Published by :
http://www.ijert.org
International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 5 Issue 08, August-2016

On the Non-uniform Torsion of Trapezoidal Thin Wings

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***Abstract*—The trapezoidal thin wing with symmetrical airfoil (e.g. double wedge--airfoil) is of considerable importance in aviation industry. We consider a non-uniform torsion issue that arises in the study of the torsion analysis of a trapezoidal thin wing. Motivated by this new engineering theory, i.e. the plate-beam theory put forward by Prof. W.F. Zhang, this paper will firstly base upon the Kirchhoff's thin plate theory and Vlasov's rigid section assumption to derive two mechanical models, i.e. the energy variational model and differential equation model for the problem of the non-uniform torsion for the trapezoidal thin wing with double symmetrical airfoil. Then the approximate and exact analytical solutions of twist angle for the cantilever case of a trapezoidal thin wing under toque applied at the free end are derived and presented by making use of the energy variational model and differential equation model, respectively. Finally, the correctness of the analytical solutions obtained from the non-uniform torsion theory of the trapezoidal thin wing is verified by the FEM numerical simulations.**

Keywords: Trapezoidal Thin Wing; Non-Uniform Torsion; Energy Variational Model; Differential Equation Model; Analytical Solution

I. INTRODUCTION

The trapezoidal thin wing with symmetrical airfoil (e.g. double wedge-airfoil) has been widely used in aviation industry. Therefore, the torsion performance of this kind of wing is of considerable importance in practical aerospace engineering[1,2].

As is known to all, the theory of the non-uniform torsion of a thin wing is the design fundamental of the flutter or aerodynamic instability for high-speed aircraft and hence many researchers have devoted themselves to solving these kinds of problems. For instance, in the aspect of rectangular thin wing, Timoshenko[3,4] obtained the decay index of stress by energy varaional method based upon the Foppl's work on the stress of the non-uniform torsion, and derived the tip rotation angle of narrow rectangular plate under non-uniform torsion; F.V. Chang[5] utilized the double triangle series to derive the tip rotation angle and deflection of a narrow rectangular plate under non-uniform torsion based upon the Kirchhoff's plate theory and the principle of superposition. However, their theoretical derivations are not universal, such as the concept of decay index of stress, and hence cannot be used in the case of a non-rectangular thin wing. This hinders the development of the theory of the non-uniform torsion of thin wings.

Moreover, from the literature review, it is found that few works has been published on the theory of non-uniform torsion of a trapezoidal thin wing.

The current unsatisfied status of research may be due to the complexity of the problem of the non-uniform torsion, coupled with the great difficulty of its mathematics and mechanics. Therefore, the theoretical study on non-uniform torsion (i.e. restrained torsion) member has been developed slowly[6].

In fact, from the point of view on the nature of mechanics, non-uniform torsion exits in any twisted member, namely, the non-uniform torsion and uniform torsion are co-exist in the torsion problem of any shape of cross section, but under extreme conditions, some of the cross section, for example, the torsion of a member with circular section will be dominated by uniform torsion. In other words, in the actual project, "pure torsion" members does not exist. Consequently, developing new theory is of great importance for the theoretical research on the issue of the non-uniform torsion.

Recently, a new non-uniform torsion theory, i.e. the plate-beam theory has been put forward by Prof. W.F. Zhang[7-13], which is not only simple and easy-to-use, but also powerful and universal. Motivated by this new theory, this paper will firstly present two mechanical models of the non-uniform torsion for the trapezoidal thin wing with double symmetrical airfoil, then the exact and approximate analytical solutions are derived and verified by the FEM simulations.

II. NON-UNIFORM TORSION THEORY OF TRAPEZOIDAL THIN WING

A. Problem Descripsions and Asssumptions

1) Problem Descripsions:

Generally, the research object in this paper is a trapezoidal thin wing with double symmetrical airfoil (e.g. double wedge- airfoil). For simplicity, this type of wing may be thought of as a tapered plate with rectangular cross section. One end of the tapered plate is free and subjected to an applied torque M_t. The other end is assumed to be fixed and cannot warp. Moreover, the distributed torque m_z is also applied along the centroid of the tapered plate. In this case the thin wing will produce torsion deformation, and can be simplified as a cantilevered, tapered plate subjected to a tip torque and distributed torque as shown in Fig.1a.

