmatrix} [\tau] \begin{bmatrix} 0 & \dfrac{\partial N_i}{\partial x} \\ \left(-\alpha\dfrac{{N_kQ_k}^2}{{N_mA_m}^2}+\dfrac{\beta}{2\rho A_o}\sqrt{N_mA_m}\right)\dfrac{\partial N_i}{\partial x} & \left(2\alpha\dfrac{N_kQ_k}{N_mA_m}\right)\dfrac{\partial N_i}{\partial x} \end{bmatrix} dx \begin{Bmatrix} A \\ Q \end{Bmatrix} \\
&\qquad+\int_x \begin{bmatrix} 0 & \dfrac{\partial N_j^T}{\partial x}\left(-\alpha\dfrac{{N_kQ_k}^2}{{N_mA_m}^2}+\dfrac{\beta}{2\rho A_o}\sqrt{N_mA_m}\right) \\ \dfrac{\partial N_j^T}{\partial x} & \dfrac{\partial N_j^T}{\partial x}\left(2\alpha\dfrac{N_kQ_k}{N_mA_m}\right) \end{bmatrix} [\tau] \begin{bmatrix} 0 & 0 \\ 0 & \kappa\dfrac{N_kQ_k}{N_mA_m}N_i \end{bmatrix} dx \begin{Bmatrix} A \\ Q \end{Bmatrix} \\
&\qquad+\left(\begin{bmatrix} 0 & N_j^TN_i \\ N_j^T\dfrac{\beta}{3\rho A_o}\sqrt{N_mA_m}\,N_i & N_j^T\,\alpha\dfrac{N_kQ_k}{N_mA_m}\,N_i \end{bmatrix}\right)\Bigg|_{x=0}^{x=L} \begin{Bmatrix} A \\ Q \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}
\end{aligned}
\tag{42}
$$

where

$$
\tau = \left(\begin{bmatrix} 0 & 1 \\ -\alpha\dfrac{{N_kQ_k}^2}{{N_mA_m}^2}+\dfrac{\beta}{2\rho A_o}\sqrt{N_mA_m} & 2\alpha\dfrac{N_kQ_k}{N_mA_m} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -\alpha\dfrac{{N_kQ_k}^2}{{N_mA_m}^2}+\dfrac{\beta}{2\rho A_o}\sqrt{N_mA_m} & 2\alpha\dfrac{N_kQ_k}{N_mA_m} \end{bmatrix}\right)^{-\frac{1}{2}}
\tag{43}
$$

$p - A$ Model 2

With the descriptions given by Eq. (38) to (41), expanded expression of Eq. (37) can be given for $p - A$ Model 2 as

