On The Second Order Time-Varying Systems

Vanita Jain
Professor,
Bharati Vidyapeeth's College of Engineering,
Paschim Vihar, New Delhi-110063.

B. K. Lande
Principal,
Shah and Anchor Kutchhi Engineering College
Chembur, Mumbai – 400 088

Abstract: *This paper presents a method to find the solution of second order time-varying autonomous systems. The methodology uses well defined Cayley-Hamilton theorem widely used to find the solution of linear time-invariant systems. To demonstrate the methodology a general form of second-order time-varying system with periodic coefficients is considered.*

Key Words: *Periodic, Autonomous Systems, Time-Varying*

1. INTRODUCTION

Most human activities involve vibration in one form or another. For example, we hear because our eardrums vibrate and see because light waves undergo vibrations. Breathing is associated with the vibration of lungs and walking involves (periodic) oscillatory motion of legs and hands. We speak due to the oscillatory motion of larynges (and tongues) [1, 2].Early scholars in the field of vibration concentrated their efforts on understanding the natural phenomena and developing mathematical theories to describe the vibration of physical systems.

In [3], the phenomenon's, the revolution of Moon around the Earth, the rotation of Earth around its axis, the swinging movement of the pendulum of the clock, the wheel of a moving car, an engine in the working order, the effect of alternating current etc are described as almost periodic motions and Floquet theorem is used to find the fundamental matrix of homogeneous linear systems with periodic coefficients. In [4], many general classes of methods have been described to compute the exponential of a matrix but the systems considered are defined by linear constant coefficient ordinary differential equations. Some formations to find the state transition matrix of continuous time-varying systems are given in [5,6,7,8]. In [9], Peano Baker series method is given to define the transition matrix of second order time-varying system but it says that computation of solution via Peano Baker series is a frightening prospect, though calm calculations is profitable in the simplest cases. For an overview of recently developed numerical algorithms for the analysis and design of periodic systems see [10 and references therein].

In this paper, we are extending the well-defined Cayley Hamilton theorem used for linear time-invariant systems [11] to calculate the solution of autonomous systems whose parameters are periodic in nature. To illustrate the methodology, a general second order periodic system is considered however, the presented method is applicable to any order of periodic system.

2. METHODOLOGY

Consider the autonomous system

$$\dot{x}(t) = A(t)x(t) \tag{1}$$

where $A(t)$ is a continuous periodic 2×2 matrix function of t; i.e., when there is a constant $T > 0$ such that $A(t+T) = A(t)$ for every $t \in \Re$. When this condition is satisfied, we say, more precisely, that $A(t)$ is T periodic. If T is the smallest positive number for which this condition holds, we say that T is the minimal period of $A(t)$

$$\text{Let } A(t) = \begin{bmatrix} a_1\omega\cos\omega t & a_2\omega\cos\omega t \\ a_3\omega\cos\omega t & a_4\omega\cos\omega t \end{bmatrix} \tag{2}$$

where a_1, a_2, a_3 and a_4 are constants and $\omega\cos\omega t$ is time-varying factor. For the simplicity in calculations let a_2 is equal to zero.

The solution of the time-varying varying system (1) under the zero initial conditions is

$$x(t)=\phi(t,0)x(0) \qquad (3)$$

where $\phi(t,0)$ is the state transition matrix of (1). According to Cayley-Hamilton technique

$\phi(t,0)=e^{B}$ where

$$B=\int_0^t A(\tau)d\tau=\begin{bmatrix} a_1\sin\omega t & a_2\sin\omega t \\ a_3\sin\omega t & a_4\sin\omega t \end{bmatrix}$$

Using Cayley-Hamilton technique, we say

$e^{B}=\alpha_1 I+\alpha_2 B$ and

$e^{\lambda_i}=\alpha_1+\alpha_2\lambda_i$ for $i=1,2$

Solving for λ_1 and λ_2

$$\det(\lambda I-B)=\det\begin{bmatrix} \lambda-a_1\sin\omega t & \lambda-a_2\sin\omega t \\ \lambda-a_3\sin\omega t & \lambda-a_4\sin\omega t \end{bmatrix}$$

$$(\lambda-a_1\sin\omega t)(\lambda-a_4\sin\omega t)=0$$

$$\lambda_1=a_1\sin\omega t \quad \lambda_2=a_4\sin\omega t$$

Solving for α_1 and α_2

$$\alpha_1=\frac{a_1\exp(a_1\sin\omega t)}{a_1-a_4}(1-\exp(-a_1\sin\omega t))-\frac{a_4\exp(a_1\sin\omega t)}{a_1-a_4}$$

$$\alpha_2=\frac{\exp((a_1-a_4)\sin\omega t)}{(a_1-a_4)\sin\omega t}$$

$$\phi(t,0)=\alpha_1 I+\alpha_2 B=\begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

where

$$p=\exp(a_1\sin\omega t)$$

$$q=0$$

$$r=\frac{a_3}{(a_1-a_4)}\exp((a_1-a_4)\sin\omega t)$$

$$s=\exp(a_1\sin\omega t)\left(1-\exp(-a_4\sin\omega t)\right)$$

The solution of (1) under zero initial conditions is

$$x(t)=\begin{bmatrix} \exp(a_1\sin\omega t) & 0 \\ \frac{a_3}{(a_1-a_4)}\exp((a_1-a_4)\sin\omega t) & \exp(a_1\sin\omega t)\left(1-\exp(-a_4\sin\omega t)\right) \end{bmatrix} \qquad (4)$$

3. EXAMPLE

Example 1. Consider the system (1) with [12]

$A(t)=\begin{bmatrix} -6t^2 & 3t^5 \\ 0 & -3t^2 \end{bmatrix}, \quad t>0$ with zero initial conditions.