Known: The length of trapezoidal thin wing is L, the width of root section is h_w, the thickness is t_w; The elastic modulus of wing is E_s, shear modulus is G_s, Poisson ration is μ_s. When the tapered plate twists, the torsion angle of cross section is assumed to be $\theta(z)$ as shown in Fig.1b.

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
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2) Asssumptions:

a) Ignore the shear strain caused by out-plane bending: This assumption is similar to that of the "plane section" of Euler beam.

b) Ignore the deformation in the mid-plane caused by out-plane bending, namely, the mid-plane cannot be streched: By this assumption, the bending and streching problem of a thin plate can be solved seprately. This is the deformation decomposition hypothesis, which is the theoretical basis of the plate-beam theory[7-13].

c) normal stress and strain in the normal direction can ommited.

d) Rigid section assumption: This assumption is similar to that of Vlasov[6], which is widely used in the therotical derivation of thin-walled structures.

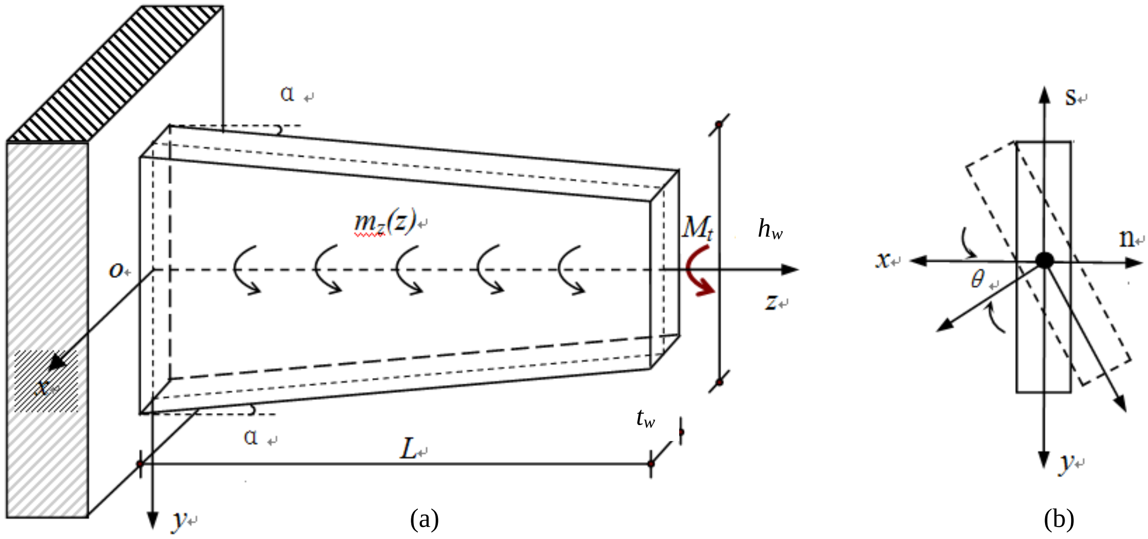

Fig.1 Calculation diagram and its deformation of narrow tapered thin-plate,

B. Mechanical Models of Trapezoidal Thin Wing

1) Energy variational model:

To facilitate the description of the deformation, the plate -beam theory introduced two sets of coordinates, i.e. the global coordinate system *xyz* and the local coordinate system *nsz*, as shown in Fig.1b. The centroid coordinates of the cross section in the global coordinate system are assumed to be (0,0). In the local coordinate system, the coordinates of arbitrary point in the cross section are denoted as (*n*,*s*). While in the global coordinate system, the coordinates of arbitrary point on the cross section are denoted as (-*n*,-*s*).

According to the assumption (d), the displacements of arbitrary point are[6]

$$\alpha = \frac{3\pi}{2}: \quad x - x_0 = -n; \quad y - y_0 = -s \tag{1}$$

$$\begin{Bmatrix} r_s \\ r_n \end{Bmatrix} = \begin{bmatrix} \sin\frac{3\pi}{2} & -\cos\frac{3\pi}{2} \\ \cos\frac{3\pi}{2} & \sin\frac{3\pi}{2} \end{bmatrix} \begin{Bmatrix} -n \\ -s \end{Bmatrix} = \begin{Bmatrix} n \\ s \end{Bmatrix} \tag{2}$$

$$\begin{Bmatrix} v_s \\ v_n \\ \theta \end{Bmatrix} = \begin{bmatrix} \cos\frac{3\pi}{2} & \sin\frac{3\pi}{2} & n \\ \sin\frac{3\pi}{2} & -\cos\frac{3\pi}{2} & -s \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta \end{Bmatrix} = \begin{Bmatrix} n\theta \\ -s\theta \\ \theta \end{Bmatrix} \tag{3}$$

