On the Solution of Fractional Differential Equation using Laplace & Sumudu Transform Method

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Abstract -- **In the present paper we discussed the application of Sumudu transform as well as Laplace transform to solve some fractional order differential using Caputo differential operator. Sumudu method is found to be fast and accurate whereas Laplace transform will allow us to transform fractional differential equations into algebraic equations and then by solving these algebraic equations, we can obtain the unknown function by using the Inverse Laplace transform. The results presented here are in compact and elegant expressed in term of Mittag-Leffler function and generalized Mittag-Leffler function which are suitable for numerical computation. Illustrative examples are included to demonstrate the validity and applicability of the presented technique. Solving some problems show that the Sumudu and Laplace transforms are powerful and efficient techniques for obtaining analytic solution of linear fractional differential equations.**

Keywords ---- Laplace Transform; Sumudu Transform; Fractional Differential equation

1. INTRODUCTION

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity during the last three decades or so. Being a generalization of classical calculus it preserves many of the basic properties and became an intensively developing area of calculus which provides useful tools for solving differential and integral equations, and various other problems of mathematical physics.

The origin of fractional calculus can be traced back to Leibniz's alone, who wrote a letter to L'Hospital in 1695, To this L'Hospital replied with a question of his own: "What if the order will be $\frac{1}{2}$?" To this, Leibniz said: "It will lead to a paradox, from which one day useful consequences will be drawn" Fractional calculus plays a vital role in the analysis of scientific problems in abroad array of fields such as physics, engineering, biology and economics. Some of the areas of present day applications of fractional calculus includes Fluid flow, Rheology, Dynamical process in Self-Similar and porous structure, diffusive transport and diffusion, Electrical networks, Probability and Statistics, Control theory of dynamical system, Viscoelasticity, Electrochemistry of corrosion, Chemical physics, optics and Signal processing and so on[2].

In this paper, we begin by introducing some necessary definitions of The Riemann-Liouville and Caputo fractional derivatives. In section 2, the Laplace transform, inverse Laplace transform, Sumudu Transform and some useful Lemmas are discussed in details. In section 3, Illustrative examples are included to demonstrate the procedure of solution of fractional differential equation using Laplace and Sumudu transformation.

Definition 1

The Gamma function denoted by $\Gamma(z)$,is a generalization of factorial function$n!$ For complex argument with positive real part it is defined as [1]

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \qquad \mathrm{Re}(z) > 0$$

By analytic continuation the function is extended to whole complex plane except for the points{ 0, -1, -2, -3, . . .} where it has simple poles.

$$\Gamma(z+1) = \Gamma(z),\ \ \Gamma(n) = (n-1)!\ \ n \in N$$

Definition: 2

While the Gamma function is a generalization of factorial function, the Mittag-Leffler function is a generalization of exponential function, first introduced as a one parameter function by the series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad \alpha > 0,\ \ \alpha \in R,\ \ z \in C$$

Later the two parameter generalization introduced by Agarwal

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha, \beta > 0,\ \ \alpha\ , \beta \in R,\ z \in C$$

For special choices of the values of the parameters α, β we obtain well-known classical functions.

Definition: 3

Suppose that $\alpha > 0, t > a, \alpha,\ a,\ t \in R.$ Then fractional operator

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha+1-n}}\, dx;$$

$$n-1 \quad < \alpha < n$$

is called the Riemann-Liouville fractional derivative or Riemann-Liouville fractional differential operator of order α.

Definition : 4

Another definition that can be used to compute a differ integral was introduced by Caputo in the 1960s. The benefit of using the Caputo definition is that it not only allows for the consideration of easily interpreted initial conditions, but it is also bounded, meaning that the derivative of a constant is equal to 0.

Suppose that $\alpha > 0, t > a, \alpha, a, t \epsilon R$. Then fractional operator[3]

$$D_*{}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha+1-n}}\, dx$$

$$n-1 < \alpha < n$$

is called the Caputo fractional derivative or Caputo fractional differential operator of order α

2. **Preliminaries of Laplace and Sumudu Transform**

2.1 Laplace Transform [4]

Let f(t) be a function of a variable t such that the function $e^{-st} f(t)$is integrable in $[0,\infty)$for some domain of values of s. The Laplace transform of the function $f(t)$ is defined for above domain values of s and it is denoted by

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

We also recall that the Laplace transform of function $f(t) = t^n$ is given as

$$L\{f(t)\} = L\{t^n\} = \frac{n!}{s^{n+1}} \quad s > 0$$

2.2 Laplace Transform of the basic fractional operator

Suppose that p > 0, and F(s) is the Laplace transform of f(t) then following statements holds

(a) The Laplace transform of Riemann-Liouville Fractional differential operator of order α is given by

$$L\{D^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{n-k-1} [D^k I^{n-\alpha} f(t)]_{t=0}$$

$$n-1 < \alpha < n$$

(b) The Laplace transform of Caputo Fractional differential operator of order α is given by[4] $L\{D_*{}^{\alpha} f(t)\} =$

$$s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0),$$

$$n-1 < \alpha \leq n \in N$$

Which can also be obtain in the form

$$L\{D_*{}^{\alpha} f(t)\} = \frac{s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots f^{(n-1)}(0)}{s^{n-\alpha}}$$