$$
\begin{aligned}
&-\int_x \begin{bmatrix} 0 & \dfrac{\partial N_j^T}{\partial x}N_i \\ \dfrac{\partial N_j^T}{\partial x}\dfrac{K_p}{\rho}\left(\dfrac{n_1e^{n_1\ln\left(\frac{N_mA_m}{A_0}\right)}}{n_1+1}-\dfrac{n_2e^{-n_2\ln\left(\frac{N_mA_m}{A_0}\right)}}{n_2-1}\right)N_i & \dfrac{\partial N_j^T}{\partial x}\,\alpha\dfrac{N_kQ_k}{N_mA_m}N_i \end{bmatrix} dx \begin{Bmatrix} A \\ Q \end{Bmatrix} + \int_x \begin{bmatrix} 0 & 0 \\ 0 & N_j^T\dfrac{\kappa}{N_mA_m}N_i \end{bmatrix} dx \begin{Bmatrix} A \\ Q \end{Bmatrix} \\
&+\int_x \begin{bmatrix} 0 & \dfrac{\partial N_j^T}{\partial x}\left(\dfrac{K_p}{\rho}\left[\left(\dfrac{N_mA_m}{A_0}\right)^{-n_2}n_2+\left(\dfrac{N_mA_m}{A_0}\right)^{n_1}n_1\right]-\dfrac{\alpha{N_kQ_k}^2}{{N_mA_m}^2}\right) \\ \dfrac{\partial N_j^T}{\partial x} & \dfrac{\partial N_j^T}{\partial x}\left(2\alpha\dfrac{N_kQ_k}{N_mA_m}\right) \end{bmatrix} [\tau] \\
&\begin{bmatrix} 0 & \dfrac{\partial N_i}{\partial x} \\ \left(\dfrac{K_p}{\rho}\left[\left(\dfrac{N_mA_m}{A_0}\right)^{-n_2}n_2+\left(\dfrac{N_mA_m}{A_0}\right)^{n_1}n_1\right]-\dfrac{\alpha{N_kQ_k}^2}{{N_mA_m}^2}\right)\dfrac{\partial N_i}{\partial x} & \left(2\alpha\dfrac{N_kQ_k}{N_mA_m}\right)\dfrac{\partial N_i}{\partial x} \end{bmatrix} dx \begin{Bmatrix} A \\ Q \end{Bmatrix} \\
&\qquad+\int_x \begin{bmatrix} 0 & \dfrac{\partial N_j^T}{\partial x}\left(\dfrac{K_p}{\rho}\left[\left(\dfrac{N_mA_m}{A_0}\right)^{-n_2}n_2+\left(\dfrac{N_mA_m}{A_0}\right)^{n_1}n_1\right]-\dfrac{\alpha{N_kQ_k}^2}{{N_mA_m}^2}\right) \\ \dfrac{\partial N_j^T}{\partial x} & \dfrac{\partial N_j^T}{\partial x}\left(2\alpha\dfrac{N_kQ_k}{N_mA_m}\right) \end{bmatrix} [\tau] \begin{bmatrix} 0 & 0 \\ 0 & \kappa\dfrac{N_kQ_k}{N_mA_m}N_i \end{bmatrix} dx \begin{Bmatrix} A \\ Q \end{Bmatrix} \\
&\qquad+\left(\begin{bmatrix} 0 & N_j^TN_i \\ N_j^T\dfrac{K_p}{\rho}\left(\dfrac{n_1e^{n_1\ln\left(\frac{N_mA_m}{A_0}\right)}}{n_1+1}-\dfrac{n_2e^{-n_2\ln\left(\frac{N_mA_m}{A_0}\right)}}{n_2-1}\right)N_i & N_j^T\,\alpha\dfrac{N_kQ_k}{N_mA_m}\,N_i \end{bmatrix}\right)\Bigg|_{x=0}^{x=L} \begin{Bmatrix} A \\ Q \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}
\end{aligned}
\tag{44}
$$

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where

$$\tau = \left( \begin{bmatrix} 0 & 1 \\ \frac{K_p}{\rho}\left[\left(\frac{N_m A_m}{A_0}\right)^{-n_2} n_2 + \left(\frac{N_m A_m}{A_0}\right)^{n_1} n_1\right] - \frac{\alpha N_k {Q_k}^2}{N_m {A_m}^2} & 2\alpha \frac{N_k Q_k}{N_m A_m} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{K_p}{\rho}\left[\left(\frac{N_m A_m}{A_0}\right)^{-n_2} n_2 + \left(\frac{N_m A_m}{A_0}\right)^{n_1} n_1\right] - \frac{\alpha N_k {Q_k}^2}{N_m {A_m}^2} & 2\alpha \frac{N_k Q_k}{N_m A_m} \end{bmatrix} \right)^{-\frac{1}{2}} \tag{45}$$

*5.2 Nonlinear Solver*

This study employs Newton-Raphson as the nonlinear solver. For this, Eqn.(42) or Eqn.(44) can be arranged in matrix form as

$$\begin{bmatrix} [k_{11}] & [k_{12}] \\ [k_{21}] & [k_{22}] \end{bmatrix} \begin{Bmatrix} A \\ Q \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{46}$$

which can be further simplified into the general form

$$[K(U)]\{U\} = 0 \tag{47}$$

The residual can be expressed as

$$\{R(U)\} \equiv [K(U)]\{U\} \tag{48}$$

Expanding Eq. (48) by Taylor's series about the known $r^{th}$ solution gives

$$\{R(U)\} = 0 = \{R(U)^r\} + \frac{\partial\{R(U)^r\}}{\partial\{U\}^r}\{\Delta U\} \tag{49}$$

where the series has been truncated up to linear terms only. Rearranging Eq. (49) gives

$$[T(U)^r]\{\Delta U\} = -\{R(U)^r\} \tag{50}$$

where $[T(U)^r]$ is thus the tangent stiffness given as

$$[T(U)^r] = \frac{\partial\{R(U)^r\}}{\partial\{U\}^r} \tag{51}$$

For a vessel with *n* nodes and for residual $\{R\}$ expressed as (from Eq. (24) and (25))

$$\{R\} = \{F\} + \{B\} = \begin{bmatrix} Q \\ \frac{\alpha Q^2}{A} + \int \frac{A}{\rho}\frac{\partial p}{\partial A} dA \end{bmatrix} + \begin{bmatrix} 0 \\ \kappa \frac{Q}{A} \end{bmatrix} = \begin{Bmatrix} f_i \\ g_i \end{Bmatrix} \tag{52}$$