Accordingly we can see that, when considering only the torsion problem, the displacements of the centroid of cross section are

$$u_{w0} = 0; \quad v_{w0} = 0; \quad w_{w0} = 0; \quad \theta_{w0} = \theta(z) \tag{4}$$

Based on deformation decomposition hypothesis (b), then the tapered plate only occur out-plane bending. At this point the lateral displacement of arbitrary point caused by out-plane bending deformation are

$$u_w(s,z) = -s\theta \; ; \; v_w(n,z) = n\theta \tag{5}$$

While the longitudinal displacement of the cross section should be determined based on assumptions (a) [7-13], that is

$$w_w(n,s,z) = -n\left(\frac{\partial u_w}{\partial z}\right) = ns\left(\frac{\partial \theta}{\partial z}\right) \tag{6}$$

Geometric equations (i.e. linear strain) [6-12]

$$\varepsilon_z^L = \frac{\partial w_w}{\partial z} = ns\left(\frac{\partial^2 \theta}{\partial z^2}\right) \tag{7}$$

$$\varepsilon_s^L = \frac{\partial v_w}{\partial s} = 0 \tag{8}$$

$$\gamma_{sz}^L = \frac{\partial w_w}{\partial s} + \frac{\partial v_w}{\partial z} = 2n\left(\frac{\partial \theta}{\partial z}\right) \tag{9}$$

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
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Physical equations (i.e. constitutive equations) [7-13]

Based on the assumption (3) and (4), for the classical Kirchhoff 's plate model, there are

$$\sigma_z = \frac{E_s}{1-\mu_s^2}(\varepsilon_z^L); \quad \tau_{sz} = G_s \gamma_{sz}^L \tag{10}$$

Using the above derived relationship, the torsional strain energy of a trapezoidal thin wing can be obtained easily.

Firstly, with the following expression of strain energy

$$U = \frac{1}{2}\iiint_{V_{wc}} \left[\sigma_z \varepsilon_z^L + \tau_{sz}\gamma_{sz}^L \right] dndsdz \tag{11}$$

and substituting Eq. (7) - Eq. (10) into Eq.(11), we can get the torsional strain energy of the trapezoidal thin wing as follows[7-13]

$$U = \frac{1}{2}\iiint_{V_{wc}} \left[\sigma_z \varepsilon_z^L + \tau_{sz}\gamma_{sz}^L \right] dndsdz$$
$$= \frac{1}{2}\iiint_{V_{wc}} \left[\frac{E_s}{1-\mu_s^2}\left[ns\left(\frac{\partial^2\theta}{\partial z^2}\right)\right]^2 + G_s\left(2n\left(\frac{\partial\theta}{\partial z}\right)\right)^2 \right] dndsdz$$

If setting

$$EI_\omega = \frac{E_s}{1-\mu_s^2} I_s^\omega = \frac{E_s}{1-\mu_s^2}\iint_{A_{wc}} n^2 s^2 dnds$$
$$= \frac{E_s}{1-\mu_s^2}\left(\frac{t_c^3 h_w^3}{144}\right)\left[1-2\left(\frac{z}{h_w}\right)\tan\alpha\right]^3 \tag{12}$$

as the non-uniform torsional rigidity or warping torsion rigidity of the trapezoidal thin wing, and

$$GJ_k = G_s J_s = 4G_s \iint_{A_{wc}} n^2 dnds$$
$$= G_s\left(\frac{t_c^3 h_w}{3}\right)\left[1-2\left(\frac{z}{h_w}\right)\tan\alpha\right] \tag{13}$$

as the uniform torsional rigidity of the trapezoidal thin wing, now the torsional strain energy of can be simplified as[7-13]

$$U = \frac{1}{2}\int_0^L \left[EI_\omega\left(\frac{\partial^2\theta}{\partial z^2}\right)^2 + GJ_k\left(\frac{\partial\theta}{\partial z}\right)^2 \right] dz \tag{14}$$