Thus $\phi(t,0)=e^{B}$ where

$$B=\int_0^t A(\tau)d\tau=\begin{bmatrix} -2t^3 & \frac{t^6}{2} \\ 0 & -t^3 \end{bmatrix}$$

Using the Cayley-Hamilton technique where we assume that

$e^{B}=\alpha_1+\alpha_2 B$

$e^{\lambda_i}=\alpha_1+\alpha_2\lambda_i$ for $i=1$ and 2

Solving for λ_1 and λ_2

$\lambda_1=-2t^3$ and $\lambda_2=-t^3$

Solving for α_1 and α_2 yields

$$\alpha_1=2e^{-t^3}-e^{-2t^3}, \quad \alpha_2=\frac{e^{-t^3}-e^{-2t^3}}{t^3}$$

Then $\phi(t,0)=\begin{bmatrix} e^{-2t^3} & \frac{t^3}{2}(e^{-t^3}-e^{-2t^3}) \\ 0 & e^{-t^3} \end{bmatrix}$

Using (3), the solution of the given system under zero initial conditions is

$$x(t)=\begin{bmatrix} e^{-2t^3} & \frac{t^3}{2}(e^{-t^3}-e^{-2t^3}) \\ 0 & e^{-t^3} \end{bmatrix}$$

Example 2: Consider the system (1) with

$A(t)=\begin{bmatrix}1 & 0\\ 0 & 2t\end{bmatrix}, \quad t>0$ and non-zero initial conditions.

Thus $\phi(t,t_0)=e^B$

where $B=\int_{t_0}^{t}A(\tau)d\tau=\begin{bmatrix}t-t_0 & 0\\ 0 & t^2-t_0^2\end{bmatrix}$

Using the Cayley-Hamilton technique where we assume that

$$e^B=\alpha_1+\alpha_2 B$$

$e^{\lambda i}=\alpha_1+\alpha_2\lambda_i$ for $i=1$ and 2

$\lambda_1=t-t_0$ and $\lambda_2=t^2-t_0^2$

Solving for α_1 and α_2 yields

$$\alpha_1=e^{t-t_0}-\left(\frac{e^{t-t_0}-e^{t^2-t_0^2}}{1-t+t_0}\right), \quad \alpha_2=\frac{e^{t-t_0}-e^{t^2-t_0^2}}{t-t_0-t^2+t_0^2}$$

$$\phi(t,t_0)==\begin{bmatrix}e^{t-t_0} & 0\\ 0 & e^{t^2-t_0^2}\end{bmatrix}$$

4. CONCLUSION

The paper presents a method to find the solution of autonomous second order time-varying systems. The method has been described for general second-order systems though the approach in general can be applied to system of any order. The application of the method has been demonstrated through examples.

REFERENCES

[1] Miller D.C, “Anecdotal History of the Science of Sound” Macmillan, New York, 1935

[2] Nayfeh A.H., “Perturbation Methods” Wiley, New York,1973

[3] Farkas M., “Periodic Motions” Appl. Math. Sci., Vol. 104, Springer- Verlag, Berlin and New York, 1994.

[4] Moler C., and Loan C.V., “Nineteen dubious ways to compute the exponential of a matrix” SIAM Review, Vol. 20, No. 4, pp. 801-833,1978

[5] Wu M.Y., “Solution of certain classes of linear time-varying systems” International Journal of Control, Vol. 31, No. 1, pp.11-20 , 1980

[6] Kloet P.V., and Neerhoff F.L., “Diagonalization algorithms for linear time-varying dynamic systems”,International Journal of System Sciences, Vol. 31, No. 8, pp.1053-1057, 2000

[7] Kloet P.V., and Neerhoff F.L., “Normalized dynamic eigen values for scalar time-varying systems” Proc. NDES 2002, Izmir, Turkey, 21-23 June 2002

[8] Zhu A., and Johnson C.D., “New results in the reduction of linear time- varying dynamical systems” SIAM Journal of Control and Optimization, Vol. 27, No. 3, pp. 476-494 ,1989

[9] Rugh,W.J. , “Linear System Theory” Second Edition, Prentice Hall, Information and System Sciences, New Jersey,1996

[10] Varga A., and VanDooren P., “Computational methods for periodic systems- an overview” Prepr. IFAC Workshop on Periodic Control Systems, Como, Italy, 2001

[11] Chen C., “Linear System Theory and Design” Orlando, Florida: Holt: Rinehart and Winston Inc., 1984

[12] Wu M.Y., “ Some new results in linear time-varying systems” IEEE Transaction on Automatic Control,Vol.20 ,No. 1 , pp. 159-161,1975