On the Euler's Summability of a Differentiated Fourier Series

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***Abstract*— In this paper we have proved a theorem "On the Euler Summability of a differentiated Fourier series which generalizes known results however our theorem is as follows**

Theorem: If $\int_t^{\pi} \frac{|\emptyset(u)|}{u} du = o(\log\frac{1}{t})$ as $t \to 0$ and $\{nq_n\}$ is a monotonic convergence sequence

Such that $\frac{q(x)}{(1-x)^2 q'(x)} \to \infty$ and

$\int_{(1-x)}^{\pi} \frac{E(n,t)}{t^3} dt = o\left(\frac{q(x)}{(1-x)^2 q'(x)}\right)$ As $x \uparrow 1$, then the series $\sum_{n=1}^{\infty} nB_n(t)$, at $t = x$ is summable (E, q) for $q > 0$ to the value s.

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1. Definition and Notations: Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with partial sums s_n. The sequence Euler's transform of a sequence $\{s_n\}$is defined by

1.1 $\quad t_n = (q+1)^{-n} \sum_{k=0}^{\infty} \binom{n}{k} q^{n-k} s_k$, where $q > 0$, if $t_n \to s$ As $n \to \infty$, we say that $\{s_n\}$ or $\sum_{n=0}^{\infty} a_n$ are summable (E, q) $(q > 0)$, to s or symbolically we write $\{s_n\} \in s$ (E, q) for $(q > 0)$, to s or symbolically we write$\{s_n\} \in s(E, q)$, for $q > 0$, Hardy[3],

It is evident that that $(E, 0)$ is equivalent to convergence.

1.2 Let $f(t)$ be a periodic function with period 2π and integrable in the Lebesgue sense over the interval $(-\pi, \pi)$ and let

1.2.1. $f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + \sin nt) = \sum_{n=1}^{\infty} A_n(t)$ be the Fourier series of f(t).Then the differentiated series of (1.2.1) at $t = x$, is

1.2.2. $\sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} nB_n(x)$

Throughout we use the following notations for $0 < q < 1$.

1.2.3 $\psi(t) = f(x+t) - f(x-t)$

1.2.4. $P(q,t) = 1 + q^2 + 2q \cos t$

1.2.5. $Q(q,t) = \tan^{-1}\left\{\frac{\sin t}{q + \cos t}\right\}$

1.2.6. $E(n,t) = \sum_{k=0}^{\infty} \binom{n}{k} q^{n-k} \sin\left(k + \frac{1}{2}\right) t$

1.2.7. $P(t) = \frac{\psi(t)}{A \sin\left(\frac{3}{2}\right)t} - C$, where C is constant K denotes an absolute constant not necessarily the same in each occurrence.

1.3. Introduction: The following theorem on Euler summability of derived Fourier series is due to Ray {7}.

Theorem A: If g (t) is of bounded variation to the right of t=0 and g (t) = O (1) as $t \to 0$,then derived Fourier series at $t = x$, is summable (E, q) for $q > 0$,to the value s, where $g(t) = \psi(t)\left(4\sin\frac{t}{2}\right)^{-1} - s$ and

$\psi(t) = f(x+t) - f(x-t)$.

Chandra[1] generalized above theorem for Euler summability of Fourier series in the following form:

Theorem B: If $\emptyset(t)\log\frac{1}{t} = o(1)$, $t \to 0$, then, $\sum_{n=0}^{\infty} A_n(x) \in s(E, q)$ for $q > 0$.The object of this paper is to generalize Theorem A and B by establishing the following theorem.

1.4. We assert the following main theorem

Theorem:

1.4.1. If $\int_t^{\pi} \frac{|\psi(u)|}{u} du = o(\log\frac{1}{t})$ as $t \to 0$

and $\{nq_n\}$ is a monotonic convex sequence such that

1.4.2. $\frac{q(x)}{(1-x)^2 q'(x)} \to \infty$,

and $\int_{(1-x)}^{\pi} \frac{E(n,t)}{t^3} dt = o\left(\frac{q(x)}{(1-x)^2 q'(x)}\right)$

as $x \uparrow 1$, then the series $\sum_{n=1}^{\infty} nB_n(t)$, at $t = x$ is summable (E, q) for $q > 0$ to the value S.

