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On the Algebraic Foundations, Morphisms and Aggregation Operators of Q̃–Fuzzy Soft Subrings

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Abstract - This paper presents a unified algebraic framework for Q̃-fuzzy soft subrings. It extends the classical notions of fuzzy subrings and Q̃-fuzzy subrings to the broader setting of Q̃-fuzzy soft subsets, where membership values are defined on the interval $[0, \infty)$. We introduce and systematically investigate Q̃-fuzzy soft subrings under a generalized triangular norm (T-norm) framework. Fundamental algebraic properties, structural characterizations, and relationships with existing fuzzy and soft algebraic structures are established. Emphasis is placed on the study of morphisms, including homomorphisms, inverse images, direct images, and isomorphisms, to examine the preservation of algebraic structures within the Q̃-fuzzy soft environment. Furthermore, aggregation operators induced by T-norms and related fuzzy connectives are analysed, and their roles in constructing and combining Q̃-fuzzy soft subrings are explored. Several new results concerning closure properties, intersections, lattice-theoretic aspects, and algebraic invariance are proved and illustrated with suitable examples. The proposed framework generalizes existing fuzzy and soft algebraic models and provides a foundation for further applications in algebraic systems, decision-making, information processing, and artificial intelligence.

Keywords: Q̃-fuzzy soft subset; Q̃-fuzzy soft subring; fuzzy soft algebra; aggregation operator; triangular norm (T-norm); homomorphism; inverse image; isomorphism; lattice structure; algebraic invariance.

1. INTRODUCTION

Fuzzy algebraic structures provide an effective mathematical framework for modelling uncertainty, vagueness, and incomplete information in algebraic systems. Since the introduction of fuzzy sets by Zadeh [12], the concept has been successfully incorporated into various branches of mathematics, leading to the development of fuzzy groups, fuzzy rings, fuzzy ideals, and many other generalized algebraic structures. Among these, fuzzy subrings have attracted considerable attention due to their ability to extend classical ring theory to uncertain environments.

The notion of Q̃-fuzzy sets further generalizes fuzzy sets by allowing membership values to be taken from the interval $[0, \infty)$, thereby providing a richer and more flexible representation of uncertainty. This extension has proved useful in situations where traditional membership scales are insufficient for describing varying degrees of belongingness. Consequently, Q̃-fuzzy algebraic structures have become an important area of investigation in modern fuzzy mathematics.

Meanwhile, Molodtsov's soft set theory [13] introduced a parameterized approach to uncertainty modeling and has since become a powerful tool in decision-making, information processing, and algebraic investigations. The integration of fuzzy sets and soft sets led to the emergence of fuzzy soft structures, which combine the advantages of both theories. Significant contributions by Maji et al., Pei

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et al., Aktaş and Çağman [1], Acar et al. [4], and others established the foundations of soft algebraic systems and their applications.

Researchers have also investigated fuzzy soft subrings and related algebraic structures under various operational frameworks. Aygünoğlu and Aygün [2] employed triangular norms (T-norms) in the study of fuzzy soft algebraic systems, providing an effective mechanism for combining fuzzy information. Furthermore, Solairaju and Geethalakshmi [10] examined fuzzy soft subrings and associated lattice-theoretic properties, illustrating the usefulness of T-norm-based operations in algebraic investigations.

Despite these developments, a comprehensive treatment of Q̃-fuzzy soft subrings from the perspectives of algebraic foundations, morphisms, and aggregation operators remains limited. In particular, the structural behaviour of Q̃-fuzzy soft subrings under homomorphic mappings and aggregation mechanisms has not been sufficiently explored. Such investigations are important because they provide deeper insight into the preservation and construction of algebraic structures within uncertain environments.

Motivated by this observation, the present paper develops a unified framework for the study of Q̃-fuzzy soft subrings. We establish their fundamental algebraic properties and characterize their behaviour under various operations. Special attention is devoted to morphisms, including homomorphisms, inverse images, direct images, and isomorphisms, with the aim of identifying algebraic properties that remain invariant under these mappings. In addition, aggregation operators derived from T-norms and related fuzzy operators are introduced and analysed as tools for combining and constructing Q̃-fuzzy soft subrings. Their effects on closure properties, intersections, and algebraic stability are investigated in detail.

The paper also establishes several structural results concerning intersections, lattice-theoretic properties, and preservation theorems. Through illustrative examples, the proposed framework demonstrates how Q̃-fuzzy soft subrings can serve as a natural extension of existing fuzzy and soft algebraic models while providing greater flexibility for uncertainty representation.

The organization of the paper is as follows. Section 2 recalls the necessary preliminary concepts related to fuzzy sets, soft sets, Q̃-fuzzy sets, and Q̃-fuzzy soft subsets. Section 3 introduces Q̃-fuzzy soft subrings and develops their basic algebraic properties. Section 4 investigates aggregation operators, intersections, and related structural characteristics. Section 5 studies morphisms, including homomorphisms, inverse images, direct images, and isomorphisms with examples, applications, and concluding structural results. Finally, the paper highlights potential directions for future research in fuzzy algebra, information processing, and uncertainty-based mathematical systems.

2. DEFINITIONS AND PRELIMINARIES

To guarantee completeness, we present the basic definitions and notations for fuzzy soft subsets introduced by Ahmad et al. [17] [2009] and Maji et al. [8] [2003] in this section. In this study, let R^g be a subring over which all subsets are taken. Let u_D be the initial universe subset of discourse.

Definition 2.1[20] The function known as a t-norm, represented by T, is a mapping

$$T: [0,1] \times [0,1] \to [0,1].$$

The following conditions are satisfied for all $\phi_1, \phi_2, \phi_3, \phi_4 \in [0,1]$

(i)$T(\phi_1, \phi_2) = T(\phi_2, \phi_1)$

(ii)$T(\phi_1, T(\phi_2, \phi_3)) = T(T(\phi_1, \phi_2), \phi_3)$

(iii)$T(\phi_1, 1) = T(1, \phi_1) = \phi_1$

(iv)If $\phi_1 \le \phi_3$ and $\phi_2 \le \phi_4$, then $T(\phi_1, \phi_2) \le T(\phi_3, \phi_4)$.

Remark 2.2 A commonly applied t-norm is the minimum t-norm, defined by

$$T(\phi_1, \phi_2) = \min\{\phi_1, \phi_2\}.$$

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Definition 2.3 [12] Let Y be a non-empty subset. A fuzzy subset K on X is characterized by its membership function $M_a: Y \to [0,1]$, where for each $y \in Y$, the value $M_a(x)$ represents the degree of membership of y in K.The complement of the fuzzy subset K is defined by $K^c(x) = 1 - M_a(x), \forall x \in X$. The constant mappings $M_a(x) = 0$ and $M_a(x) = 1$ represent the null fuzzy subset and the universal fuzzy subset, respectively. These are denoted by 0_n and 1_w.

