On the degree of Approximation of function Belonging to the $Lip(\alpha, r)$ Class by (C,2) (E,1)

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Abstract

In this paper we obtained a theorem on the degree of approximation of functions belonging to $Lip(\alpha, r)$ class by (C,2)(E,1) product mean of its Fourier series.
Keywords: Cesaro matrices, Euler matrices, degree of approximation.

1 Introduction

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue. The Fourier series of $f(t)$ is given by

$$f(t) \approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + bn \sin nt) \tag{1.1}$$

A function $f \in Lip\ \alpha$ if

$$|f(x+t) - f(x)| = \mathrm{O}(|t|^{\alpha}) \text{ for } 0 < \alpha \leq 1 \tag{1.2}$$

And a function $f \in Lip\ (\alpha, r)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{1/r} = 0(|t|^{\alpha}) \text{ for } 0 < \alpha \leq |f\ r \geq | \tag{1.3}$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial tn of order n is defined by (Zygmund [3])

$$||tn - f||_{\infty} = \sup\{|tn(x) - f(x)| : x \in R\} \tag{1.4}$$

if

$$E_n^1 = 2^{-n}\sum_{k=0}^{n}\binom{n}{k} s_k \to s \text{ as } n \to \infty \tag{1.5}$$

then an infinite series $\sum_{k=0}^{n} a_k$ with the partial sums s_n is said to be summable (E,1) to the S, (Hardy [2]). The (C,2) transform of the (E,1) transform E_n^1 defines the (C,2) (E,1) transform of the partial sums s_n of the series $\sum_{k=0}^{\infty} a_k$. Thus if

$$(C_2, E)_n^1(x) = \sum_{k=0}^{n} \frac{2(n-k+1)}{(n+1)(n+2)} E_k^1 \to s \text{ as } n \to \infty$$

where E_n^1 denotes the (E,1) transform of s_n, then the series $\sum_{k=0}^{\infty} a_k$ is said to be summable by (C,2) (E,1) means to s. We shall use the following notations:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

2 Main Result

We shall generalize the theorem of Albayrak [1].

Theorem 2.1. If $f : R \to R$ is 2π periodic, Lebesgue integrabble on $[-\pi, \pi]$ and belonging to *Lip* (α, r) class then the degree of approximation of f by the (C,2) (E,1) product means of its Fourier series satisfies for $n = 0, 1, 2, \ldots$

$$||(C_2E)_n^1 - f||_\infty = \mathrm{O}\left(\frac{1}{(n+1)^{\frac{\alpha}{2}-\frac{1}{2r}}}\right) \text{ for } 0 < \alpha \leq 1 \;\&\; r \geq 1$$

3 Proof

The n$^{\text{th}}$ partial sum $S_n(x)$ of the series (1.1) at $t = x$ is defined as

$$S_n(x) = f(x) + \frac{1}{2\pi} \int_0^{\pi} \phi(t) \frac{\sin(n+1/2)t}{\sin t/2} dt$$

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So the (E,1) means of the series (1.1) are

$$
\begin{aligned}
E_n^1 &= 2^{-n} \sum_{k=0}^{\infty} \binom{n}{k} s_k(x) \quad \text{for } n = 0, 1, 2, \ldots \\
&= f(x) + \frac{1}{2^{n+1}\pi} \int_0^{\pi} \frac{\phi(t)}{\sin t/2} \left(\sum_{k=0}^{n} \binom{n}{k} \sin\left(k + \frac{1}{2}\right) t \right) dt \\
&= f(x) + \frac{1}{2^{n+1}\pi} \int_0^{\pi} \frac{\phi(t)}{\sin t/2} Im \left(e^{\frac{it}{2}} (1 + e^{it})^n \right) \\
&= f(x) + \frac{1}{2^{n+1}\pi} \int_0^{\pi} \frac{\phi(t)}{\sin t/2} Im \ (e^{\frac{it}{2}})(1 + \cos t + \sin t)^n \ dt \\
&= f(x) + \frac{1}{2^{n+1}\pi} \int_0^{\pi} \frac{\phi(t)}{\sin t/2} Im \left\{ e^{\frac{it}{2}} 2^n \cos^n \frac{t}{2} \left(\cos \frac{t}{2} + i \sin \frac{t}{2} \right)^n \right\} dt \\
&= f(x) + \frac{1}{2^{n+1}\pi} \int_0^{\pi} \frac{\phi(t)}{\sin t/2} 2^n \cos^n \left| \frac{t}{2} \right| Im \left\{ \left(\cos \frac{t}{2} + i \sin \frac{t}{2} \right) \left(\cos \frac{nt}{2} + i \sin \frac{nt}{2} \right) \right\} dt \\
&= f(x) + \frac{1}{2\pi} \int_0^{\pi} \frac{\phi(t)}{\sin t/2} \cos^n \frac{t}{2} \sin(n+1) \frac{t}{2} dt
\end{aligned}
$$