(c) Let $\alpha, \beta, \lambda \in R$, $\alpha, \beta > 0$, $m \in N$. Then the Laplace transform of the two parameter function of Mittag-Leffler type is given by

$$L\{t^{\alpha m+\beta-1} E_{\alpha,\beta}{}^{(m)}(\pm\lambda t^{\alpha})\} = = \frac{m!\, s^{\alpha-\beta}}{(s^{\alpha} \mp \lambda)^{m+1}}, \quad Re(s) > |\lambda|^{\frac{1}{\alpha}}$$

This formula is mainly used for solving FDEs.

2.3 Inverse Laplace Transform

If $L\{f(t)\} = \varphi(s)$ then inverse Laplace transform of $\varphi(s)$ is defined as

$$L^{-1}\{\varphi(s)\} = f(t)$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \varphi(s) ds, \qquad c = Re(s) > c_0$$

Where c_0 *lies* in the right half plane

2.4 Sumudu Transform[5, 7]

The sumudu transform over the set of functions,

A = $\{ f(t) / \ \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{|t|/\tau_j}, \ if \ t \in (-1)^j \times [0,\infty)\}$

is defined by

$$G(u) = S[f(t)] = \int_0^{\infty} f(ut) e^{-t} dt, \quad u \in (-\tau_1, \tau_2)$$

We also recall that the Sumudu transform of function $f(t) = t^{n-1}$ is given as

$$S\{f(t)\} = S\left\{\frac{t^{n-1}}{\Gamma(n)}\right\} = u^{n-1}, \qquad n > 0$$

Let $f(t)$ and $g(t)$ be in A having Sumudu Transforms M(u) and N(u) respectively then the Sumudu Transform of, $f * g$ the convolution of f *and* g is given by,

$$S[(f*g)(t)] = u\, M(u) N(u)$$

2.5 Definition: The Sumudu Transform of the fractional derivative introduced by Caputo is given by[6]

$$S[D_*{}^{\alpha} f(t)] = \frac{G(u)}{u^{\alpha}} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{\alpha-k}},$$

$$where\ G(u) = S[f(t)]$$

Which can also be written as

$$S[D_*{}^{\alpha} f(t)] = \frac{G(u) - f(0) - u f'(0) - u^2 f''(0) - -u^{\alpha-1} f^{(n-1)}(0)}{u^{\alpha}}$$

2.6 Main Results

Lemma 1: Let $a \in R$, $\alpha \geq \beta > 0$, $s^{\alpha-\beta} > |a|$ we have following inverse Laplace transform formula [2]

$$L^{-1}\left\{\frac{1}{(s^{\alpha}+as^{\beta})^{n+1}}\right\}$$

$$= t^{\alpha(n+1)-1}\sum_{k=0}^{\infty}\frac{(-a)^k\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha)}t^{k(\alpha-\beta)}$$

Proof :

$$\frac{1}{(s^{\alpha}+as^{\beta})^{n+1}} = \frac{1}{(s^{\alpha})^{n+1}\left(1+\frac{a}{s^{\alpha-\beta}}\right)^{n+1}}$$

$$= \frac{1}{(s^{\alpha})^{n+1}}\sum_{k=0}^{\infty}\binom{n+k}{k}\left(\frac{-a}{s^{\alpha-\beta}}\right)^k$$

$$since\ \left[\frac{1}{(1+x)^{n+1}} = \sum_{k=0}^{\infty}\binom{n+k}{k}(-x)^k\right]$$

Applying inverse Laplace transform

$$= L^{-1}\left\{\frac{1}{(s^{\alpha})^{n+1}}\right\}\sum_{k=0}^{\infty}\binom{n+k}{k}(-a)^k L^{-1}\left\{\frac{1}{s^{k(\alpha-\beta)}}\right\}$$

$$= t^{\alpha(n+1)-1}\sum_{k=0}^{\infty}\frac{(-a)^k\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha)}t^{k(\alpha-\beta)}$$

Lemma 2: $\alpha \geq \beta,\ \alpha > \gamma,\ a \in R,\ s^{\alpha-\beta} > |a|,\ |s^{\alpha}+as^{\beta}| > |b|$

$$L^{-1}\left\{\frac{s^{\gamma}}{(s^{\alpha}+as^{\beta}+b)}\right\}$$

$$= t^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^n(-a)^k\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha-\gamma)}t^{k(\alpha-\beta)+n\alpha}$$

Proof: using lemma 1, it can be easily established by expanding the binomial function,

Lemma 3: The following results are satisfied to inverse Sumudu Transform

$\alpha \geq \beta,\ \alpha > \gamma,\ a \in R,\ u^{\alpha-\beta} > |a|,$
$|u^{\alpha}+au^{\beta}| > |b|$