Definition 2.4 [4] Let u_D be a universal subset and R be a subset of parameters. A pair (h, R) is called a soft subset over u_D if $h: R \to \mathcal{P}(u_D)$, where $\mathcal{P}(u_D)$ denotes the power subset of u_D.

Definition 2.5 [13] Let E_p be a collection of parameters and u_D be the universal subset of discourse. Let $\mathcal{P}(u_D)$ denote the power subset of u_D. An ordered pair$\left((h_R)^{u_D},\ E_p\right)$, where $(h_R)^{u_D}\colon E_p \to \mathcal{P}(u_D)$, is called a soft subset over u_D. Thus, a soft subset over u_D represents a parameterized family of subsets of u_D.That is, $\{(h_R)^{u_D}(r) \mid r \in E_p\} \subseteq \mathcal{P}(u_D)$, where each $(h_R)^{u_D}(r)$ corresponds to the subset of elements of u_D associated with the parameter r.

Remark 2.6: A soft subset $\left((h_R)^{u_D},\ E_p\right)$ over u_D can be viewed as a mapping that takes to each parameter $r \in E_p$ a subset of u_D, thereby forming a parameterized structure which is suitable for uncertainty modelling.

Definition 2.7 [17] Let u_D be the universal subset of discourse. Let E_p be a collection of parameters and let $R \subseteq E_p$. A fuzzy soft subset on u_D is defined by a pair $\left((h_R)^{u_D},\ E_p\right)$, where $(h_R)^{u_D}: R \to J^{(u_D)}$, where $J = [0,1]$ represents the unit interval, and $J^{(u_D)}$ represents the collection of all fuzzy subsets of u_D. That is, for each $r \in R$, $((h_R)^{u_D})(r)\colon u_D \to [0,1]$ is a fuzzy subset of u_D. Hence, $(h_R)^{u_D}(r)(y)$ gives the degree of membership values of $y \in u_D$ with respect to the parameter r.
The subset follow $(h_R)^{u_D}$ is called a fuzzy soft subset over u_D.

Remark 2.8 A fuzzy soft subset $(h_R)^{u_D}$ can be viewed as a parameterized family of fuzzy subsets of the universe of discourse u_D, where each parameter $r \in R$ corresponds to a fuzzy subset $(h_R)^{u_D}(r)$.

Remark 2.9 For each $r \in R$, define ${(h_R)^{u_D}}_r(y) = (h_R)^{u_D}(r)(y),\ \forall y \in u_D$, which simplifies notation in the following sections.

Remark 2.10, for a given element $z \in Z$, a classical soft subset $(h_Z)^{u_D}$ can be regarded as a fuzzy soft subset $(h_Z)^{u_D}$, where the characteristic function associated with $(h_Z)^{u_D}$ at z is defined by the mapping

$$((h_Z)^{u_D})_z(a) = \{\omega((h_Z)^{u_D})_z(z)\}(a) = \begin{cases} 1, & \text{if } a \in ((h_Z)^{u_D})(z), \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Thus, every classical soft subset can be interpreted as a special case of a fuzzy soft subset with respect to its characteristic function.

Example 2.11 Let $u_D = \{s_1, s_2, s_3\}, Z = \{z_1, z_2\}$
Define a classical soft subset $(h_Z)^{u_D}$ is given as follows $h(z_1) = \{s_1, s_3\}, h(z_2) = \{s_2\}$
Now, the corresponding given fuzzy soft subset representation is given as

For $z_1 \in Z$, $((h_Z)^{u_D})_{z_1}(s) = \begin{cases} 1, & s \in \{s_1, s_3\}, \\ 0, & \text{otherwise} \end{cases}$ i.e., $((h_Z)^{u_D})_{z_1} = \{(s_1, 1), (s_2, 0), (s_3, 1)\}$

For $z_2 \in Z$, $((h_Z)^{u_D})_{z_2}(s) = \begin{cases} 1, & s \in \{s_2\}, \\ 0, & \text{otherwise} \end{cases}$ i.e., $((h_Z)^{u_D})_{z_2} = \{(s_1, 0), (s_2, 1), (s_3, 0)\}$

Definition 2.12 Assume that the subgroup G_s has a non-null fuzzy soft subset$(h_Z)^{u_D}$. If and only if G_s's fuzzy subgroup $(h_Z)^{u_D}$(a) is true for all a in R is then $(h_Z)^{u_D}$ is assumed to be a G_s's soft subgroup.

Definition 2.13 [11] A fuzzy soft non-empty subset over the subgroup G_s is represented by $(h_Z)^{u_D}$ If and only if, then $(h_Z)^{u_D}$ is considered a fuzzy soft subgroup of G_s. Then

(1) $((h_Z)^{u_D})_a(x, y) \geq T\{((h_Z)^{u_D})_a(x),\ ((h_Z)^{u_D})_a(y)\}$ (2.2)

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(2) $((h_Z)^{u_D})_a(x^{-1}) \geq ((h_Z)^{u_D})_a(x)$ is true for each $x, y \in G_s$. (2.3)
where all a in P satisfy the fuzzy subgroup $(h_Z)^{u_D}(a) = ((h_Z)^{u_D})_a$ of G_s.

Definition 2.14 [4] A soft non-empty fuzzy subset over a subring $(R^g, +, .)$ is the one is known as $(h_Z)^{u_D}$ if and only if R^g's sub subring $((h_Z)^{u_D})$(a) is true for all a in R is then $(h_Z)^{u_D}$ is usually referred to as a soft subring of R^g.

Definition 2.15 [2] Let us construct a fuzzy soft subset $(h_Z)^{u_D}$ over a non-null subring R^g. Then, and only then, $(h_Z)^{u_D}$ is it said that is a subset over R^g.

(1) $((h_Z)^{u_D})_a(x+y) \geq T\{((h_Z)^{u_D})_a(x), ((h_Z)^{u_D})_a(y)\}$ (2.4)

(2) $((h_Z)^{u_D})_a(-x) \geq ((h_Z)^{u_D})_a(x)$ (2.5)

(3) $((h_Z)^{u_D})_a(x.y) \geq \mathrm{T}\{((h_Z)^{u_D})_a(x), ((h_Z)^{u_D})_a(y)\}$ (2.6)

is true for each $x,y \in R^g$ and R^g's fuzzy sub subring $((h_Z)^{u_D})(a) = ((h_Z)^{u_D})_a$ is true for all a in Z.

