On Solution Of Some Double Integrals Involving H-Function

By

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Abstract

In this paper, we shall establish some double integrals involving H-Function.

Keyword

Argument, contour.

Introduction:

Integrals are useful in the study of certain boundary value problems. It is also helpful in obtaining the expansion formulae. Integrals are also used in the study of statistical distribution, probability and integral equation. Here, we have developed and solved integrals involving hyper-geometric functions.

Gauss hyper geometric function ${}_2F_1$ [a, b; c; z] has been generalized by the introduction of p parameters of nature of a, b and q parameters of the nature of c. This ensuring series

$$ {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p; \\ \\ b_1, \ldots, b_q; \end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots\ldots (a_p)_n \; z^n}{(b_1)_n \ldots\ldots (b_q)_n \; n!} $$

$$ = \sum_{n=0}^{\infty} \frac{\prod^{p} (a_i)_n \, z^n}{\prod^{q} (b_j)_n \; n!}, $$

is known as the generalized hypergeometric series and the function ${}_pF_q$ is called generalized hypergeometric function of variable z. ${}_pF_q$ is not defined if any denominator parameter b_q is a negative integer or zero. If any numerator parameter a_p is zero or a negative integer, the series terminates. If ${}_pF_q$ does not terminate, it converges

(i) for all finite z if $p \leq q$;

(ii) for |z| < 1 if p = q + 1;

(iii) for |z| = 1 if p = q + 1 and $R\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0$

and diverges for all z ≠ 0 if p > q + 1.

The class of the hypergeometric series and functions considered above are of single variable. The great success of the theory of hypergeometric series in one variable has stimulated the development of corresponding theory in two and more than two variables.

In this section, we will establish the following double integrals:

$$\int_0^{\pi/2}\int_0^{\pi/2} \frac{2^{\alpha+\beta+1}}{\pi} e^{i(\alpha-\beta)x}(\cos x)^{\alpha+\beta} e^{i(\sigma+\rho)y}(\sin y)^{\sigma-1}(\cos y)^{\rho-1} \times$$

$$\times H_{p,q}^{m,n}\left[z\left(2e^{i(x+y)}\cos x \sin y\right)^{\lambda}\left(2e^{i(y-x)}\cos x \cos y\right)^{\mu}\right]dxdy$$

$$= \frac{\pi e^{\frac{i\pi\sigma}{2}}}{2^{\alpha+\beta+1}} H_{p+3,q+3}^{m,n+3}\left[\frac{ze^{\frac{i\pi\lambda}{2}}}{2^{\lambda+\mu}}\middle|\begin{matrix}(-\alpha-\beta,\lambda+\mu),(1-\sigma,\lambda),(1-\rho,\mu),(a_j,\alpha_j)_{1,p}\\ (b_j,\beta_j)_{1,q},(1-\sigma-\rho,\lambda+\mu),(-\alpha,\lambda),(-\beta,\mu),\end{matrix}\right], \quad (1)$$

Provided that Re $(\alpha+\beta)$ > -1, Re(σ) > 0, Re(ρ) > 0, $\lambda \geq 0$ and $\mu \geq 0$, |arg z| < ½ πA , where A is given in equation

$$\sum_{j=1}^{n}\alpha_i - \sum_{j=n+1}^{p}\alpha_i + \sum_{j=1}^{m}\beta_i - \sum_{j=m+1}^{q}\beta_i \equiv A > 0,$$

$$\sum_{j=1}^{p}\alpha_i - \sum_{j=1}^{q}\beta_i < 0$$

$$\int_0^{t}\int_0^{\pi/2} \frac{2^{\alpha+\beta+1}}{\pi} x^{\rho-1}(t-x)^{\sigma-1} e^{i(\alpha-\beta)y}(\cos y)^{\alpha+\beta} \times$$

$$\times H_{p,q}^{m,n}\left[z\left(2xe^{iy}\cos y\right)^{\lambda}\left(2(t-x)e^{-iy}\cos y\right)^{\mu}\right]dxdy$$

$$= \frac{\pi t^{\rho+\sigma-1}}{2^{\alpha+\beta+1}} H^{m,n+3}_{p+3,q+3}[z(t/2)^{\lambda+\mu}|^{(-\alpha-\beta,\lambda+\mu),(1-\sigma,\lambda),(1-\rho,\mu),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},(1-\sigma-\rho,\lambda+\mu),(-\alpha,\lambda),(-\beta,\mu),}] \,, \qquad (2)$$