The expanded form of the tangent stiffness given by Eq. (51) can be given as

$$\mathbf{T} = [T(U)^r] = \begin{bmatrix} \frac{\partial f_1}{\partial A_1} & \cdots & \frac{\partial f_1}{\partial A_n} & \frac{\partial f_1}{\partial Q_1} & \cdots & \frac{\partial f_1}{\partial Q_n} \\ \frac{\partial g_1}{\partial A_1} & & \frac{\partial g_1}{\partial A_n} & \frac{\partial g_1}{\partial Q_1} & & \frac{\partial g_1}{\partial Q_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial A_1} & \cdots & \frac{\partial f_n}{\partial A_n} & \frac{\partial f_n}{\partial Q_1} & \cdots & \frac{\partial f_n}{\partial Q_n} \\ \frac{\partial g_n}{\partial A_1} & & \frac{\partial g_n}{\partial A_n} & \frac{\partial g_n}{\partial Q_1} & & \frac{\partial g_n}{\partial Q_n} \end{bmatrix} \tag{53}$$

The problem is solved by first solving for the change of variable, $\Delta\mathbf{U}$ symbolically given as

$$\Delta\mathbf{U} = -\mathbf{T}^{-1}\mathbf{R} \tag{54}$$

Then the variables are updated by

$$\{\mathbf{U}\}^{r+1} = \{\mathbf{U}\}^r + \{\Delta\mathbf{U}\} \tag{55}$$

The above process will be iterated until the satisfaction of some specified convergence criteria is attained.

## 6. VALIDATION OF FORMULATIONS

In this study, once the formulations are established, the corresponding source codes are written in Matlab. Results obtained are then verified against the analytical solutions. The following subsections detailed such verifications according to the constitutive laws.

$p - A$ Model 1

Figures 3(a), (b) and (c) show the plotting of results for the cross sectional area, pressure and flow rate distribution, respectively, along the vessel. The plots are given for various pressure difference, $\Delta P$. Based on the figures, it can be seen that, for relatively lower values of pressure differences, (i.e. $\Delta P$<1800Pa), no oscillations are observed. On the other hand, for $\Delta P$=1800Pa, slight oscillation is observed for Bubnov Galerkin formulation represented by the wiggling-like curve which becomes greater for higher pressure differences (i.e. $\Delta P$=2200Pa, $\Delta P$=2500Pa). However, these oscillations vanish when SUPG formulation is employed hence the attainment of the stabilized solutions. This observation marks the success of this study.