therefore (C,2) (E,1) product means of the series (1.1) are

$$
\begin{aligned}
(C_2, E)_n^1(x) &= \sum_{n=0}^{n} \frac{2(n-k+1)}{(n+1)(n+2)} E_k^1(x) \\
&= \frac{2(n+1)}{(n+1)(n+2)} \sum_{k=0}^{n} E_k^1(x) - \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} k \cdot E_k^1(x) \\
&= f(x) + \frac{1}{\pi(n+2)} \int_0^{\pi} \frac{\phi(t)}{\sin \frac{t}{2}} \left(\sum_{k=0}^{n} k \cdot \cos^k \left(\frac{t}{2} \right) \sin(k+1) \frac{t}{2} \right) dt \\
&\quad - \frac{1}{\pi(n+1)(n+2)} \int_0^{\pi} \frac{\phi(t)}{\sin t/2} \left(\sum_{k=0}^{n} k \cdot \cos^k \frac{t}{2} \sin(k+1) \frac{1}{2} \right) dt \\
&= f(x) + I_1 - I_2 \qquad (3.1)
\end{aligned}
$$

Now

$$
\begin{aligned}
I_2 &= \frac{1}{\pi(n+1)(n+2)} \int_0^{\pi} \frac{\phi(t)}{\sin \frac{t}{2}} \left(\sum_{k=0}^{n} k \cdot \cos^k \left(\frac{t}{2} \right) \sin(k+1) \frac{t}{2} \right) dt \\
&= \frac{1}{\pi(n+1)(n+2)} \left(\int_0^{\frac{1}{\sqrt{n+1}}} + \int_{\frac{1}{\sqrt{n+1}}}^{\pi} \right) \frac{\phi(t)}{\sin \frac{t}{2}} \left(\sum_{k=0}^{n} k \cdot \cos^k \left(\frac{t}{2} \right) \sin(k+1) \frac{t}{2} \right) dt \\
I_2 &= I_{21} + I_{22} \qquad (3.2)
\end{aligned}
$$

Applying the fact that $f \in \; Lip\,(\alpha, r)$ and $\sin\frac{t}{2} \geq \frac{t}{\pi}$. We have

$$\begin{aligned} |I_{21}| &\leq \frac{n}{2\pi(n+2)} \int_0^{\frac{1}{\sqrt{n+1}}} \left| \frac{\mathrm{O}|t|}{\sin\frac{t}{2}} \right| dt \\ &\leq \frac{n}{2(n+2)} \int_0^{\frac{1}{\sqrt{n+1}}} \left| \frac{\phi|t|}{t} \right| dt \end{aligned}$$

Using Hölder inequality.

$$\begin{aligned} |I_{21}| &\leq \frac{n}{2(n+2)} \left\{ \left[\int_0^{\frac{1}{\sqrt{n+1}}} \left| \frac{\phi(t)}{t} \right|^r dt \right]^{\frac{1}{r}} \right\} \left\{ \int_0^{\frac{1}{\sqrt{n+1}}} (1)^s dt \right\}^{\frac{1}{s}} \\ &\leq \mathrm{O}\left(\frac{1}{(n+1)^{\alpha/2}} \right) \cdot \frac{1}{(n+1)^{1/2s}} \\ &= \mathrm{O}\left(\frac{1}{(n+1)^{\alpha/2}} \right) \left(\frac{1}{(n+1)^{\frac{1}{2}-\frac{1}{2r}}} \right) \\ &\leq \mathrm{O}\left(\frac{1}{(n+1)^{\frac{\alpha}{2}+\frac{1}{2}-\frac{1}{2r}}} \right) \end{aligned} \tag{3.3}$$

By using $k \cdot \cos^k\left(\frac{t}{2}\right) = \frac{-2\left(\cos^k\left(\frac{t}{2}\right)\right)}{\sin\frac{t}{2}} \cdot \cos\frac{t}{2}$, we estimate I_{22}.

$$\begin{aligned} |I_{22}| &\leq \frac{1}{\pi(n+1)(n+2)} \int_{\frac{1}{\sqrt{n+1}}}^{\pi} \left| \frac{\phi(t)}{\sin\frac{t}{2}} \right| \left| \sum_{k=0}^{n} \frac{-2(cos^k\left(\frac{t}{2}\right))}{\sin\frac{t}{2}} \right| \cdot \cos\frac{t}{2} \left| \sin(k+1)\frac{t}{2} \right| dt \\ &\leq \frac{-2\pi}{(n+1)(n+2)} \int_{\frac{1}{\sqrt{n+1}}}^{\pi} t^{\alpha-2} \left(\frac{\cos^{n+1}\frac{t}{2} - 1}{\cos\frac{t}{2} - 1} \right) dt \\ &= \frac{-2\pi}{(n+1)(n+2)} \int_{\frac{1}{\sqrt{n+1}}}^{\pi} t^{\alpha-2} \left(\frac{\cos^{n+1}\frac{t}{2} - 1}{\cos\frac{t}{2} - 1} \right) dt \\ &= \frac{2}{\pi(n+1)(n+2)} \int_{\frac{1}{\sqrt{n+1}}}^{\pi} t^{\alpha-2} \sin t \frac{(\cos^{n+1} t - 1)}{(\cos t - 1)^2} dt \\ &= A - B. \end{aligned} \tag{3.4}$$