Where Re $(\alpha+\beta)$ > - 1, Re(σ) > 0, Re(□ρ) > 0, $\lambda \geq 0$ and $\mu \geq 0$, |arg z| < ½ πA , where A is given in equation

$$\sum_{j=1}^{n} \alpha_i - \sum_{j=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_i - \sum_{j=m+1}^{q} \beta_i \equiv A > 0,$$

$$\sum_{j=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_i < 0$$

$$\int_0^\infty \int_0^{\pi/2} \frac{2^{\alpha+\beta+1}}{\pi} x^{\rho-1} e^{-x} e^{i(\alpha-\beta)y} (\cos y)^{\alpha+\beta} \times$$

$$\times H^{m,n}_{p,q}[z\left(2xe^{iy}\cos y\right)^{\lambda}\left(2e^{-iy}\cos y\right)^{\mu}]dxdy$$

$$= \frac{\pi}{2^{\alpha+\beta+1}} H^{m,n+2}_{p+2,q+2}[\frac{z}{2^{\lambda+\mu}}|^{(-\alpha-\beta,\lambda+\mu),(1-\sigma,\lambda),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},(-\alpha,\lambda),(-\beta,\mu),}], \qquad (3)$$

Provided that Re $(\alpha+\beta)$ > -1, Re(σ) > 0, $\lambda \geq 0$ and $\mu \geq 0$, |arg z| < ½ πA, where A is given in equation

$$\sum_{j=1}^{n} \alpha_i - \sum_{j=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_i - \sum_{j=m+1}^{q} \beta_i \equiv A > 0,$$

$$\sum_{j=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_i < 0$$

$$\int_0^t \int_0^{\pi/2} x^{\rho-1} (t-x)^{\sigma-1} e^{i(\alpha+\beta)y} (\sin y)^{\alpha-1} (\cos y)^{\beta-1} \times$$

$$\times H^{m,n}_{p,q}[z\left(2xe^{iy}\sin y\right)^{\lambda}\left((t-x)e^{iy}\cos y\right)^{\mu}]dxdy$$

$$= e^{\frac{i\pi\alpha}{2}} t^{\rho+\sigma-1} H_{p+4,q+3}^{m,n+4}\left[zt^{\lambda+\mu} e^{\frac{i\pi\lambda}{2}} \Big|_{(b_j,\beta_j)_{1,q},(1-\sigma-\rho,\lambda+\mu),(1-\alpha-\beta,\lambda+\mu)}^{(1-\alpha,\lambda),(1-\beta,\mu),(1-\rho,\lambda),(1-\sigma,\mu),(a_j,\alpha_j)_{1,p}}\right], \quad (4)$$

where Re $(\alpha) > 0$, Re $(\beta) > 0$, Re$(\sigma) > 0$, Re$(\rho) > 0$, $\lambda \geq 0$ and $\mu \geq 0$, |arg z| < ½ πA , where A is given in equation

$$\sum_{j=1}^{n} \alpha_i - \sum_{j=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_i - \sum_{j=m+1}^{q} \beta_i \equiv A > 0,$$

$$\sum_{j=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_i < 0$$

$$\int_0^{\infty} \int_0^{\pi/2} x^{\rho-1}\, e^{-x} e^{i(\alpha+\beta)y} (\sin y)^{\alpha-1} (\cos y)^{\beta-1} \times$$

$$\times\, H_{p,q}^{m,n}\left[z\left(xe^{iy}\sin y\right)^{\lambda}\left(e^{iy}\ \cos y\right)^{\mu}\right]dxdy$$

