On Sums And Products Of K-Tripotent Matrices

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Abstract: In this paper, the sums and products of k-Tripotent matrices are discussed and some related results are obtained.

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1 Introduction

R.D.Hill and S.R.Waters [1] have developed a theory of k-real and k-hermitian matrices as a generalization of secondary real and secondary hermitian matrices. S.Krishnamoothy and P.S.Meenakshi [2] have studied the basic concepts of k-Tripotent matrices as generalization of k-Tripotent matrices. J.A.Erdos [3] has initiated the study on products of idempotent matrices.

Throughout the paper, let A∈ $\mathcal{C}_{n\times n}$, $\mathcal{C}_n$denote the unitary space of order n and $\mathcal{C}_{n\times n}$be the space all complex n×n matrices. Let 'k' be a fixed product of disjoint transportation in S_n the set of all permutation on {1,2…n}. Hence it is involutory (i.e. $K^2=I$ identity permutation). If 'K'is the associated permutation matrix of 'k' then it clearly satisfies the following properties.

$K^2=I$ and $K= K^T = \overline{K} = K^*$

2 Sums and Products of k-Tripotent matrices

Theorem 2.1. Let A and B be two k-Tripotent matrices then $A+B$ is k-Tripotent if and only if $A=-B$

Proof. Let A and B be k-Tripotent matrices, Therefore $KA^3K = A \text{ and } KB^3K = B$.
Assume that $A=-B$

Now $A+B = KA^3K + KB^3K$

$$= K\left(A^3 + B^3\right)K$$

$$= K(A+B)^3K \qquad \text{if } A = -B$$

Hence $A+B$ is k-Tripotent matrices.

Conversely, Assume $A+B$ is k-Tripotent matrices.

$$A+B = K\left(A+B\right)^3 K$$

$$= K\left(A^3 + B^3 + 3AB\left[A+B\right]\right)K$$

$$= KA^3K + KB^3K + K\left(3AB\left[A+B\right]\right)K$$

$$= A+B+ K\left(3AB\left[A+B\right]\right)K$$

Hence $K\left(3AB\left[A+B\right]\right)K = 0$, which implies that $A = -B$.

Generalization. Let $A_1A_2\ldots\ldots\ldots\ldots A_n$ be k-Tripotent matrices then $\sum_{i=1}^{n} A_i$ is k-Tripotent if and only if $\sum_{i\neq j\neq t} A_iA_jA_t = 0$ for i , j and t in {1,2…….n}.

Proof. Let

$$K\left(\sum_{i=1}^{n} A_i\right)^3 K = K\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{n} A_iA_jA_t\right)K$$

$$= K\left(\sum_{i=1}^{n} A_i^3\right)K + K\left\{\sum_{i=1}^{n} A_i\left[\sum_{s=1}^{n} A_s^2 + \sum_{\substack{j,t=1\\ j\neq t}}^{n} A_jA_t\right]\right\}K$$

$$+K\left[\sum_{j=1}^{n} A_j \sum_{\substack{i,t=1\\ i\neq t}}^{n} A_iA_t\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{\substack{i,j=1\\ i\neq j}}^{n} A_iA_j\right]K$$

$$= \sum_{i=1}^{n} KA_i^3K + K\left[\sum_{i=1}^{n} A_i \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{i=1}^{n} A_i \sum_{\substack{j,t=1\\ j\neq t}}^{n} A_jA_t\right]K + K\left[\sum_{j=1}^{n} A_j \sum_{s=1}^{n} A_s^2\right]K$$

$$+K\left[\sum_{j=1}^{n} A_j \sum_{\substack{i,t=1\\ i\neq t}}^{n} A_iA_t\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{\substack{i,j=1\\ i\neq j}}^{n} A_iA_j\right]K$$

Since A_i s are k-Tripotent matrices, we have

$$K\left(\sum_{i=1}^{n} A_i\right)^3 K = \sum_{i=1}^{n} A_i + K\left[\sum_{i=1}^{n} A_i \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{i=1}^{n} A_i \sum_{\substack{j,t=1\\ j\neq t}}^{n} A_j A_t\right]K + K\left[\sum_{j=1}^{n} A_j \sum_{s=1}^{n} A_s^2\right]K$$
$$+K\left[\sum_{j=1}^{n} A_j \sum_{\substack{i,t=1\\ i\neq t}}^{n} A_i A_t\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{\substack{i,j=1\\ i\neq j}}^{n} A_i A_j\right]K$$

………………………(2.1.)