Now

$$\begin{aligned} |A| &= \left| \frac{2}{\pi(n+2)} \int_{\frac{1}{\sqrt{n+1}}}^{\pi} t^{\alpha-2} \frac{\cos^n t \cdot \sin t}{\cos t - 1} dt \right| \\ &\leq \frac{c1}{(n+2)} \int_{\frac{1}{\sqrt{n+1}}}^{\pi} t^{\alpha-3} dt = \frac{C1}{(n+2)} \int_{\frac{1}{\sqrt{n+1}}}^{\pi} \frac{\phi(t)}{t^3} dt \end{aligned}$$

Using h older inequality.

$$|A| \leq \frac{C1}{(n+2)} \left[\int_{\frac{1}{\sqrt{n+1}}}^{\pi} \left(\frac{\phi(t)}{t^3} \right)^r dt \right]^{1/r} \cdot \left[\int_{\frac{1}{\sqrt{n+1}}}^{\pi} (1)^s dt \right]^{1/s}$$

$$\leq \frac{C1}{(n+2)} \mathrm{O} \left(\frac{1}{(n+1)^{\frac{\alpha}{2}-\frac{3}{2}}} \right) \cdot \frac{1}{(n+1)^{\frac{1}{2s}}}$$

$$\leq \frac{C1}{(n+2)} \mathrm{O} \left(\frac{1}{(n+1)^{\frac{\alpha}{2}-\frac{3}{2}}} \right) \cdot \frac{1}{(n+1)^{\frac{1}{2}-\frac{1}{2r}}}$$

$$\leq \mathrm{O} \left(\frac{1}{(n+1)^{\frac{\alpha}{2}-\frac{1}{2r}}} \right) \tag{3.5}$$

where $C1$ is positive constant.

$$|B| = \left| \frac{2}{\pi(n+1)(n+2)} \int_{\frac{1}{\sqrt{n+1}}}^{\pi} t^{\alpha-2} \frac{\sin t(\cos^{n+1} t - 1)}{(\cos t - 1)^2} dt \right|$$

$$\leq \frac{C2}{(n+1)(n+2)} \int_{\frac{1}{\sqrt{n+1}}}^{\pi} \frac{\phi(t)}{t^3} dt$$

Using Hölder inequality

$$|B| \leq \frac{C2}{\pi(n+1)(n+2)} \left[\int_{\frac{1}{\sqrt{n+1}}}^{\pi} \left| \frac{\phi(t)}{t^3} \right|^r dt \right]^{1/r} \cdot \left[\int_{\frac{1}{\sqrt{n+1}}}^{\pi} (1)^s dt \right]^{1/s}$$

$$\leq \frac{C2}{(n+1)(n+2)} \int_{\frac{1}{\sqrt{n+1}}}^{\pi} (t^{\alpha-3})^r dt \cdot \left(\frac{1}{(n+1)^{\frac{1}{2s}}} \right)$$

$$|B| \leq \frac{C2}{(n+1)(n+2)} \mathrm{O} \left(\frac{1}{(n+1)^{\frac{\alpha}{2}-\frac{3}{2}}} \right) \cdot \left(\frac{1}{(n+1)^{\frac{1}{2}-\frac{1}{2r}}} \right)$$

$$|B| \leq \mathrm{O} \left(\frac{1}{(n+1)^{\frac{\alpha}{2}+1-\frac{1}{2r}}} \right) \tag{3.6}$$

In similar way, we can obtain

$$I_1 = \mathrm{O} \left(\frac{1}{(n+1)^{\frac{\alpha}{2}-\frac{1}{2r}}} \right) \tag{3.7}$$

from (3.1), (3.2), (3.3), (3.4), (3.5), (3.6) and (3.7) we have

$$||(C_2, E)_n^1 - f(x)||_\infty = \mathrm{O} \left(\frac{1}{(n+1)^{\frac{\alpha}{2}-\frac{1}{2r}}} \right)$$

This completes the proof of the theorem.

4 Corollary

If $r = \infty$ then above theorem reduces to Inci (Albayrak [1]).

$$||(C_2, E)^1_n - f(x)||_\infty = \mathrm{O}\left(\frac{1}{(n+1)^{\frac{\alpha}{2}}}\right).$$

References

[1] Albayrak Inci etc; On the degree of approximation of function belonging to the *Lip* α by (C,2) (E,1) Int. J. of Math. Analysis, Vol. 4, 2010.

[2] Hardy, G.H.; Divergent Series, First Ed. Oxford Uni. Press, 1949.

[3] Zygmund, A.; Trigonometric series, 2nd rev. Ed. Cambridge Uni. Press, Cambridge. 1968.