$$= e^{\frac{i\pi\alpha}{2}} H_{p+3,q+1}^{m,n+3}\left[ze^{\frac{i\pi\lambda}{2}} \Big|_{(b_j,\beta_j)_{1,q},(-\alpha-\beta,\lambda+\mu)}^{(1-\sigma,\lambda),(1-\alpha,\lambda),(1-\beta,\mu)(a_j,\alpha_j)_{1,p}}\right], \quad (5)$$

provided that Re $(\alpha) > 0$, Re $(\beta) > 0$, Re$(\sigma) > 0$, $\lambda \geq 0$ and $\mu \geq 0$, |arg z| < ½ πA, where A is given in equation

$$\sum_{j=1}^{n} \alpha_i - \sum_{j=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_i - \sum_{j=m+1}^{q} \beta_i \equiv A > 0,$$

$$\sum_{j=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_i < 0$$

Proof: To prove (1), we express the H-function of one variable on the left hand side as contour integral. Then interchanging the order of integration (which is justifiable due to given condition), we get

$$I = \frac{1}{2\pi i} \int_L \ \theta(s)\, z^s \times$$

$$\times \left[\int_0^{\frac{\pi}{2}} \frac{2^{(\alpha+\lambda s)+(\beta+\mu s)+1}}{\pi} e^{ix[(\alpha+\lambda s)-(\beta+\mu s)]} (\cos x)^{[(\alpha+\lambda s)+(\beta+\mu s)]} dx\right]$$

$$\times \left[\int_0^{\frac{\pi}{2}} e^{iy[(\sigma+\lambda s)+(\rho+\mu s)]} (\sin y)^{(\sigma+\lambda s)-1} (\cos y)^{(\rho+\mu s)-1} dy\right] ds$$

Now using the results (5), (6) and interpreting it with the help of

$$H_{p,q}^{m,n}\left[x\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right] = \left(\frac{1}{2\pi i}\right)\int_L \theta(s)x^5 ds\ ,\ i = \sqrt{-1},$$

$$\theta(s) = \frac{\prod^m \Gamma(b_j - \beta_j s)\prod^n \Gamma(1 - a_j + \alpha_j s)}{\prod^q \Gamma(1 - b_j + \beta_j s)\prod^p \Gamma(a_j - \alpha_j s)}$$

we get right hand side of (1).

Proceeding as above we can prove (2) to (5) with the help of A, B, C, and D

$$\int_0^{\pi/2} \frac{2^{\alpha+\beta+1}}{\pi} e^{i(\alpha-\beta)\theta} (\cos\theta)^{\alpha+\beta} d\theta = \frac{\pi\,\Gamma(\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}, \qquad \text{(A)}$$

$\mathrm{Re}(\alpha+\beta) > -1.$

$$\int_0^{\pi/2} e^{i(\alpha+\beta)\theta} (\sin\theta)^{\alpha-1} (\cos\theta)^{\beta-1} d\theta = \frac{e^{\pi i\alpha/2}\,\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \text{(B)}$$

$\mathrm{Re}(\alpha) > 0, Re(\beta) > 0.$

$$\int_0^{\infty} x^{\alpha-1} e^{-x} dx = \Gamma(\alpha), \qquad \text{(C)}$$

$$\int_0^{t} x^{\rho-1}(t-x)^{\sigma-1} dx = t^{\rho+\sigma-1}\frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}, \qquad \text{(D)}$$

$\mathrm{Re}(\rho) > 0, Re(\sigma) > 0.$

PARTICULAR CASE:

Collecting real and imaginary part of (4), we get

$$\int_0^t \int_0^{\pi/2} x^{\rho-1}(t-x)^{\sigma-1}\cos(\alpha+\beta)\, y\, (\sin y)^{\alpha-1}(\cos y)^{\beta-1} \times$$

$$\times H_{p,q}^{m,n}[z(2xe^{iy}\sin y)^{\lambda}((t-x)e^{iy}\cos y)^{\mu}]dxdy$$

$$= \cos\frac{\pi\alpha}{2} t^{\rho+\sigma-1} H_{p+4,q+3}^{m,n+4}\left[zt^{\lambda+\mu}e^{\frac{i\pi\lambda}{2}}\Big|_{(b_j,\beta_j)_{1,q},(1-\sigma-\rho,\lambda+\mu),(1-\alpha-\beta,\lambda+\mu)}^{(1-\alpha,\lambda),(1-\beta,\mu),(1-\rho,\lambda),(1-\sigma,\mu),(a_j,\alpha_j)_{1,p}}\right] \quad (6)$$

and

$$\int_0^t \int_0^{\pi/2} x^{\rho-1}(t-x)^{\sigma-1}\sin(\alpha+\beta)\, y\, (\sin y)^{\alpha-1}(\cos y)^{\beta-1} \times$$

$$\times H_{p,q}^{m,n}[z(2xe^{iy}\sin y)^{\lambda}((t-x)e^{iy}\cos y)^{\mu}]dxdy$$

$$= \sin\frac{\pi\alpha}{2} t^{\rho+\sigma-1} H_{p+4,q+3}^{m,n+4}\left[zt^{\lambda+\mu}e^{\frac{i\pi\lambda}{2}}\Big|_{(b_j,\beta_j)_{1,q},(1-\sigma-\rho,\lambda+\mu),(1-\alpha-\beta,\lambda+\mu)}^{(1-\alpha,\lambda),(1-\beta,\mu),(1-\rho,\lambda),(1-\sigma,\mu),(a_j,\alpha_j)_{1,p}}\right] \quad (7)$$

INTEGRAL REPRESENTATION:

$$H_{p,q:p_1+1,q_1:\ldots:p_r+1,q_r}^{0,n:m_1,n_1+1:\ \ldots:\ m_r,n_r+1}\left[\begin{matrix}x_1\\ \vdots\\ x_r\end{matrix}\Bigg|\begin{matrix}(a_i;A_i^{(1)},\ldots,A_i^{(r)})_{1\,n}:(1-\beta_1,1),(c_i^{(1)},C_i^{(1)})_{1\,.01}:\ldots(1-\beta_r,1),\ (c_i^{(r)},C_i^{(r)})_{1\,.0r}\\ (b_i;B_i^{(1)},\ldots,B_i^{(r)})_{1\,n}:,(d_i^{(1)},D_i^{(1)})_{1\,.01}:\ldots\ (d_i^{(r)},CD_i^{(r)})_{1\,.0r}\end{matrix}\right]$$

$$\prod_{i=1}^{r}\frac{1}{\Gamma(\nu_i-\beta_i)}\int_0^1 \ldots\ldots \int_0^1 \prod_{i=1}^{r}(1-t_i)^{\nu_i-\beta_i-1}(t_i)^{\beta_i-1}$$

$$H_{p,q:p_1+1,q_1:\ldots:p_r+1,q_r}^{0,n:m_1,n_1+1:\ \ldots:\ m_r,n_r+1}\left[\begin{matrix}x_1t_1\\ \vdots\\ x_rt_r\end{matrix}\Bigg|\begin{matrix}(a_i;A_i^{(1)},\ldots,A_i^{(r)})_{1\,n}:(1-\nu_1,1),(c_i^{(1)},C_i^{(1)})_{1\,.01}:\ldots(1-\nu_r,1),\ (c_i^{(r)},C_i^{(r)})_{1\,.0r}\\ (b_i;B_i^{(1)},\ldots,B_i^{(r)})_{1\,n}:,(d_i^{(1)},D_i^{(1)})_{1\,.01}:\ldots\ (d_i^{(r)},CD_i^{(r)})_{1\,.0r}\end{matrix}\right]$$

$$dt_1 \ldots .\, dt_r \quad (8)$$

Proof:

We consider

$$\frac{\delta^{\beta_1-\nu_1+\ \ldots+\ \beta_r-\nu_r}}{\delta x_1^{\beta_1-\nu_1}\ \ldots\ldots\ldots\ \delta x_r^{\beta_r-\nu_r}}\left\{x_1^{\beta_1-1}\ \ldots\ldots\ x_r^{\beta_r-1} H_{p,q:p_1+1,;\ p_r+1,q_r}^{0,n:m_1,n_1+1:\ldots:m_r,n_r+1}\right\}$$

$$\left[\begin{matrix} x_1 t_1 \\ \vdots \\ x_r t_r \end{matrix} \middle| \begin{matrix} \left(a_i;A_i^{(1)},\ldots,A_i^{(r)}\right)_{1\,n} :(1-\nu_1,1),\left(c_i^{(1)},C_i^{(1)}\right)_{1\,.01} :\ldots\ (1-\nu_r,1),\ \left(c_i^{(r)},C_i^{(r)}\right)_{1\,.0r} \\ \left(b_i;B_i^{(1)},\ldots,B_i^{(r)}\right)_{1\,n} :,\left(d_i^{(1)},D_i^{(1)}\right)_{1\,.01} :\ldots\ \left(d_i^{(r)},D_i^{(r)}\right)_{1\,.0r} \end{matrix}\right]$$

$$= \frac{1}{(2\pi i)^r} \int_{L_1} \ldots \int_{L_r} \psi(\xi_i, \ldots, \xi_r) \prod_{i=1}^{r} \phi_i(\xi_i)\Gamma(\nu_i + \xi_i)$$

$$\frac{\delta^{\beta_1-\nu_1+\ldots+\beta_r-\nu_r}}{\delta x_1^{\beta_1-\nu_1} \ldots\ldots\ldots\ \delta x_r^{\beta_r-\nu_r}} \left\{ {x_1}^{\beta_1+\xi_1-1} \ldots\ldots\ {x_r}^{\beta_r+\xi_1-1} \right\} d\xi_1 \ldots\ldots\ d\xi_r$$

$$\prod_{i=1}^{r} x_i^{\nu_i-1}\, H^{0,n:m_1,n_1+1:\ \ldots:\ m_r,n_r+1}_{p,q:p_1+1,q_1:\ldots:p_r+1,q_r}$$

$$\left[\begin{matrix} x_1 \\ \vdots \\ x_r \end{matrix} \middle| \begin{matrix} \left(a_i;A_i^{(1)},\ldots,A_i^{(r)}\right)_{1\,n} :(1-\beta_1,1),\left(c_i^{(1)},C_i^{(1)}\right)_{1\,.01} :\ldots\ (1-\beta_r,1),\ \left(c_i^{(r)},C_i^{(r)}\right)_{1\,.0r} \\ \left(b_i;B_i^{(1)},\ldots,B_i^{(r)}\right)_{1\,n} :,\left(d_i^{(1)},D_i^{(1)}\right)_{1\,.01} :\ldots\ \left(d_i^{(r)},CD_i^{(r)}\right)_{1\,.0r} \end{matrix}\right]$$

which on using

$$\frac{\partial^{\nu} U}{\partial (x-a)^{\nu}} = \frac{1}{\Gamma(-\nu)} \int_a^x (x-y)^{-\nu-1} U(y)dy \qquad \text{(E)}$$

gives

$$H^{0,n:m_1,n_1+1:\ \ldots:\ m_r,n_r+1}_{p,q:p_1+1,q_1:\ldots:p_r+1,q_r} \begin{matrix} x_1 \\ \vdots \\ x_r \end{matrix} \middle| \begin{matrix} \left(a_i;A_i^{(1)},\ldots,A_i^{(r)}\right)_{1,p} :(1-\beta_1,1),\left(c_i^{(1)},C_i^{(1)}\right)_{1\,.p1} :\ldots\ (1-\beta_r,1),\ \left(c_i^{(r)},C_i^{(r)}\right)_{1\,.pr} \\ \left(b_i;B_i^{(1)},\ldots,B_i^{(r)}\right)_{1,q} :,\left(d_i^{(1)},D_i^{(1)}\right)_{1\,.q1} :\ldots\ \left(d_i^{(r)},CD_i^{(r)}\right)_{1\,.qr} \end{matrix}$$