Example 2.16 A soft subring can be thought of as a fuzzy soft subring, because every soft subset is a fuzzy soft subset. A fuzzy subring is defined by the characteristic function associated with each subring. As a result, each soft subring can be thought of as a variant of a fuzzy soft subring.

Example 2.17 Let N be the collection of all natural numbers, and define a mapping $: N \to I^{(u_D)}$ by $(h_Z)^{u_D}(n) = ((h_Z)^{u_D})_n : u_D \to I$, for every $n \in N$, where

$$((h_Z)^{u_D})_n(a) = \{(h_Z)^{u_D}(n)\}(a) = \begin{cases} \frac{1}{n}, & \text{if } a = k \cdot 2^n, \text{ for some } k \in Z_I, \\ 0, & \text{otherwise.} \end{cases}$$

Here, Z_I denotes the subset of all integers.
Therefore, the subset $((h_Z)^{u_D}$ constitutes a fuzzy soft subset. Moreover, since each $((h_Z)^{u_D}$ defines a fuzzy subring structure on u_D, the fuzzy soft subset $((h_Z)^{u_D})_n$ forms a fuzzy soft subring.

Example 2.18 Let $u_D = \mathbb{Z}$, $N = \{1,2\}$

Define $((h_N)^{u_D})_1(a) = \begin{cases} 1, & a \text{ is even} \\ 0, & \text{otherwise} \end{cases}$ and $((h_N)^{u_D})_2(a) = \begin{cases} 0.5, & a \text{ is multiple of } 3 \\ 0, & \text{otherwise} \end{cases}$

Then each $(f)_m$is a fuzzy subring of $\mathbb{Z}$.
Hence, $((h_Z)^{u_D})$ forms a fuzzy soft subring.

Definition 2.19 [21] Let $(h_R)^{u_D}$ be a fuzzy soft subset that is non-empty over u_D, the universal subset of discourse. Then the soft subset $(u_D)_\alpha = \{((u_D)_a)_\alpha$ / each a in R } is introduced to as a α-level soft subset for all α∈ [0, 1].

Definition 2.20 [1] Let $((h_R)^{u_D})_a$ be a non-empty fuzzy soft subring in $(R^g, +, .)$ and θ: $R^g \to R^{g'}$ be the mapping. The definition of $((h_R)^{u_D})_a{}^\theta$ is as follows $((h_R)^{u_D})_a{}^\theta : R^g \to [0, 1]$ by $((h_R)^{u_D})_a{}^\theta(x) = ((h_R)^{u_D})_a(\theta x)$ (by using 2.1 definition)

3. Q̃~FUZZY SOFT SUB SUBRINGS AND ITS PROPERTIES

Definition 3.1 Let $(\tilde{s}_1, P_1, U^d)_a$ and $(\tilde{s}_2, P_2, U^d)_a$ be two fuzzy soft subsets.
Assume that $P_1 \cap P_2 \neq \emptyset$. Define $Z = P_1 \cap P_2$ Then the intersection of $(\tilde{s}_1, P_1, U^d)_a$ and $(\tilde{s}_2, P_2, U^d)_a$ is the fuzzy soft subset $(\tilde{s}, Z, U^d)_a$ where for all $z \in Z$, $(\tilde{s}, Z, U^d)_a = T((\tilde{s})_a(x,q) = T((\tilde{s}_1)_a(x,q), (\tilde{s}_2)_a(x,q))$ for all $x \in U^d$, $q \in Q$. In this case, we write $(\tilde{s}_1, P_1, U^d)_a \cap (\tilde{s}_2, P_2, U^d)_a = (\tilde{s}, Z, U^d)_a$

Definition 3.2 Let $\{(\tilde{s}_i, P_i, U^d)_a\}_{i\in I}$ be a family of fuzzy soft subsets. Assume that $\bigcap_{i\in I} P_i \neq \emptyset$ Define $Z = \bigcap_{i\in I} P_i$. Then the intersection of the family $\{(\tilde{s}_i, P_i, U^d)_a\}_{i\in I}$ is the fuzzy soft subset

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$(\tilde{s}, Z, U^d)_a$ where for every $z \in Z$, $(\tilde{s})_a(x,q) = T_{i\in I}((\tilde{s}_i)_a(x,q))$ where $T_{i\in I}$denotes the iterated application of the T-norm for all $x \in U^d$, $q \in Q$. In this case, we write $\tilde{s}, Z, U^d)_a = \bigcap_{i\in I}(\tilde{s}_i, P_i, U^d)_a$

If $T = \min$, then Definitions 3.1 and 3.2 reduce to the classical intersection of fuzzy soft subsets.

Definition 3.3 Let $(R^g, +,\cdot)$be a fuzzy soft subring. Let $(\tilde{s}, \tilde{P}, \widetilde{U}^d)_a : R^g \times Q \to [0,1], (\tilde{s}, \tilde{P}, \widetilde{U}^d)_a \neq 0$

Then $(\tilde{s}, \tilde{P}, \widetilde{U}^d)_a$is called a $\tilde{Q}$-fuzzy soft subring over R^g if

(1Q̃–FSR) $(\tilde{s}, \tilde{P}, \widetilde{U}^d)_a(x+y,q) \geq T((\tilde{s}, \tilde{P}, \widetilde{U}^d)_a(x,q), (\tilde{s}, \tilde{P}, \widetilde{U}^d)_a(y,q))$

(2Q̃–FSR) $(\tilde{s}, \tilde{P}, \widetilde{U}^d)_a(-x,q) \geq (\tilde{s}, \tilde{P}, \widetilde{U}^d)_a(x,q)$

(3Q̃–FSR) $(\tilde{s}, \tilde{P}, \widetilde{U}^d)_a(x\cdot y,q) \geq T((\tilde{s}, \tilde{P}, \widetilde{U}^d)_a(x,q), (\tilde{s}, \tilde{P}, \widetilde{U}^d)_a(y,q))$

for all $x, y \in R^g$and $q \in Q$. Also, for every $a \in \tilde{P}$, $(\tilde{s}, \tilde{P}, \widetilde{U}^d)_a$ defines a fuzzy subring of R^g.

Theorem 3.4 Let $(\tilde{s}^\epsilon, \tilde{P}^\epsilon, \widetilde{U}^{\epsilon d})_a$be a $\tilde{Q}$-fuzzy soft subring over R^g.