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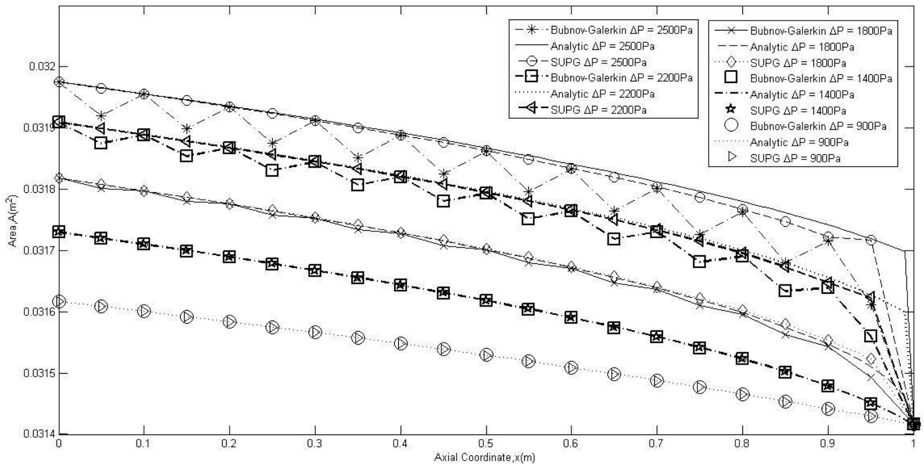
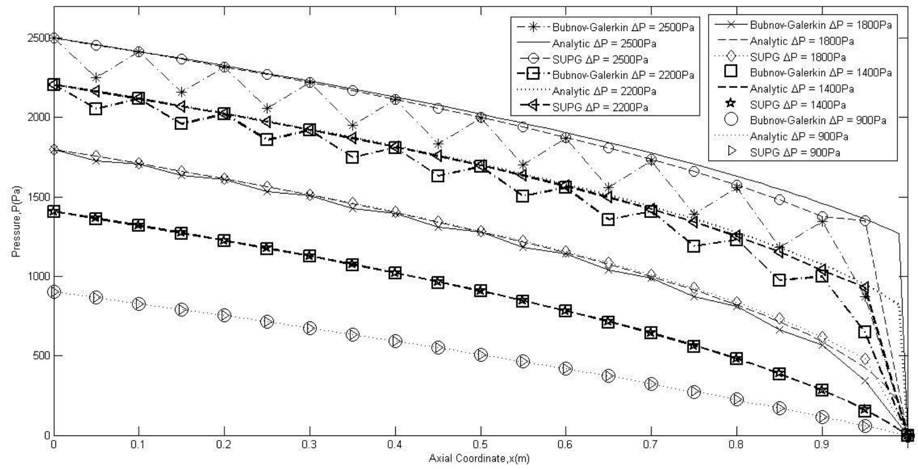
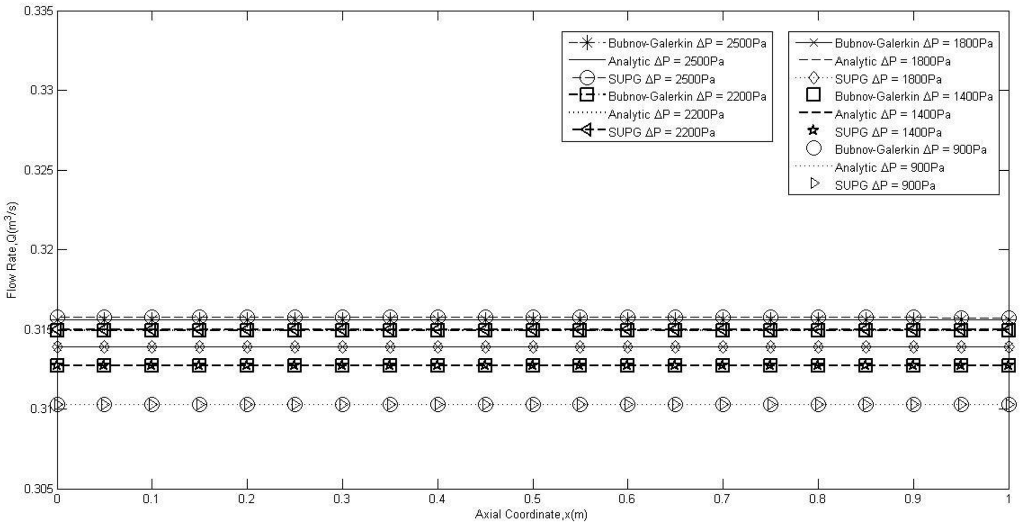



(a) Area




(b) Pressure




(c) Flow Rate

**Fig. 3** Stabilization of solutions with the employment of SUPG formulation for $p - A$ Model 1

Another trend that can be observed is that, despite the wiggling found in the solutions of pressure and area distributions, flow rate solutions seem not being affected. This highlights that pressure and area solutions are more sensitive to instability than the flow rate solution.

Also, it can be observed that somehow both Bubnov and SUPG formulations diverge from the analytical solutions towards the outlet as the pressure difference increases. This is a typical phenomenon (hence shortcoming) of both finite element formulation which occurs due to sharp internal and boundary layers as identified and studied in Hughes *et al*. [20] and Tezduyar and Park [21].

For future reference, numerical data of pressure taken at $x = 0.45m$ related to the employment of constitutive relation of $p - A$ Model 1 are given in Table 1.