In the case of a concentrated torque is applied at the free end and a distributed torque is applied along the length of a cantilevered, trapezoidal thin wing , the corresponding load potential energy is

$$V = -\left[M_t\theta\right]_{z=L} - \int_0^L m_z(z)\theta\, dz \tag{15}$$

Now the total potential energy of the torsion problem of the trapezoidal thin wing can be expressed as[7-13]

$$\Pi = \frac{1}{2}\int_0^L \left[\begin{array}{l} \left(EI_\omega\right)_{comp}\left(\frac{\partial^2\theta}{\partial z^2}\right)^2 + \\ \left(GJ_k\right)_{comp}\left(\frac{\partial\theta}{\partial z}\right)^2 - m_z(z)\theta \end{array} \right] dz \tag{16}$$
$$- M_t\,\theta\big|_{z=L}$$

This is the total potential energy for the non-uniform torsion problem of the trapezoidal thin wing. It is consistent with the result of traditional torsion theory in the form. However, the derivation of this paper is more natural, and only the Kirchhoff's thin plate theory and Vlasov's rigid section assumption are used.

Using expression of the total potential energy (16), the non-uniform torsion problem of the trapezoidal thin wing subjected to torsion loads can be translated into such an energy variational model: Within the range, $0 \leqslant z \leqslant L$, looking for a function $\theta(z)$ to make it satisfy the specified geometric boundary conditions, i.e. the endpoint constraints, and the energy functional defined by the following formula is minimum.

$$\Pi = \int_0^L F\left(\theta', \theta''\right) dz \tag{17}$$

Where

$$F\left(\theta', \theta''\right) = \frac{1}{2}\left[EI_\omega \theta''^2 + GJ_k \theta'^2\right] \tag{18}$$

2) Differential equation model:

The differential equation, along with boundary conditions, can be readily obtained by the principle of energy variational, which states that the true state of deformation is distinguished from all other statically correct state of deformation by the condition that the energy functional be a minimum, i.e.

$$\delta\Pi = 0 \tag{19}$$

Then we get

$$\int_0^L \left[\begin{array}{l} EI_\omega \theta'' \delta\theta'' + \\ GJ_k \theta' \delta\theta' - m_z(z)\delta\theta \end{array} \right] dz - M_t\,\delta\theta\big|_{z=L} = 0 \tag{20}$$

Use integration by parts, we get

$$\int_0^L \left[\begin{array}{l} \frac{\partial^2}{\partial z^2}\left[EI_\omega\theta''\right] \\ -\frac{\partial}{\partial z}\left[GJ_k\theta'\right] - m_z(z) \end{array} \right] \delta\theta\, dz$$
$$+\left[\left[EI_\omega\theta''\right]\delta\theta'\right]_{z=0}^{z=L} \tag{21}$$
$$+\left[\left[-\frac{\partial}{\partial z}\left[EI_\omega\theta''\right] + GJ_k\theta'\right]\delta\theta\right]_{z=0}^{z=L}$$
$$-M_t\,\delta\theta\big|_{z=L} = 0$$

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 5 Issue 08, August-2016

Due to the arbitrariness of $\delta\theta$ in the above formula, we could get the following differential equation

$$\frac{\partial^2}{\partial z^2}\left[EI_\omega\theta''\right]-\frac{\partial}{\partial z}\left[GJ_k\theta'\right]-m_z(z)=0 \quad (22)$$

and the corresponding boundary conditions

- Fixed end (cross section cannot freely rotate, nor freely warp)

$$\theta=0;\quad \theta'=0 \quad (23)$$

- Free end (with tip toque, cross section can freely warp)

$$\left.\begin{aligned}-\frac{\partial}{\partial z}\left[EI_\omega\theta''\right]+GJ_k\theta'-M_t&=0\\ EI_\omega\theta''&=0\end{aligned}\right\} \quad (24)$$

Thus, the non-uniform torsion issue of the trapezoidal thin wing can also be expressed as follows: Within the range, $0\le z\le L$, looking for a function θ (z), it satisfies the differential equation (i.e. the equilibrium equation) (22), and at the same time meets the boundary conditions (25) and (26).

It should be noted that in the equilibrium equation (22) given herein, the first term is the internal twisting moment caused by the non-uniform torsion, and the second term is the internal twisting moment caused by the uniform torsion, i.e. the free torsion (St. Venant torsion). It can be seen that, according to the proposed plate-beam theory, two types of torsion, i.e. non-uniform torsion and uniform torsion, could be integrated in one mixed torsion equation naturally, namely, the "separated" traditional torsion theory is included in one theoretical framework, Therefore, the new mixed torsion theory has important theoretical and practical value.