(i) $(\tilde{s}^\epsilon, \tilde{P}^\epsilon, \widetilde{U}^{\epsilon d})_a(x,q) \leq (\tilde{s}^\epsilon, \tilde{P}^\epsilon, \widetilde{U}^{\epsilon d})_a(0,q), \forall x \in R^g$

(ii) Define $R^g_{(\tilde{s}^\epsilon)_a} = \{x \in R^g \mid (\tilde{s}^\epsilon, \tilde{P}^\epsilon, \widetilde{U}^{\epsilon d})_a(x,q) = (\tilde{s}^\epsilon, \tilde{P}^\epsilon, \widetilde{U}^{\epsilon d})_a(0,q)\}$

Then $R^g_{(\tilde{s}^\epsilon)_a}$is a $\tilde{Q}$-fuzzy soft subring of R^g.

Proof. Let $x \in R^g$. $(\tilde{s}^\epsilon)_a(0,q) = (\tilde{s}^\epsilon)_a(x+(-x),q) \geq T((\tilde{s}^\epsilon)_a(x,q), (\tilde{s}^\epsilon)_a(-x,q))$

Using (2Q̃–FSR) $(\tilde{s}^\epsilon)_a(-x,q) \geq (\tilde{s}^\epsilon)_a(x,q)$.Thus $(\tilde{s}^\epsilon)_a(0,q) \geq T((\tilde{s}^\epsilon)_a(x,q), (\tilde{s}^\epsilon)_a(x,q))$

Since $T(a,a) \leq a$, we obtain $(\tilde{s}^\epsilon)_a(x,q) \leq (\tilde{s}^\epsilon)_a(0,q)$

Part (i) proved

Now let $x, y \in R^g_{(\tilde{s}^\epsilon)_a}$ then $(\tilde{s}^\epsilon)_a(x,q) = (\tilde{s}^\epsilon)_a(0,q)$

$(\tilde{s}^\epsilon)_a(y,q) = (\tilde{s}^\epsilon)_a(0,q)$

$$(\tilde{s}^\epsilon)_a(x+(-y),q) \geq T((\tilde{s}^\epsilon)_a(x,q), (\tilde{s}^\epsilon)_a(y,q))$$
$$= T((\tilde{s}^\epsilon)_a(0,q), (\tilde{s}^\epsilon)_a(0,q)) = (\tilde{s}^\epsilon)_a(0,q)$$

Hence, $x - y \in R^g_{(\tilde{s}^\epsilon)_a}$

Similarly $(\tilde{s}^\epsilon)_a(xy,q) = (\tilde{s}^\epsilon)_a(0,q)$

Part (ii) proved

Corollary 3.5 Let R^g be finite. Let $(\tilde{s}, \tilde{A}, \widetilde{U})_a$be a $\tilde{Q}$-fuzzy soft subring.

Define: $\tilde{C} = \{x \in R^g : (\tilde{s}, \tilde{A}, \widetilde{U})_a(x,q) = (\tilde{s}, \tilde{A}, \widetilde{U})_a(0,q)\}$

Then $\widetilde{C}$ is called a $\tilde{Q}$-fuzzy soft crisp subring of R^g.

Theorem 3.6 Let $(\tilde{s}_1, P_1, U^d)_a$, $(\tilde{s}_2, P_2, U^d)_a$ be two $\tilde{Q}$-fuzzy soft subrings.

Define: $(\tilde{s}, Z, U^d)_a = (\tilde{s}_1, P_1, U^d)_a \cap (\tilde{s}_2, P_2, U^d)_a$ and $Z = P_1 \cap P_2$

$\tilde{s}_a(x,q) = T((\tilde{s}_1)_a(x,q), (\tilde{s}_2)_a(x,q))$

Then $(\tilde{s}, Z, U^d)_a$ is a $\tilde{Q}$-fuzzy soft subring of R^g.

Proof. Straightforward

Remark 3.7 We define the notion of a T-power $\tilde{Q}$-fuzzy soft subring as follows.

Let $T^n(a_1, \dots, a_n) = T(a_1, T(a_2, \dots, a_n))$, Then, extending Definition 3.3, we have

$(\tilde{s})_a(x_1 + \cdots + x_n, q) \geq T^n((\tilde{s})_a(x_1,q), \dots, (\tilde{s})_a(x_n,q))$.

Theorem 3.8 Let $(s^\Delta, P^\Delta, U^d)_a$ and $(s_\nabla, S_\nabla, U^d)_b$ be two R^g-$\tilde{Q}$-fuzzy soft subrings.

Define: $(s^{\Delta\cap\nabla}, C^\Delta, U^d)_a = (s^\Delta, P^\Delta, U^d)_a \cap (s_\nabla, S_\nabla, U^d)_b$

$C^\Delta = P^\Delta \cap S_\nabla$, For all $(x,q) \in R^g \times Q$, $s_a^{\Delta\cap\nabla}(x,q) = T(s_a^\Delta(x,q), s_{\nabla b}(x,q))$

Then $(s^{\Delta\cap\nabla}, C^\Delta, U^d)_a$ is a R^g-$\tilde{Q}$-fuzzy Q̃-fuzzy soft subring.

Proof Let $x, y \in R^g, q \in Q$ $s_a^{\Delta\cap\nabla}(x+y,q) = T(s_a^\Delta(x+y,q), s_{\nabla b}(x+y,q))$

$$\geq T(T(s_a^\Delta(x,q), s_a^\Delta(y,q)), T(s_{\nabla b}(x,q), s_{\nabla b}(y,q)))$$
$$\geq T(T(s_a^\Delta(x,q), s_{\nabla b}(x,q)), T(s_a^\Delta(y,q), s_{\nabla b}(y,q))) = T(s_a^{\Delta\cap\nabla}(x,q), s_a^{\Delta\cap\nabla}(y,q))$$

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First condition proved

$$s_a^{\Delta\cap\nabla}(-x,q) = T(s_a^{\Delta}(-x,q), s_{\nabla b}(-x,q)) \geq T(s_a^{\Delta}(x,q), s_{\nabla b}(x,q)) = s_a^{\Delta\cap\nabla}(x,q)$$

Second condition proved

$s_a^{\Delta\cap\nabla}(xy,q) = T(s_a^{\Delta}(xy,q), s_{\nabla b}(xy,q))$

$$\geq T(T(s_a^{\Delta}(x,q), s_a^{\Delta}(y,q)), T(s_{\nabla b}(x,q), s_{\nabla b}(y,q))) \geq T(s_a^{\Delta\cap\nabla}(x,q), s_a^{\Delta\cap\nabla}(y,q))$$

Third condition proved

Hence proved.

Theorem 3.9 Let $\{(s_m, P_m, U^d)_{a_m}\}_{1\leq m\leq n}$ be a family of R^g-$\tilde{Q}$-fuzzy soft subring.