Table 1 Numerical data of pressure distributions taken at $x = 0.45m$ for $p - A$ Model 1

| Inlet Pressure, $P_{in}$ (Pa) | Outlet Pressure, $P_{ou}$ (Pa) | Pressure, P -Present (Analytical) (Pa) | Pressure, P -Present (Bubnov) (Pa) | Pressure, P -Present (SUPG) (Pa) |
|---|---|---|---|---|
| 400 | 0 | 219.0113 | 221.8988 | 221.9219 |
| 900 | 0 | 544.7454 | 550.6878 | 551.0865 |
| | 400 | 679.5921 | 683.4366 | 683.4965 |
| | 800 | 856.4529 | 857.0487 | 857.0489 |
| 1400 | 0 | 956.2520 | 960.1980 | 964.4542 |
| | 400 | 1021.7960 | 1028.1233 | 1028.7451 |
| | 800 | 1147.6963 | 1152.2351 | 1152.3028 |
| | 1200 | 1311.3055 | 1312.8957 | 1312.9524 |
| 1800 | 0 | 1332.9789 | 1309.1983 | 1343.5355 |
| | 400 | 1349.2886 | 1353.3251 | 1357.4480 |
| | 800 | 1418.0922 | 1424.6723 | 1425.1408 |
| | 1200 | 1540.6901 | 1545.3126 | 1545.3213 |
| 2200 | 0 | 1753.8284 | 1631.7019 | 1756.4283 |
| | 400 | 1739.3268 | 1713.7936 | 1750.2536 |
| | 800 | 1755.8363 | 1760.0212 | 1763.9044 |
| | 1200 | 1821.4187 | 1828.0692 | 1828.3641 |
| 2500 | 0 | 2067.5523 | 1831.6639 | 2057.8781 |
| | 400 | 2042.1463 | 1951.5272 | 2050.8800 |
| | 800 | 2033.6771 | 2022.2246 | 2041.8587 |
| | 1200 | 2061.0896 | 2067.0662 | 2069.0130 |

$p - A$ Model 2

Figures 4(a), (b) and (c) give the plotting of results for $p - A$ Model 2. Based on the figures, similar trends are observed where the oscillations are greater for higher pressure difference for Bubnov formulation which is stabilized when SUPG formulation is employed. Again, no oscillation is observed for flow-rate solution thus confirms the insensitivity of the variable to instability problem, at least in the range of pressure differences considered. The typical phenomenon of divergence of the numerical formulations from the analytical solution due to sharp boundary is also observed.

For future reference, numerical data of pressure taken at $x = 0.045m$ related to the employment of constitutive relation of $p - A$ Model 2 are given in Table 2.

Table 2 Numerical data of pressure distributions taken at $x = 0.045m$ for $p - A$ Model 2

| Inlet Pressure, $P_{in}$ (mmHg) | Outlet Pressure, $P_{ou}$ (mmHg) | Pressure, $P$ Present (Analytical) (mmHg) | Pressure, $P$ Present (Bubnov) (mmHg) | Pressure, $P$ Present (SUPG) (mmHg) |
|---|---|---|---|---|
| 70 | 60 | 65.7656 | 65.7697 | 65.7696 |
| 80 | 60 | 72.9638 | 72.9336 | 72.9593 |
| 90 | 60 | 81.7944 | 81.1998 | 81.6302 |
| | 80 | 85.6916 | 85.6948 | 85.6949 |
| 100 | 60 | 91.5684 | 88.4568 | 90.7133 |
| | 80 | 92.4244 | 92.4131 | 92.4241 |
| 105 | 60 | 96.6077 | 91.5647 | 95.2215 |
| | 80 | 96.3631 | 96.3132 | 96.3560 |