1.5. Proof of the theorem: note that

$\Omega B_r(x) = \frac{1}{\pi}\int_0^{\pi} \psi(t)\Omega \sin(\Omega+1)\, dt$ so that

$$S_n(x) = -\frac{1}{\pi}\int_0^{\pi} \psi(t) \frac{d}{dt}\left(\frac{1}{2} + \sum_{\Omega=1}^{n} \cos(\Omega+1)\right) dt$$

$$= -\frac{1}{\pi}\int_0^{\pi} \psi(t) \frac{d}{dt}\left(\frac{\sin\left(n+\frac{3}{2}\right)t}{2\sin\left(\frac{3}{2}\right)t}\right) dt$$

$$= -\frac{1}{\pi}\int_0^{\pi} \psi(t) \left(\frac{2n \sin(3/2)t \cos\left(n+\frac{3}{2}\right)t - \sin(n+1)t}{4\sin^2(3/2)t}\right) dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\psi(t)\sin(n+1)}{4\sin^2\left(\frac{3}{2}\right)t}dt - \frac{2n}{\pi}\int_0^{\pi} \frac{\psi(t)\cos\left(n+\frac{3}{2}\right)t}{4\sin\left(\frac{3}{2}\right)t}dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{P(t)\sin(n+1)t}{\sin\left(\frac{3}{2}\right)t}dt - \frac{2n}{\pi}\int_0^{\pi} P(t)\cos\left(n+\frac{3}{2}\right)tdt$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{P(t)}{t}\sin(n+1)t\,dt$$

$$-\frac{2}{\pi}\int_0^{\pi} P(t)n\cos\left(n+\frac{3}{2}\right)tdt + o(1)$$

Therefore,

$$Q(x) = \frac{2}{\pi q(x)}\int_0^{\pi}\frac{P(t)}{t}\{\sum_{n=1}^{\infty} q_n x^n \sin(n+1)t\}dt$$

$$-\frac{2}{\pi q(x)}\int_0^{\pi} P(t)\int_{n=1}^{\infty} nq_n x^n\cos(n+3/2)t\,dt + o(1)$$

Or

$$Q(x) - o(1) = \frac{2}{\pi q(x)}\int_0^{\pi}\frac{P(t)}{t}E(n,t)dt$$

$$-\frac{2}{\pi q(x)}\int_0^{\pi} P(t)P(q,t)\,dt$$

$= I_1 - I_2$, say

Now, $E(n,t) = I(\sum_{n=1}^{\infty} q_n x^{n+1} e^{int})$

$$= I\{\sum_{n=1}^{\infty}\Delta q_n \sum_{\Omega=1}^{n} e^{i\Omega t}x^{\Omega+1}\}$$

$$= I\left[\sum_{n=1}^{\infty}\Delta q_n\left\{\frac{xe^{it}(1-x^{n+1}e^{int})}{1-xe^{it}}\right\}\right]$$

$$= -I\sum_{n=1}^{\infty}\Delta q_n\left\{\frac{x^{n+2}e^{i(n+1)t}}{1-xe^{it}}\right\}$$

$$= -\sum_{n=1}^{\infty}\Delta q_n\left\{\frac{x^{n+2}e^{i(n+1)t}}{1-xe^{it}}\right\}$$

$$= -\frac{1}{Q(q,t)}\left[\sum_{n=1}^{\infty}\Delta q_n x^{n+2}\sin(n+2)t - x^2\sum_{n=1}^{\infty}\Delta q_{n+1}x^{n+2}\sin(n+2)t\right]$$

1.5.1.=

$$-\frac{1}{Q(q,t)}[\sum_{n=1}^{\infty}\Delta^2 q_n x^{n+2}\sin(n+2)t + (1-x^2)\sum_{n=1}^{\infty}\Delta q_{n+1}x^{n+2}\sin(n+2)t]$$

We have

1.5.2.

$$\sum_{n=1}^{\infty}\Delta q_n x^{n+2} = -(1-x)\sum_{n=1}^{\infty}P_n x^{n+1} = -(1-x)q(x)$$