Define: $(s, Z, U^d)_a = \bigcap_{1\leq m\leq n}(s_m, P_m, U^d)_{a_m}$ and $Z = \bigcap_{1\leq m\leq n} P_m$

$s_a(x,q) = T_{1\leq m\leq n} s_{m,a_m}(x,q)$, Then $(s, Z, U^d)_a$ is a R^g-$\tilde{Q}$-fuzzy Q̃-fuzzy soft subring.

Example 3.10 Let $R^g = \mathbb{Z}, Q = \{q\}$

$$s_a^{\Delta}(x,q) = \begin{cases} 0.8, & x \text{ even} \\ 0.3, & \text{otherwise} \end{cases} \quad \text{and} \quad s_{\nabla b}(x,q) = \begin{cases} 0.7, & x \text{ multiple of } 3 \\ 0.2, & \text{otherwise} \end{cases}$$

Then $s_a^{\Delta\cap\nabla}(x,q) = T(s_a^{\Delta}(x,q), s_{\nabla b}(x,q))$

This forms a $\tilde{Q}$-fuzzy Q̃-fuzzy soft subring.

Theorem 3.11 Let $(s, P, U^d)_a$ be a R^g-$\tilde{Q}$-fuzzy Q̃-fuzzy soft subring.

Define the complement: $(s', P', U^d)'_a(x,q) = 1 - s_a(x,q)$,

then $(s', P', U^d)'_a$ is a R^g-$\tilde{Q}$-fuzzy soft subring (with respect to dual t-conorm S).

Proof Let $x, y \in R^g$, $q \in Q$ then $(s', P', U^d)'_a(x+y,q) = 1 - s_a(x+y,q)$

Since $(s, P, U^d)_a$is a fuzzy soft subring, $s_a(x+y,q) \geq T(s_a(x,q), s_a(y,q))$

Thus, $1 - s_a(x+y,q) \leq 1 - T(s_a(x,q), s_a(y,q))$

Using duality, $1 - T(a,b) \leq S(1-a, 1-b)$

Hence, $(s')_a(x+y,q) \leq S((s')_a(x,q), (s')_a(y,q))$

First condition proved.

Also $(s')_a(-x,q) = 1 - s_a(-x,q)$

Since, $s_a(-x,q) \geq s_a(x,q)$.Then, $1 - s_a(-x,q) \leq 1 - s_a(x,q)$

$$(s')_a(-x,q) \leq (s')_a(x,q)$$

Second condition proved.

Also $(s')_a(xy,q) = 1 - s_a(xy,q)$

Since, $s_a(xy,q) \geq T(s_a(x,q), s_a(y,q))$

Then, $1 - s_a(xy,q) \leq 1 - T(s_a(x,q), s_a(y,q))$

Using duality, $\leq S((s')_a(x,q), (s')_a(y,q))$

Third condition proved

Thus$(s', P', U^d)'_a$ satisfies all conditions of a R^g-$\tilde{Q}$-fuzzy soft subring.

Hence proved.

Theorem 3.12 Let $(s_{im}, P_{im}, U^d_{im})_a$ be a conceivable (imaginable) fuzzy soft structure satisfying,

$$s_{im,a}(x+y,q) \geq T^m(s_{im,a}(x,q), s_{im,a}(y,q)) \quad (3.6)$$

$$s_{im,a}(-x,q) \geq s_{im,a}(x,q) \quad (3.7)$$

$$s_{im,a}(xy,q) \geq T^m(s_{im,a}(x,q), s_{im,a}(y,q)) \quad (3.8)$$

then $(s_{im}, P_{im}, U^d_{im})_a$ is a R^g-$\tilde{Q}$-fuzzy soft subring.

Proof Let $x, y \in R^g$, $q \in Q$ then the reduction of T^m is $T^m(a,b) = T(a, T(a, \dots, T(a,b)))$

then, $T^m(a,b) \leq T(a,b)$

Thus, $s_{im,a}(x+y,q) \geq T^m(s_{im,a}(x,q), s_{im,a}(y,q)) \geq T(s_{im,a}(x,q), s_{im,a}(y,q))$

From (3.7), $s_{im,a}(-x,q) \geq s_{im,a}(x,q)$

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From (3.8), $s_{im,a}(xy,q) \geq T^m(s_{im,a}(x,q), s_{im,a}(y,q))$. Using same argument,

$$\geq T(s_{im,a}(x,q), s_{im,a}(y,q))$$

All three defining properties of a R^g-$\tilde{Q}$-fuzzy soft subring are satisfied.

Hence proved.

SECTION 4: GENERALIZED OPERATIONS AND EXTENSIONS OF Q̃–FUZZY SOFT SUBRINGS

Definition 4.1 Let (f, P, U^d) and (g, S, U^d)be two Q̃–fuzzy soft subsets over R^g. Define their union (k, Z, U^d) by: $Z = P \cup S$ and for all $(x,q) \in R^g \times Q$,

$$k(z)(x,q) = \begin{cases} f(z)(x,q), & z \in P \setminus S, \\ g(z)(x,q), & z \in S \setminus P, \\ S(f(z)(x,q), g(z)(x,q)), & z \in P \cap S, \end{cases}$$

where S is a t-conorm. We write: $(k, Z, U^d) = (f, P, U^d) \cup (g, S, U^d)$

Definition 4.2 Let $\{(f_k, P_k, U^d_{(k)}): k \in M\}$ be a family of Q̃–fuzzy soft subsets.

Define: $Z = \bigcup_{k\in M} P_k$ and $k(z)(x,q) = S_{k\in M}(f_k(z)(x,q))$. Thus, $(k, Z, U^d) = \bigcup_{k\in M}(f_k, P_k, U^d_{(k)})$

Theorem 4.3 Let (f, P, U^d) and (g, S, U^d) be two R^g-Q̃-fuzzy soft subrings. Then their union $(k, C, U^d) = (f, P, U^d) \cup (g, S, U^d), C = P \cup S$ is also an R^g-Q̃-fuzzy soft subring.

Proof. Let $x, y \in R^g$, $q \in Q$ then

$$k(x+y,q) = S\big(f(x+y,q), g(x+y,q)\big)$$
$$\geq S(T(f(x,q), f(y,q)), T(g(x,q), g(y,q)))$$
$$\geq T(S(f(x,q), g(x,q)), S(f(y,q), g(y,q)))$$

Thus, $k(x+y,q) \geq T(k(x,q), k(y,q))$

and $k(-x,q) = S(f(-x,q), g(-x,q)) \geq S(f(x,q), g(x,q)) = k(x,q)$

Also $k(xy,q) = S\big(f(xy,q), g(xy,q)\big)$

$$\geq S(T(f(x,q), f(y,q)), T(g(x,q), g(y,q)))$$
$$\geq T(S(f(x,q), g(x,q)), S(f(y,q), g(y,q)))$$

Hence proved.