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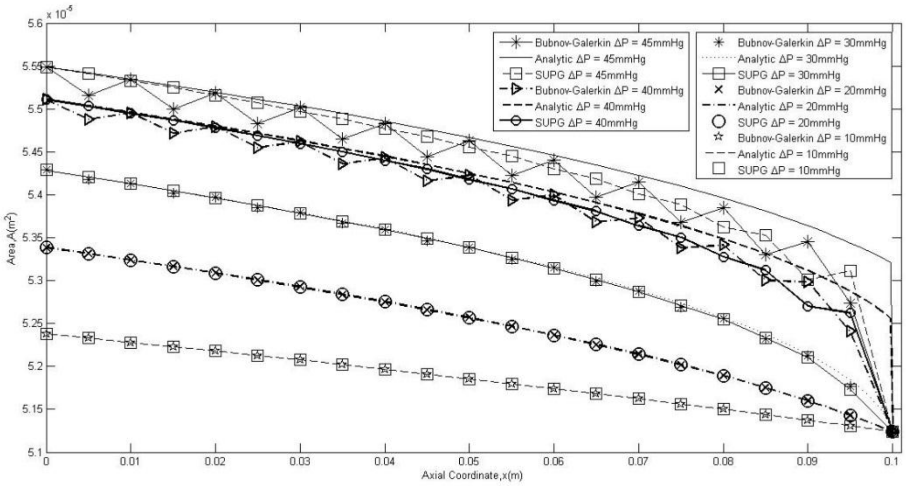
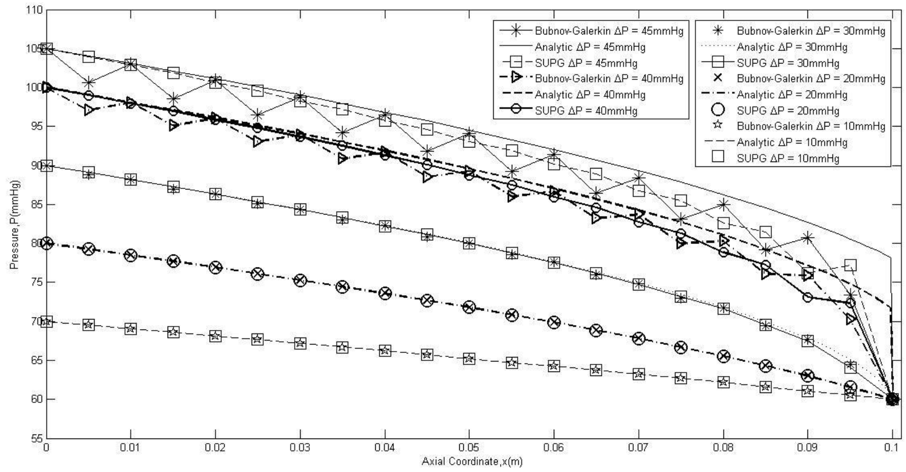
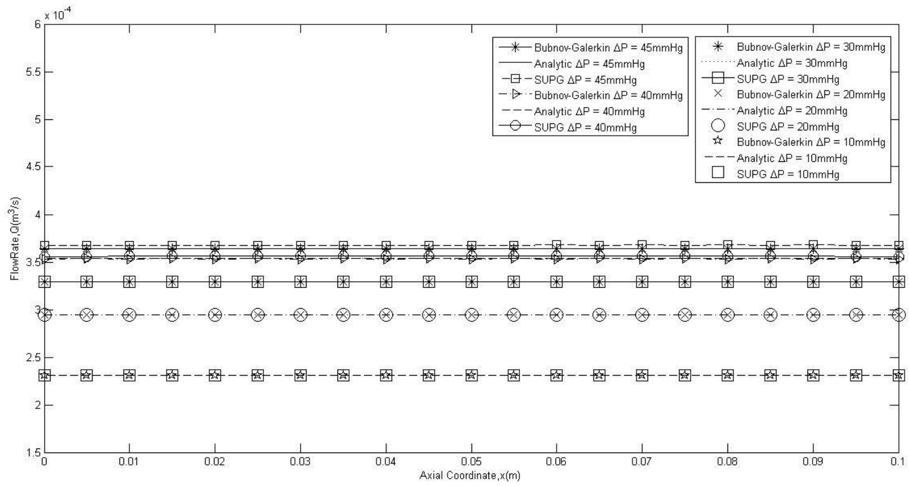



(a) Area




(b) Pressure




(c) Flow Rate

**Fig. 4** Stabilization of solutions with the employment of SUPG formulation for $p - A$ Model 2

## 7. SUMMARY AND CONCLUSIONS

In this study, SUPG formulation has been developed for the one-dimensional fluid-structure-interaction (1D-FS1) steady flow that employs pressure-area constitutive relation to complement the mass and the momentum equations of Navier-Stokes. For validation purposes, an analytical solution is derived for one of the constitutive relation. From the study, it was found that SUPG able to provide stable solutions to the problem which otherwise would wiggle due to numerical instability. This study is important as it provides the first SUPG formulation and numerical data for future reference for the specific problem of 1D-FSI employing pressure-area constitutive relation.

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