III. ANALYTICAL SOLUTION OF TRAPEZOIDAL THIN WING SUBJECTED TO TIP TORQUE

A. Approximate Analytical Solution Based on Energy Variation Model

In this section, we will base on the energy variation model to examine the non-uniform torsion issue of the trapezoidal thin wing. For the purposes of simplifying this discussion, we consider here the case where only concentrated torque is applied in the free end torque. In this case the total potential energy simplifies to

$$\Pi=\frac{1}{2}\int_0^L\left[EI_\omega\left(\frac{\partial^2\theta}{\partial z^2}\right)^2+GJ_k\left(\frac{\partial\theta}{\partial z}\right)^2\right]dz-M_t\,\theta\Big|_{z=L} \quad (25)$$

In order to obtain an approximate analytic solution, we can choose the trail function for the rotation of cross section as follows

$$\theta(z)=\frac{A_0}{2}\left(\frac{x}{L}\right)^2\left[3-\left(\frac{x}{L}\right)\right] \quad (26)$$

Obviously, the above trial function satisfies the following boundary conditions of the cantilever plate

$$\theta(0)=\theta'(0)=0\ ;\ \theta''(L)=0 \quad (27)$$

If Eq.(26) is substituted into Eq.(25), then we get

$$\Pi=\frac{1}{2}\int_0^L\left[\begin{aligned}&\frac{E_c}{1-\mu_c^2}\left(\frac{t_c^3h_w^3}{144}\right)\left[1-2(\frac{z}{h_w})\tan\alpha\right]^3\\&\left(\frac{\partial^2}{\partial z^2}\left(\frac{A_0}{2}\left(\frac{x}{L}\right)^2\left[3-\left(\frac{x}{L}\right)\right]\right)\right)^2\\&+G_c\left(\frac{t_c^3h_w}{3}\right)\left[1-2(\frac{z}{h_w})\tan\alpha\right]\\&\left(\frac{\partial}{\partial z}\left(\frac{A_0}{2}\left(\frac{x}{L}\right)^2\left[3-\left(\frac{x}{L}\right)\right]\right)\right)^2\end{aligned}\right]dz-M_t\frac{A_0}{2}\left(\frac{x}{L}\right)^2\left[3-\left(\frac{x}{L}\right)\right]\Bigg|_{z=L}$$

and its integration result is

$$\Pi=\frac{1}{2}\left(\frac{6A_0^2}{5L}-\frac{33tan\alpha A_0^2}{20h}\right)(GJ_k)_E+\frac{1}{2}\left(\begin{aligned}&\frac{3A_0^2}{L^3}-\frac{9tan\alpha A_0^2}{2hL^2}+\\&\frac{18tan^2\alpha A_0^2}{5h^2L}-\frac{6tan^3\alpha A_0^2}{5h^3}\end{aligned}\right)(EI_\omega)_E-A_0M_t \quad (28)$$

Where

$$(EI_\omega)_E=\frac{E_c}{1-\mu_c^2}\left(\frac{t_c^3h_w{}^3}{144}\right);\quad (GJ_k)_E=G_c\left(\frac{t_c^3h_w}{3}\right) \quad (29)$$

Based upon energy variational method, there must be

$$\frac{\partial\Pi}{\partial A_0}=0 \quad (30)$$

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
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After arrangement, we get

$$A_0\left(\begin{pmatrix}3+\dfrac{18}{5}tan^2\alpha\left(\dfrac{L}{h}\right)^2\\-\dfrac{9}{2}tan\alpha\left(\dfrac{L}{h}\right)\\-\dfrac{6}{5}tan^3\alpha\left(\dfrac{L}{h}\right)^3\end{pmatrix}K_E^2+\left(\dfrac{6}{5}-\dfrac{33}{20}tan\alpha\left(\dfrac{L}{h}\right)\right)\right)-\frac{M_tL}{(GJ_k)_E}=0 \quad (31)$$