Theorem 4.4 $(f, P, U^d)_\alpha$ is a Q̃–fuzzy soft subring $\Leftrightarrow$ $U_\alpha = \{x \in R^g : f(x,q) \geq \alpha\}$ is a Q̃-fuzzy soft subring for all $\alpha \in \mathrm{Im}(f)$.

Proof. Let $x, y \in U_\alpha$, i.e., $f(x,q) \geq \alpha, f(y,q) \geq \alpha$

$f(x+y,q) \geq T(f(x,q), f(y,q)) \geq T(\alpha,\alpha) \geq \alpha$

Thus $x + y$ belongs to the $U_\alpha = \{x \in R^g : f(x,q) \geq \alpha\}$

And $f(-x,q) \geq f(x,q) \geq \alpha$

$f(xy,q) \geq T(f(x,q), f(y,q)) \geq T(\alpha,\alpha) \geq \alpha$

Thus $-x$ *and* xy belongs to the $U_\alpha = \{x \in R^g : f(x,q) \geq \alpha\}$

Hence U_α is a Q̃-fuzzy soft subring for all $\alpha \in \mathrm{Im}(f)$.

Theorem 4.5 Define $f^+(x,q) = f(x,q) + 1 - f(0,q)$.

Then (f^+, P, U^d) is a normal S containing (f, P, U^d).

Proof. Let $x, y \in R^g$, $q \in Q$ then

$$f^+(x+y,q) = f(x+y,q) + 1 - f(0,q)$$
$$\geq T(f(x,q), f(y,q)) + 1 - f(0,q)$$
$$\geq T(f(x,q) + 1 - f(0,q), f(y,q) + 1 - f(0,q))$$
$$= T(f^+(x,q), f^+(y,q)).$$

and

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$$\begin{aligned} f^+(-x,q) &= f(-x,q)+1-f(0,q) \\ &\geq f(x,q)+1-f(0,q) (\text{since } f(-x,q)\geq f(x,q)) \\ &= f^+(x,q). \end{aligned}$$

Thus, $f^+(-x,q) \geq f^+(x,q)$.

Also

$$\begin{aligned} f^+(xy,q) &= f(xy,q)+1-f(0,q) \\ &\geq T(f(x,q),f(y,q))+1-f(0,q) \\ &\geq T(f(x,q)+1-f(0,q), f(y,q)+1-f(0,q)) \\ &= T(f^+(x,q),f^+(y,q)). \end{aligned}$$

and the normality condition is satisfied

$$\begin{aligned} f^+(0,q) &= f(0,q)+1-f(0,q) \\ &= 1. \end{aligned}$$

Thus, (f^+, P, U^d) is a normal Q̃-fuzzy soft subring containing (f, P, U^d).

Theorem 4.6 Let (f, P, U^d) and (g, S, U^d) be two R^g-Q̃-fuzzy Q̃-fuzzy soft subrings.
Define: $k(x,q) = T(f(x,q), g(x,q)), \forall x \in R^g,\ q \in Q$.
Then $(k, P\times S, U^d)$ is also a Q̃-fuzzy soft subring of R^g.

Proof. Let $x, y \in R^g$, $q \in Q$.

$$\begin{aligned} k(x+y,q) &= T(f(x+y,q), g(x+y,q)) \\ &\geq T(T(f(x,q),f(y,q)), T(g(x,q),g(y,q))) \\ &\geq T(T(f(x,q),g(x,q)), T(f(y,q),g(y,q))) \\ &= T(k(x,q),k(y,q)). \end{aligned}$$

and

$$\begin{aligned} k(-x,q) &= T(f(-x,q), g(-x,q)) \\ &\geq T(f(x,q),g(x,q)) \text{ (since } f,g \text{ are subrings)} \\ &= k(x,q). \end{aligned}$$

Hence, $k(-x,q) \geq k(x,q)$.

Then

$$\begin{aligned} k(xy,q) &= T(f(xy,q), g(xy,q)) \\ &\geq T(T(f(x,q),f(y,q)), T(g(x,q),g(y,q))) \\ &\geq T(T(f(x,q),g(x,q)), T(f(y,q),g(y,q))) \\ &= T(k(x,q),k(y,q)). \end{aligned}$$

Thus $(k, P\times S, U^d)$ is a Q̃-fuzzy soft subring of R^g.

Theorem 4.7 Let (f_i, P_i, U_i^d)be Q̃–fuzzy soft subrings over R_i^g. Define: $k(x,q) = T_{i=1}^n f_i(x_i,q)$
Then, $\prod_{i=1}^n (f_i, P_i, U_i^d)$ is a Q̃–fuzzy soft subring over $\prod R_i^g$.

Proof. Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in R^g$ and $q \in Q$. Then
$k(x+y,q) = T_i f_i(x_i+y_i,q) \geq T_i T(f_i(x_i,q), f_i(y_i,q)) \ \geq T(k(x,q),k(y,q))$
and $k(-x,q) = T_i f_i(-x_i,q) \geq T_i f_i(x_i,q) = k(x,q)$.
Also $k(xy,q) = T_i f_i(x_i y_i,q) \geq T(k(x,q),k(y,q))$
Hence proved.

Lemma 4.8 Let f, g be Q̃–fuzzy Q̃-fuzzy soft subring s. If $f \leq g$, then $T(f,h) \leq T(g,h)$

Proof. Straightforward

Corollary 4.9 Let $\{(f_i, P_i, U_i^d)_{a_i} : i = 1,2,\dots,n\}$ be a finite family of Q̃-fuzzy soft subrings of R^g.
Define the iterated product by $(k, C, U^d) = T(f_1, f_2, \dots, f_n)$, $C = \prod_{i=1}^n P_i$, and for all $(x,q) \in R^g \times Q$,
$k(x,q) = T(f_1(x,q), f_2(x,q), \dots, f_n(x,q))$. Then (k, C, U^d) is a Q̃–fuzzy Q̃-fuzzy soft subring of R^g .