1. $L^{-1}\left\{\frac{1}{(u^{\alpha}+au^{\beta})^{n+1}}\right\} =$
$t^{-\alpha(n+1)}\sum_{k=0}^{\infty}\frac{(-a)^k\binom{n+k}{k}}{\Gamma(k(\beta-\alpha)-(n+1)\alpha+2)}t^{k(\beta-\alpha)}$

2. $L^{-1}\left\{\frac{u^{\gamma}}{(u^{\alpha}+au^{\beta}+b)}\right\}$

$$= t^{\gamma-\alpha}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^n(-a)^k\binom{n+k}{k}}{\Gamma(k(\beta-\alpha)-(n+1)\alpha+\gamma+3)}t^{k(\beta-\alpha)-n\alpha}$$

3 Illustration

3.1 Laplace Transform Method

Example Consider the fractional differential equation of the form

$$D_*^{\alpha}y(t) = f(t)$$

With initial conditions

$$y^{(k)}(0) = c_k$$

$$k = 0, 1, 2, \ldots\ n-1$$

where D_*^{α} denotes Caputo derivative and n is the smallest integer greater than α

Solution:

Suppose that $f(t)$ is a sufficiently good function i.e. Laplace transform of $f(t)$existApplying the Laplace transform on both the sides of equation we have,

$$s^{\alpha}Y(s) - \sum_{k=0}^{n-1}s^{\alpha-k-1}y^{(k)}(0) = F(s)$$

Since Laplace transform is linear equation can be solved with respect to Y(s) as follows

$$Y(s) = \frac{F(s)+\sum_{k=0}^{n-1}s^{\alpha-k-1}y^{(k)}(0)}{s^{\alpha}}$$

By using initial conditions

$$Y(s) = \frac{F(s)+\sum_{k=0}^{n-1}c_k s^{\alpha-k-1}}{s^{\alpha}}$$

By Laplace transform of the two-parameter of Mittag-Leffler type as well as the linearity property it follows

$$Y(s) = \frac{F(s)}{s^{\alpha}} + \sum_{k=0}^{n-1}\frac{s^{\alpha-k-1}}{s^{\alpha}}c_k$$

$$= \frac{F(s)}{s^{\alpha}} + \sum_{k=0}^{n-1}L\{t^k E_{\alpha,k+1}(t^{\alpha})\}c_k$$

$$= \frac{F(s)}{s^{\alpha}} + L\left\{\sum_{k=0}^{n-1}c_k t^k E_{\alpha,k+1}(t^{\alpha})\right\}$$

Then using the inverse Laplace transform y(t) can be found as

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{F(s)}{s^{\alpha}} + L\left\{\sum_{k=0}^{n-1}c_k t^k E_{\alpha,k+1}(t^{\alpha})\right\}\right]$$

$$y(t) = L^{-1}\left\{\frac{F(s)}{s^{\alpha}}\right\} + \sum_{k=0}^{n-1}c_k t^k E_{\alpha,k+1}(t^{\alpha})$$

$$= t^{\alpha-1}E_{\alpha,\alpha}(t^{\alpha}) * f(t) + \sum_{k=0}^{n-1}c_k t^k E_{\alpha,k+1}(t^{\alpha})$$

which is a required solution.

3.2 Sumudu Transform Method

Example:

$D_*^{\alpha}y(t) = f(t)$ With initial conditions

$$y^{(k)}(0) = c_k;$$

$k = 0, 1, 2, \ldots n-1$

where D_*^{α} denotes Caputo derivative and n is the smallest integer greater than α

Solution: Applying Sumudu transform both side

$S\{\ D_*{}^{\alpha} y(t)\} = S\{f(t)\}$ which gives,

$$\frac{G(u) - f(0) - uf'(0) - u^2 f''(0) - - - - u^{\alpha-1} f^{(n-1)}(0)}{u^{\alpha}} = H(u)$$

$$\frac{G(u) - c_0 - uc_1 - u^2 c_2 - - - - u^{\alpha-1} c_{k-1}}{u^{\alpha}} = H(u)$$

$$G(u) = u^{\alpha} * H(u) + c_0 + uc_1 + u^2 c_2 - - - + u^{\alpha-1} c_{k-1}$$

$$G(u) = u^{\alpha} * H(u) + \sum_{n=0}^{k-1} c_n u^n$$

Applying Inverse Laplace Transform we get,

$$S^{-1}\{G(u)\} = S^{-1}\{u\, u^{\alpha-1} * H(u)\} + \sum_{n=0}^{k-1} S^{-1}\{c_n u^n\}$$

$$y(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * f(t) + \sum_{n=0}^{k-1} c_n \frac{t^n}{\Gamma(1+n)}$$

Which gives required solution.

CONCLUSION

Solving some LDEs using Laplace transform and Sumudu Transform shows that these are powerful and efficient techniques for obtaining analytic solution of linear ordinary fractional differential equations.

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