Then the solution can be obtained

$$A_0=\frac{M_tL}{(GJ_k)_E}\left(\frac{1}{\chi_1K_E^2+\chi_2}\right) \quad (32)$$

where

$$\chi_1=3+\frac{18}{5}tan^2\alpha\left(\frac{L}{h}\right)^2-\frac{9}{2}tan\alpha\left(\frac{L}{h}\right)-\frac{6}{5}tan^3\alpha\left(\frac{L}{h}\right)^3 \quad (33)$$

$$\chi_2=\frac{6}{5}-\frac{33}{20}tan\alpha\left(\frac{L}{h}\right) \quad (34)$$

$$K_E=\sqrt{\frac{(EI_\omega)_E}{(GJ_k)_E L^2}} \quad (35)$$

B. Exact Analytical Solution Based on differential equation model

In this section, we will base on the differential equation model to examine the non-uniform torsion issue of the trapezoidal thin wing. For the purposes of simplifying this discussion, we consider here the case where only concentrated torque is applied in the free end torque. In this case the simplified differential equation is

$$\frac{\partial^2}{\partial z^2}\left[(EI_\omega)_{comp}\theta''\right]-\frac{\partial}{\partial z}\left[(GJ_k)_{comp}\theta'\right]=0 \quad (36)$$

Using the first condition of the boundary condition (24) , we can obtain

$$\frac{\partial}{\partial z}\left[(EI_\omega)_E\left[1-2(\frac{z}{h_w})\tan\alpha\right]^3\theta''\right]-\left[(GJ_k)_E\left[1-2(\frac{z}{h_w})\tan\alpha\right]\theta'\right]=-M_t \quad (37)$$

This is a more complex, third-order differential equations with variable coefficients. In this paper, based on its unique mathematical structure constituted, along with the remaining three boundary conditions, its exact analytical solution is derived.

Firstly, Eq.(37) is rewritten in dimensionless form.

Setting

$$\bar{z}=1-2(\frac{z}{h_w})\tan\alpha \quad (38)$$

After arrangement, we get

$$\Lambda^2\frac{\partial}{\partial\bar{z}}\left[\bar{z}^3\left(\frac{\partial^2\theta}{\partial\bar{z}^2}\right)\right]-\left[\bar{z}\left(\frac{\partial\theta}{\partial\bar{z}}\right)\right]=\frac{M_th_w}{2\tan\alpha(GJ_k)_E} \quad (39)$$

where

$$\Lambda=K_E\left(\frac{2L}{h_w}\right)\tan\alpha\ ;\ K_E=\sqrt{\frac{(EI_\omega)_E}{(GJ_k)_E L^2}} \quad (40)$$

Then, using the theory of differential equations, the exact analytical solution can be derived as follows

$$\theta(\bar{z})=A_{h1}\left(\frac{1}{\Upsilon}\right)\bar{z}^{\Upsilon}-A_{h2}\left(\frac{1}{\Upsilon}\right)\bar{z}^{-\Upsilon}-\left(\frac{M_th_w}{2\tan\alpha(GJ_k)_E}\right)\left(\frac{1}{\Lambda^2+1}\right)\ln\bar{z}+A_0 \quad (41)$$

where, $\Upsilon=\sqrt{\dfrac{1+\Lambda^2}{\Lambda^2}}$; A_0 , A_{h1} and A_{h2} are three constants of integration, whose expression are as follows, which can be obtained according to the ends of the boundary conditions

$$\begin{cases}A_{h1}=\dfrac{M_th_w\cot\alpha}{2(1+\Lambda^2)(GJ_k)_E};\\A_{h2}=0;\\A_0=-\dfrac{M_th_w\cot\alpha}{2\Upsilon(1+\Lambda^2)(GJ_k)_E}\end{cases} \quad (42)$$

Finally, the maximum rotation (tip rotation in the free end) of the cantilevered, tapered plate can be obtained

$$\theta_{\max} = \theta(L) = \left(\frac{M_t L}{(GJ_k)_E}\right)\left[\frac{\cot\alpha}{2(1+\Lambda^2)}\left(\frac{h_w}{L}\right)\left(\frac{1}{\Upsilon}\right)\left[1-2(\frac{L}{h_w})\tan\alpha\right]^{\Upsilon} - \frac{1}{2\tan\alpha}\left(\frac{h_w}{L}\right)\left(\frac{1}{\Lambda^2+1}\right)\ln\left[1-2(\frac{L}{h_w})\tan\alpha\right] - \frac{\cot\alpha}{2\Upsilon(1+\Lambda^2)}\left(\frac{h_w}{L}\right)\right] \quad (43)$$

IV. FEM SIMULATION AND VERIFICATION

A. FEM Model

In order to verify the correctness of the above analytical solutions, the finite element software ANSYS is used to establish the model for the analysis of the non-uniform torsion of the cantilevered, tapered plate subjected to a tip torque. The elastic shell element SHELL63 with 4 nodes is chosen to simulate the trapezoidal thin wings. The elastic modulus E_s=2.06×105MPa, Poisson ratio μ_s=0.3.