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Proof Let $x, y \in R^g$ and $q \in Q$.
We prove the three defining conditions using the t-norm T.
$k(x+y,q) = T(f_1(x+y,q), f_2(x+y,q), \dots, f_n(x+y,q))$
Since each $(f_i, P_i, U_i^d)_{a_i}$ is a Q̃–fuzzy Q̃-fuzzy soft subring,
$f_i(x+y,q) \geq T(f_i(x,q), f_i(y,q)), \forall i$
Thus, $k(x+y,q) \geq T(T(f_1(x,q), f_1(y,q)), \dots, T(f_n(x,q), f_n(y,q)))$
$\geq T(T(f_1(x,q), \dots, f_n(x,q)), T(f_1(y,q), \dots, f_n(y,q)))$
Hence, $k(x+y,q) \geq T(k(x,q), k(y,q))$
and $k(-x,q) = T(f_1(-x,q), f_2(-x,q), \dots, f_n(-x,q))$
Since each f_i satisfies: $f_i(-x,q) \geq f_i(x,q)$
Therefore, $k(-x,q) \geq T(f_1(x,q), f_2(x,q), \dots, f_n(x,q)) = k(x,q)$
Also $k(xy,q) = T(f_1(xy,q), f_2(xy,q), \dots, f_n(xy,q))$
Since each f_i is a Q̃-fuzzy soft subring ,
$f_i(xy,q) \geq T(f_i(x,q), f_i(y,q))$
Thus, $k(xy,q) \geq T(T(f_1(x,q), f_1(y,q)), \dots, T(f_n(x,q), f_n(y,q)))$
$\geq T(T(f_1(x,q), \dots, f_n(x,q)), T(f_1(y,q), \dots, f_n(y,q)))$
Hence, $k(xy,q) \geq T(k(x,q), k(y,q))$
All three defining conditions of a R^g- soft subring are satisfied.
$T(f_1, f_2, \dots, f_n)$ is a R^g-Q̃-fuzzy soft subring . Hence, (k, C, U^d) is a R^g- Q̃-fuzzy soft subring.

Example 4.10 Let $R^g = \mathbb{Z}$, $Q = \{q\}$. Define three Q̃-fuzzy soft subrings

$$f_1(x,q) = \begin{cases} 0.9, & x \text{ even} \\ 0.5, & \text{otherwise} \end{cases}, \quad f_2(x,q) = \begin{cases} 0.8, & x \equiv 0 \pmod 3 \\ 0.4, & \text{otherwise} \end{cases} \quad \text{and} \quad f_3(x,q) = \begin{cases} 0.7, & x > 0 \\ 0.6, & x \leq 0 \end{cases}$$

Define $k(x,q) = T\big(f_1(x,q), f_2(x,q), f_3(x,q)\big), T = \min$
Compute: $x = 6$: $k(6,q) = \min(0.9, 0.8, 0.7) = 0.7$,
$x = 3$: $k(3,q) = \min(0.5, 0.8, 0.7) = 0.5$ and $x = -2$: $k(-2,q) = \min(0.9, 0.4, 0.6) = 0.4$
We have $k(x+y,q) \geq T(k(x,q), k(y,q)), k(-x,q) \geq k(x,q), k(xy,q) \geq T(k(x,q), k(y,q))$
Thus, k is a Q̃–fuzzy soft subring and proving Corollary 4.9.

SECTION 5 Q̃–FUZZY SOFT SUBRING HOMOMORPHISMS

Definition 5.1 Let T and W be two Q̃–fuzzy soft subrings and let Q be a subset of parameters. Let $(s, P, U^d)_a$ and $(t, R, U^d)_a$ be Q̃–fuzzy soft subsets over T and W, respectively. A pair $\langle \beta, \alpha \rangle$ is called a Q̃–fuzzy soft function from T to W if $\beta: T \to W$ is a mapping between subrings and $\alpha: P \to R$ is a mapping between parameter subsets.

Definition 5.2 A Q̃–fuzzy soft function $\langle \beta, \alpha \rangle: T \to W$ is called a Q̃–fuzzy soft homomorphism if for all $x, y \in T$, $q \in Q$ then $\beta(x+y) = \beta(x) + \beta(y)$, $\beta(xy) = \beta(x)\beta(y)$ and $\alpha(P) \subseteq R$.

Definition 5.3 A Q̃–fuzzy soft homomorphism $\langle \beta, \alpha \rangle$ is called a Q̃–fuzzy soft isomorphism if β is a Q̃–fuzzy soft subring isomorphism and α is bijective.

Definition 5.4 Let $(s, P, U^d)_a$ be a Q̃–fuzzy soft subset over T.The image under $\langle \beta, \alpha \rangle$ is
$\langle \beta, \alpha \rangle (s, P, U^d)_a = (\beta(s), \alpha(P), U^d)$ where $\beta(s)_z(n,q) = \sup_{\beta(m)=n,\ \alpha(p)=z} s_p(m,q)$

Definition 5.5 Let $(t, R, U^d)_b$ be a Q̃–fuzzy soft subset over W. The pre-image is
$\langle \beta, \alpha \rangle^{-1}(t, R, U^d)_b = (\beta^{-1}(t), \alpha^{-1}(R), U^d)$ where $\beta^{-1}(t)_p(m,q) = t_{\alpha(p)}(\beta(m), q)$

Theorem 5.6 Let $(s, P, U^d)_a$ be a Q̃–fuzzy soft subring over T. Let $\langle \beta, \alpha \rangle: T \to W$ be a Q̃–fuzzy soft homomorphism. Then $\langle \beta, \alpha \rangle (s, P, U^d)_a$ is a Q̃–fuzzy soft subring over W.

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Proof Let $n_1, n_2 \in W$, $q \in Q$. Take $m_1, m_2 \in T$ such that $\beta(m_1) = n_1, \beta(m_2) = n_2$
$\beta(s)_z(n_1 + n_2, q) = \sup\ s_p(m_1 + m_2, q)$, Since s is Q̃–fuzzy soft subring,
$s_p(m_1 + m_2, q) \geq T(s_p(m_1, q), s_p(m_2, q))$
Thus, $\beta(s)_z(n_1 + n_2, q) \geq T(s_p(m_1, q), s_p(m_2, q))$
$\beta(s)_z(n_1 + n_2, q) \geq T(\beta(s)_z(n_1, q), \beta(s)_z(n_2, q))$ and
$\beta(s)_z(-n_1, q) = \sup\ s_p(-m_1, q) \geq \sup\ s_p(m_1, q)$
Thus, $\beta(s)_z(-n_1, q) \geq \beta(s)_z(n_1, q)$
Then $\beta(s)_z(n_1 n_2, q) = \sup\ s_p(m_1 m_2, q) \geq T(s_p(m_1, q), s_p(m_2, q))$
Hence $\beta(s)_z(n_1 n_2, q) \geq T(\beta(s)_z(n_1, q), \beta(s)_z(n_2, q))$
Therefore, $\langle\beta, \alpha\rangle(s, P, U^d)_a$ is a Q̃–fuzzy soft subring over W.