Here i,j,t ϵ {1,2……….n}

Assume that $\sum_{i=1}^{n} A_i$ is k- Tripotent matrices. From (2.1.) we have,

$$\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} A_i + K\left[\sum_{i=1}^{n} A_i \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{i=1}^{n} A_i \sum_{\substack{j,t=1\\ j\neq t}}^{n} A_j A_t\right]K + K\left[\sum_{j=1}^{n} A_j \sum_{s=1}^{n} A_s^2\right]K$$
$$+K\left[\sum_{j=1}^{n} A_j \sum_{\substack{i,t=1\\ i\neq t}}^{n} A_i A_t\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{\substack{i,j=1\\ i\neq j}}^{n} A_i A_j\right]K$$

$$\sum_{i=1}^{n} A_i - \sum_{i=1}^{n} A_i = K\left[\sum_{i=1}^{n} A_i \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{i=1}^{n} A_i \sum_{\substack{j,t=1\\ j\neq t}}^{n} A_j A_t\right]K + K\left[\sum_{j=1}^{n} A_j \sum_{s=1}^{n} A_s^2\right]K$$
$$+K\left[\sum_{j=1}^{n} A_j \sum_{\substack{i,t=1\\ i\neq t}}^{n} A_i A_t\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{\substack{i,j=1\\ i\neq j}}^{n} A_i A_j\right]K$$

$$0 = K\left[\sum_{i=1}^{n} A_i \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{i=1}^{n} A_i \sum_{\substack{j,t=1\\ j\neq t}}^{n} A_j A_t\right]K + K\left[\sum_{j=1}^{n} A_j \sum_{s=1}^{n} A_s^2\right]K$$
$$+K\left[\sum_{j=1}^{n} A_j \sum_{\substack{i,t=1\\ i\neq t}}^{n} A_i A_t\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{\substack{i,j=1\\ i\neq j}}^{n} A_i A_j\right]K$$

Hence, it follows that,

$$K\left[\sum_{i=1}^{n} A_i \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{i=1}^{n} A_i \sum_{\substack{j,t=1 \\ j\neq t}}^{n} A_j A_t\right]K + K\left[\sum_{j=1}^{n} A_j \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{j=1}^{n} A_j \sum_{\substack{i,t=1 \\ i\neq t}}^{n} A_i A_t\right]K$$

$$+K\left[\sum_{t=1}^{n} A_t \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{\substack{i,j=1 \\ i\neq j}}^{n} A_i A_j\right]K = 0$$

But,

$$K\left[\sum_{i=1}^{n} A_i \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{i=1}^{n} A_i \sum_{\substack{j,t=1 \\ j\neq t}}^{n} A_j A_t\right]K + K\left[\sum_{j=1}^{n} A_j \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{j=1}^{n} A_j \sum_{\substack{i,t=1 \\ i\neq t}}^{n} A_i A_t\right]K$$

$$+K\left[\sum_{t=1}^{n} A_t \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{\substack{i,j=1 \\ i\neq j}}^{n} A_i A_j\right]K = \sum_{i\neq j\neq t} A_i A_j A_t$$

Hence $\sum_{i\neq j\neq t} A_i A_j A_t = 0$.

Conversely, Assume that $\sum_{i\neq j\neq t} A_i A_j A_t = 0$

But $\sum_{i\neq j\neq t} A_i A_j A_t = K\left[\sum_{i=1}^{n} A_i \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{i=1}^{n} A_i \sum_{\substack{j,t=1 \\ j\neq t}}^{n} A_j A_t\right]K + K\left[\sum_{j=1}^{n} A_j \sum_{s=1}^{n} A_s^2\right]K$

$$+K\left[\sum_{j=1}^{n} A_j \sum_{\substack{i,t=1 \\ i\neq t}}^{n} A_i A_t\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{\substack{i,j=1 \\ i\neq j}}^{n} A_i A_j\right]K$$

Hence $K\left[\sum_{i=1}^{n} A_i \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{i=1}^{n} A_i \sum_{\substack{j,t=1 \\ j\neq t}}^{n} A_j A_t\right]K + K\left[\sum_{j=1}^{n} A_j \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{j=1}^{n} A_j \sum_{\substack{i,t=1 \\ i\neq t}}^{n} A_i A_t\right]K$

$$+K\left[\sum_{t=1}^{n} A_t \sum_{s=1}^{n} A_s^2\right]K + K\left[\sum_{t=1}^{n} A_t \sum_{\substack{i,j=1 \\ i\neq j}}^{n} A_i A_j\right]K = 0$$

…………(2.2.)