When the model is built, the key points of the left and right ends are established, and the tapered plate is formed through these key points, using the "ESIZE" command to control the length of element, which is 50mm. Then, the freedoms of all nodes at the fixed end are restrained to simulate the fixed constraint conditions, a unit of torque is applied at the centroid of the cross section at the free end. In addition, the "CERIG" command is used in the finite element simulation in order to define rigid region for each cross section. Last, the torsion angle at the free end is extracted after the analysis. The corresponding modeling, meshing, load and constraint and torsion deformation are shown in Fig.2.

B. Comparison of theoretical and FEM solutions

There are 3 different sizes of the trapezoidal thin wings, using the above theoretical method and finite element method (FEM) to calculate the torsion angles for all wings. The comparison results are shown in Table I and Table II.

From the analysis of data of Table I and Table II, it can be seen that: (1) The results given by the exact analytical solutions are almost the same as those obtained from FEM simulations. This proves the correctness of the theory of non-uniform torsion presented herein. (2) The results given by the approximate analytical solution and those given by the finite element analysis are basically consistent, and the error is within the range of -6.45% and -2.76%. However, the expression of the approximate analytical solution is more simple than the exact one. Therefore, the approximate analytical solution is suitable for the engineering design personnel in the approximate calculation or estimation.

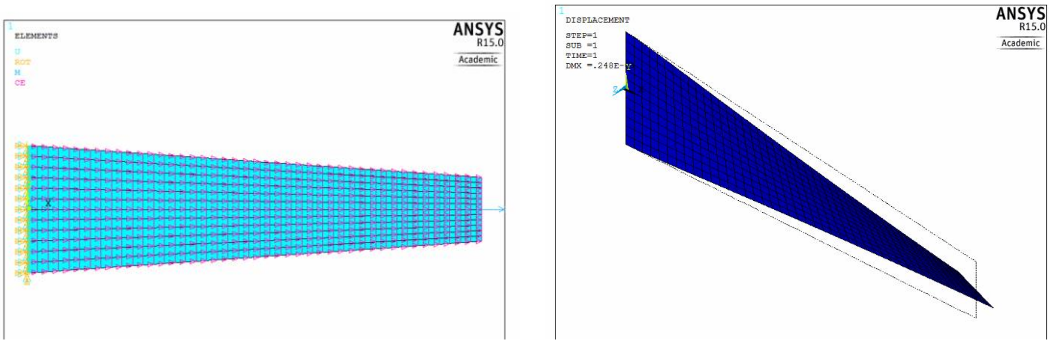

Fig.2 FEM model and its deformation

TABLE I. COMPARISON OF TORSION ANGLES BETWEEN EXACT SOLUTIONS AND FEM SOLUTIONS

Number	L(m)	h_w(m)	FEM solutions (10^{-3}rad)		Exact solutions (10^{-3}rad)	Error1 (%)
1	8.6	1.2	1.7003		1.6983	-0.12
2	4.3	1.2	0.82828		0.827094	-0.14
3	2.2	1.2	0.40093		0.401075	-0.04

TABLE II. COMPARISON OF TORSION ANGLES BETWEEN APPROXIMATE SOLUTIONS AND FEM SOLUTIONS

Number	L(m)	h_w(m)	FEM solutions (10^{-3}rad)		Approximate solutions (10^{-3}rad)	Error2 [a] (%)
1	8.6	1.2	1.7003		1.59042	-6.45
2	4.3	1.2	0.82828		0.787992	-4.86
3	2.2	1.2	0.40093		0.389851	-2.76

[a] *Erro2=(Approximate solution- FEM solution)/ FEM solution ×100%*

V. CONCLUTION

Theoretical studies and numerical simulation practice prove that:

(1) The plate-beam theory proposed by Prof. W.F. Zhang [7-13] is universal, and can be easily used to solve the non-uniform torsion problem of the trapezoidal thin wing with double symmetrical airfoil;

(2) Deriving the differential equation model from the energy variation model has a clear concept of mechanics.

(3) Approximate analytical solution presented here is simple and practical, and is basically agree with the FEM analysis results. While the exact analytical solution is almost the same as the results obtained from the FEM analysis. This proves the correctness of the solution of differential equation model.

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