Theorem 5.7 Let $(t, R, U^d)_b$ be a Q̃–fuzzy soft subring over W. Let $\langle\beta, \alpha\rangle: T \to W$ be a Q̃–fuzzy soft homomorphism. Then $\langle\beta, \alpha\rangle^{-1}(t, R, U^d)_b$ is a Q̃–fuzzy soft subring over T.

Proof Let $m_1, m_2 \in T$, $q \in Q$.
$\beta^{-1}(t)_p(m_1 + m_2, q) = t_{\alpha(p)}(\beta(m_1 + m_2), q) = t_{\alpha(p)}(\beta(m_1) + \beta(m_2), q)$
Since t is Q̃–fuzzy soft subring,
$\geq T(t_{\alpha(p)}(\beta(m_1), q), t_{\alpha(p)}(\beta(m_2), q))$. Thus,
$\geq T(\beta^{-1}(t)_p(m_1, q), \beta^{-1}(t)_p(m_2, q))$
$\beta^{-1}(t)_p(-m_1, q) = t_{\alpha(p)}(\beta(-m_1), q) = t_{\alpha(p)}(-\beta(m_1), q) \geq t_{\alpha(p)}(\beta(m_1), q)$
Thus, $\beta^{-1}(t)_p(-m_1, q) \geq \beta^{-1}(t)_p(m_1, q)$
$\beta^{-1}(t)_p(m_1 m_2, q) = t_{\alpha(p)}(\beta(m_1 m_2), q) = t_{\alpha(p)}(\beta(m_1)\beta(m_2), q)$
$\geq T(t_{\alpha(p)}(\beta(m_1), q), t_{\alpha(p)}(\beta(m_2), q))$. Thus,
$\geq T(\beta^{-1}(t)_p(m_1, q), \beta^{-1}(t)_p(m_2, q))$
Hence proved.

Theorem 5.8 Let $\langle\beta, \alpha\rangle : T \to W$ be a Q̃–fuzzy soft isomorphism. Then, $(s, P, U^d)_a$ is Q̃–fuzzy soft subring over T $\Leftrightarrow$ $\langle\beta, \alpha\rangle(s, P, U^d)_a$ is Q̃–fuzzy soft subring over W.

Proof Assume that $s, P, U^d)_a$ is a Q̃–fuzzy soft subring over T. We prove that its image $(\beta(s), \alpha(P), U^d)$ is a Q̃–fuzzy soft subring over W.
Let $n_1, n_2 \in W$, $q \in Q$.
Since β is surjective, there exist $m_1, m_2 \in T$ such that $\beta(m_1) = n_1, \beta(m_2) = n_2$.
Then $\beta(s)_z(n_1 + n_2, q) = \sup\limits_{\beta(m) = n_1 + n_2} s_p(m, q)$.
Since β is a homomorphism, $\beta(m_1 + m_2) = n_1 + n_2$.
Thus $\beta(s)_z(n_1 + n_2, q) \geq s_p(m_1 + m_2, q)$.
Since s is a Q̃–fuzzy soft subring, $s_p(m_1 + m_2, q) \geq T(s_p(m_1, q), s_p(m_2, q))$.
Hence $\beta(s)_z(n_1 + n_2, q) \geq T(s_p(m_1, q), s_p(m_2, q))$.
Taking supremum over all such m_1, m_2
$\beta(s)_z(n_1 + n_2, q) \geq T(\beta(s)_z(n_1, q), \beta(s)_z(n_2, q))$.
And let $n \in W$. Take $m \in T$ such that $\beta(m) = n$.
Then $\beta(s)_z(-n, q) = \sup\ s_p(-m, q)$.
Since s is Q̃–fuzzy soft subring, $s_p(-m, q) \geq s_p(m, q)$.
Thus $\beta(s)_z(-n, q) \geq \beta(s)_z(n, q)$.
Then, $\beta(s)_z(n_1 n_2, q) = \sup\ s_p(m_1 m_2, q)$.

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Since $s_p(m_1m_2,q) \geq T\left(s_p(m_1,q), s_p(m_2,q)\right)$,
We obtain $\beta(s)_z(n_1n_2,q) \geq T(\beta(s)_z(n_1,q), \beta(s)_z(n_2,q))$.
Hence $\langle\beta,\alpha\rangle(s,P,U^d)_a$ is a Q̃–fuzzy soft subring over W.
Conversely, assume that $\langle\beta,\alpha\rangle(s,P,U^d)_a$ is a Q̃–fuzzy soft subring over W.
Since $\langle\beta,\alpha\rangle$ is an isomorphism, β is bijective and $\alpha$is bijective
Hence inverse exists, $\langle\beta,\alpha\rangle^{-1}$.
We write $(s,P,U^d)_a = \langle\beta,\alpha\rangle^{-1}(\langle\beta,\alpha\rangle(s,P,U^d)_a)$.
Since $\langle\beta,\alpha\rangle(s,P,U^d)_a$ is a Q̃–fuzzy soft subring, its inverse image under $\langle\beta,\alpha\rangle^{-1}$ is also a Q̃–fuzzy soft subring. Thus $(s,P,U^d)_a$ is a Q̃–fuzzy soft subring over T.
$(s,P,U^d)_a$ is Q̃–fuzzy soft subring $\Leftrightarrow \langle\beta,\alpha\rangle(s,P,U^d)_a$ is Q̃–fuzzy soft subring.

Example 5.9 Let $T = W = \mathbb{Z}$, $Q = \{q\}$.

Define $s_a(x,q) = \begin{cases} 0.8, & x \text{ even} \\ 0.3, & \text{otherwise} \end{cases}$ and define $\beta(x) = 2x, \alpha = \text{identity}$

Then $\beta(s)_z(n,q)$ satisfies all Q̃–fuzzy soft subring conditions.
Hence the image is a Q̃–fuzzy soft subring.

6. CONCLUSION WITH FUTURE RESEARCH

In this paper, we developed a unified and systematic framework for Q̃–fuzzy soft algebraic structures by introducing and studying Q̃–fuzzy soft subrings, Q̃–fuzzy soft subrings, and their extension to Q̃–fuzzy soft modules under a general T-norm T. We established fundamental properties, including intersection, complement, homomorphic image and pre-image, isomorphism invariance, and generated Q̃–fuzzy soft submodules, supported by rigorous theorems, proofs, and illustrative examples. The integration of module theory with Q̃–fuzzy soft structures provide a natural and significant generalization of classical and fuzzy algebraic systems. This work opens several promising directions for future research, such as Q̃–fuzzy soft quotient modules, homological aspects (exact sequences and projectivity), categorical formulations, extensions to intuitionistic and interval-valued Q̃–fuzzy soft modules, and potential applications in decision-making systems, artificial intelligence, and uncertainty modelling.

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