1.5.3

$$\sum_{n=1}^{\infty}\Delta^2 q_n x^{n+2} = -(1-x)\sum_{n=1}^{\infty}\Delta q_n x^{n+1} = \frac{(1-x)^2}{x}q(x)$$

1.5.4.

$$\sum_{n=1}^{\infty}\Delta(nq_n)x^{n+2} = -(1-x)\sum_{n=1}^{\infty}nq_n x^{n+1}$$

$$= -x(1-x)^2 q'(x)$$

1.5.5.

$$\sum_{n=1}^{\infty}\Delta^2(nq_n)x^{n+2} = -(1-x)\sum_{n=1}^{\infty}\Delta(nq_n)x^{n+1}$$

$$= (1-x)^3 q'(x)$$

Also,

$$\Delta^2(nq_n) = (n+2)q_{n+2} - 2(n+1)q_{n+1} + nq_n$$

$$(N+1)\Delta^2 q_n = (n+1)q_{n+2} - 2(n+1)q_{n+1} + (n+1)q_n$$

This implies that

$$\Delta^2(nq_n) - (n+1)\Delta^2(q_n) = q_{n+2} - q_n$$

Or,

$$(n+1)\Delta^2(q_n) = \Delta^2(nq_n) + q_n - q_{n+2}$$

Hence,

$$\sum_{n=1}^{\infty}(n+1)\Delta^2 q_n x^{n+2}$$

$$= \sum_{n=1}^{\infty}\Delta^2(nq_n)x^{n+2} + \sum_{n=1}^{\infty}q_n x^{n+2} - \sum_{n=1}^{\infty}q_{n+2}x^{n+2}$$

$$= (1-x)^3 q'(x) - \frac{(1-x)^3}{x}q(x)$$

and

1.5.6.

$$\sum_{n=1}^{\infty}(n+1)\Delta q_{n+2}x^{n+2} = x^2 q(x) - (1-x^2)q'(x)$$

So that

1.5.7.

$$\sum_{n=1}^{\infty}(n)\Delta q_n x^{n+1} \le \frac{q(x)}{x^2} + (1-x^2)q'(x)$$

Now using (1.5.1), (1.5.6) and (1.5.7)

$$|I_1| \le \frac{2}{\pi q(x)}\left[\int_0^{(1-x)}\frac{|P(t)|}{t}dt + \int_{(1-x)}^{\pi}\frac{|P(t)|}{t}|E(n,t)|\,dt\right]$$

$$= \frac{2}{\pi q(x)}\left[\int_0^{(1-x)}\frac{|P(t)|}{t}dt + \int_{(1-x)}^{\pi}\frac{|P(t)|}{t}dt\right]\frac{1}{|Q(q,t)|}$$

$$\left|\sum_{n=1}^{\infty}\Delta^2 q_n x^{n+2}\sin(n+2)t + (1-x^3)\sum_{n=1}^{\infty}\Delta q_{n+1}x^{n+2}\sin(n+2)t\right|$$

$$\le \frac{2}{\pi q(x)}\left[\int_0^{(1-x)}\frac{|P(t)|}{t}dt + \int_{(1-x)}^{\pi}\frac{|P(t)|}{t}dt\right]\frac{1}{|Q(q,t)|}$$

$$+\left[\sum_{n=1}^{\infty}(1n+)\Delta^2 q_n x^{n+2} + (1-x^3) + \sum_{n=1}^{\infty}(n+1)\Delta q_{n+1}x^{n+2}\right]$$

$$\le \left|\sum_{n=1}^{\infty}\Delta^2 q_n x^{n+2}\sin(n+2)t(1 - x^3)\sum_{n=1}^{\infty}\Delta q_{n+1}x^{n+2}\sin(n+2)t\right|$$

$$\leq \frac{2}{\pi q(x)}\left[\int_0^{(1-x)} \frac{|P(t)|}{t} dt + \int_{(1-x)}^{\pi} \frac{|P(t)|}{t} dt\right] \frac{1}{Q(x,t)} + \left[\sum_{n=1}^{\infty}(n+1)\Delta^2 q_n x^{n+2} + (1-x^3) + \sum_{n=1}^{\infty}(n+1)\Delta q_{n+1} x^{n+2}\right]$$