Substitute (2.2.) in (2.1.), then we have ,

$$K\left(\sum_{i=1}^{n} A_i\right)^3 K = \sum_{i=1}^{n} A_i$$

Hence $\sum_{i=1}^{n} A_i$ is k-Tripotent matrices.

Example 2.2. Let $A=\begin{pmatrix}1 & 0 & 0\\ i & -1 & i\\ 0 & 0 & 1\end{pmatrix}$ and $B=\begin{pmatrix}-1 & 0 & 0\\ -i & 1 & -i\\ 0 & 0 & -1\end{pmatrix}$ clearly A and B are k-Tripotent matrices. Let K be the associated permutation matrix such as,

$K=\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$ Here $A = -B$. Hence A+B is also a k-Tripotent matrices.

Theorem 2.3. Let A and B be k-Tripotent matrices. If $AB = BA$ then AB is also k-Tripotent matrix.

Proof. Let A and B be two k-Tripotent matrices. Since $KA^3K = A \text{ and } KB^3K = B$. Assume that $AB = BA$.

$$AB = KA^3KKB^3K$$
$$= KAAAKKBBBK$$
$$= KAAABBBK$$
$$= KAABABBK \qquad if\ AB = BA$$
$$= KABABABK$$

$= K(AB)^3K$, Hence the matrix AB is k-Tripotent matrix.

Generalization. If $A_1A_2................A_n$ be k-Tripotent matrix belonging to a commuting family of matrices then $\prod_{i=1}^{n} A_i$ is a k-Tripotent matrices.

Proof. Let $A_1A_2................A_n$ be k-Tripotent matrices.

$$K\left(\prod_{i=1}^{n} A_i\right)^3 K = K\left[A_1A_2................A_nA_1A_2................A_nA_1A_2................A_n\right]$$
$$= K\left[A_1^3A_2^3...............A_n^3\right]K$$
$$= KA_1^3KKA_2^3K...............KA_n^3K$$
$$= A_1^3A_2^3...............A_n^3$$
$$= \prod_{i=1}^{n} A_i$$

Hence the matrices $\prod_{i=1}^{n} A_i$ is k-Tripotent .

Example 2.4. Let $A=\begin{pmatrix} 1 & 0 & 0 \\ i & -1 & i \\ 0 & 0 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} -1 & 0 & 0 \\ -i & 1 & -i \\ 0 & 0 & -1 \end{pmatrix}$

$K=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ where K is the associated permutation matrix.

Clearly A and B are k-Tripotent matrices.

$AB=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $BA=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ i.e $AB=BA$. Hence A B is also a k-Tripotent matrices.

Remark 2.5. If A and B are two k-Tripotent matrices then, $A+B$ is k-Tripotent if and only if $[A,B]=3AB[A+B]$. AB is k-Tripotent if $[AB]=0$. Where $[A,B]$ be the commutator of the matrices A and B.

Lemma 2.6. Let A be a k-Tripotent matrix. Then A is Tripotent if and only if $AK=KA$, where K is the associated permutation matrix of k.

Proof. Let A be a k-Tripotent matrix.

Assume that $AK=KA$

Pre-multiply by K, we have

$$KAK=A,$$

$$A^3=A$$

Hence A is Tripotent.

Conversely, Assume that A is Tripotent matrices.

$$A^3=A$$

$$KAK=A \qquad \text{if } A \text{ is k-Tripotent}$$

Pre-multiply by K, $AK=KA$.

Example 2.7. Let $A=\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$ is a $\langle 1,2 \rangle$ tripotent matrix and it also commutes with the associated permutation matrix $K=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, i.e $AK=KA$.

Note 2.8. Lemma 2.6. fails if we relax the condition of commutability of matrices A and K. A is not Tripotent then $AK \neq KA$ in such cases.

References

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