$$= \frac{2}{\pi q(x)}\left[\int_0^{(1-x)} |P(t)| dt + \int_{(1-x)}^{\pi} |P(t)| dt\right] \frac{1}{Q(q,t)} + \left[\sum_{n=1}^{\infty}(n+1)(1-x^3)\sum_{n=1}^{\infty} n\,\Delta q_n x^{n+1}\right]$$

$$\leq \frac{k}{q(x)}\left[\frac{1}{(1-x)^3}\int_0^{(1-x)} |P(t)| dt\right] + \int_{(1-x)}^{\pi} \frac{|P(t)|}{t^2} dt (1-x)^3 q'(x)(x+3)$$

$$= \frac{k(x+3)q'(x)}{q(x)}\left[\int_0^{(1-x)} |P(t)| dt + (1-x)^3 \int_{(1-x)}^{\pi} \frac{|P(t)|}{t^2} dt\right]$$

$$= \frac{2k(x+3)q'(x)(1-x)^3}{q(x)}\left[\int_{(1-x)}^{\pi} \frac{E(n,t)}{t^3} dt + \frac{E(n,\pi)}{\pi^2}\right]$$ by integrating by parts

$= o(1)$ as $x \uparrow 1$ by using (1.4.2) and (1.4.3)

Proceeding as in $E(n,t)$, we easily get,(1.5.8)

$$P(q,t) = \frac{-1}{Q(q,t)}\left\{\sum_{n=1}^{\infty}\Delta^2(nq_n)x^{n+2}\cos\left(n+\frac{5}{2}\right)t\right\} + (1-x^3)\sum_{n=1}^{\infty}\Delta(nq_n)x^{n+1}\cos\left(n+\frac{5}{2}\right)t$$

Therefore, Using (1.5.4), (1.5.5) and (1.5.8),

$$I_2 \leq \frac{2}{\pi q(x)}\left[\int_0^{(1-x)} |P(t)| dt + \int_{(1-x)}^{\pi} |P(t)| dt\right] \frac{1}{|Q(q,t)|} \left|\sum_{n=1}^{\infty}\Delta^2(nq_n)x^{n+2}\cos\left(n+\frac{5}{2}\right)t + (1-x^3)\sum_{n=1}^{\infty}\Delta(nq_n)x^{n+1}\cos\left(n+\frac{5}{2}\right)t\right|$$

$$\leq \frac{k}{q(x)}\left[\frac{1}{(1-x)^3}\int_0^{(1-x)} |P(t)dt| + \int_{(1-x)}^{\pi} \frac{|P(t)|}{t^2} dt \left(\left|\sum_{n=1}^{\infty}\Delta^2(nq_n)x^{n+2} + (1-x^3)\sum_{n=1}^{\infty}\Delta(nq_n)x^{n+1}\cos\left(n+\frac{5}{2}\right)t\right|\right)\right]$$

$$\leq \frac{k}{q(x)}\left[\frac{1}{(1-x)^3}\int_0^{(1-x)} |P(t)| dt + \int_{(1-x)}^{\pi} \frac{|P(t)|}{t^2} dt\right]\left(\left|\sum_{n=1}^{\infty}\Delta^2(nq_n)x^{n+2}\right| + (1-x^3)\left|\sum_{n=1}^{\infty}\Delta(nq_n)x^{n+1}\right|\right)$$

$$\leq \frac{k}{q(x)}\left[\frac{1}{(1-x)^3}\int_0^{(1-x)} |P(t)| dt + \int_{(1-x)}^{\pi} \frac{|P(t)|}{t^2} dt\right]\{(1-x)^3 q'(x)(x+3)\}$$

$$= \frac{k(x+3)q'(x)}{q(x)}\left\{\int_0^{(1-x)} |P(t)| dt + (1-x)^3 \int_{(1-x)}^{\pi} \frac{|P(t)|}{t^2} dt\right\}$$

$$= \frac{k(1-x)^3 q'(x)}{q(x)}\left\{\int_{(1-x)}^{\pi} \frac{E(n,t)}{t^3} dt + \frac{E(n,\pi)}{\pi^2}\right\},$$ by integration by parts

By hypothesis (1.4.2) and (1.4.3). This completes the proof of the theorems.

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