$$\prod_{i=1}^{r} x_i^{\nu_i-1}\, H^{0,n:m_1,n_1+1:\ \ldots:\ m_r,n_r+1}_{p,q:p_1+1,q_1:\ldots:p_r+1,q_r}$$

$$\prod_{i=1}^{r} x_i^{\nu_i-1} \frac{\delta^{\beta_1-\nu_1+\ldots+\beta_r-\nu_r}}{\delta x_1^{\beta_1-\nu_1} \ldots\ldots\ldots\ \delta x_r^{\beta_r-\nu_r}} \left\{ {x_1}^{\beta_1-1} \ldots\ldots\ {x_r}^{\beta_r-1} H^{0,n:m_1,n_1+1:\ldots:m_r,n_r+1}_{p,q:p_1+1,:\ p_r+1,q_r} \right\}$$

$$\left[\begin{matrix} x_1 t_1 \\ \vdots \\ x_r t_r \end{matrix} \middle| \begin{matrix} \left(a_i;A_i^{(1)},\ldots,A_i^{(r)}\right)_{1\,p} :(1-\nu_1,1),\left(c_i^{(1)},C_i^{(1)}\right)_{1\,.p1} :\ldots\ (1-\nu_r,1),\ \left(c_i^{(r)},C_i^{(r)}\right)_{1\,.pr} \\ \left(b_i;B_i^{(1)},\ldots,B_i^{(r)}\right)_{1\,q} :,\left(d_i^{(1)},D_i^{(1)}\right)_{1\,.q1} :\ldots\ \left(d_i^{(r)},D_i^{(r)}\right)_{1\,.qr} \end{matrix}\right]$$

$$\prod_{i=1}^{r} x_i^{1-\nu_i} \frac{1}{\Gamma(\nu_i-\beta_i)} \int_0^{x_1} \dots\dots \int_0^{x_r} (x_1-y_1)^{\nu_1-\beta_1-1} y_1^{\beta_1-1} \dots (x_r - y_r)^{\nu_r-\beta_r-1} y_r^{\beta_r-1}$$

$$\left[H^{0,n:m_1,n_1+1:\dots:m_r,n_r+1}_{p,q:p_1+1,q_1:\dots:p_r+1,q_r} \begin{matrix} y_1 \\ \vdots \\ y_r \end{matrix} \left| \begin{matrix} (a_i;A_i^{(1)},\dots,A_i^{(r)})_{1\,p}:(1-\nu_1,1),(c_i^{(1)},C_i^{(1)})_{1.p1}:\dots(1-\nu_r,1),\ (c_i^{(r)},C_i^{(r)})_{1.pr} \\ (b_i;B_i^{(1)},\dots,B_i^{(r)})_{1\,q}:,(d_i^{(1)},D_i^{(1)})_{1.q1}:\dots\ (d_i^{(r)},CD_i^{(r)})_{1.qr} \end{matrix} \right. \right]$$

$$dy_1 dy_r$$

Now putting $y_i = x_i t_i$, where i = 1, 2, …, r, we establish the following particular case.

PARTICULAR CASES:

(i) If we take r = 2 in (8) it gives

$$H^{0,n:m_1,n_1+1:\dots:m_2,n_r+1}_{p,q:p_1+1,q_1:\dots:p_2+1,q_2} \begin{matrix} x_1 \\ \vdots \\ x_2 \end{matrix} \left| \begin{matrix} (a_i;A_i^{(1)},\dots,A_i^{(2)})_{1\,p}:(1-\beta_2,1),(c_i^{(1)},C_i^{(1)})_{1.p1}:\dots(1-\beta_2,1),\ (c_i^{(2)},C_i^{(2)})_{1.p2} \\ (b_i;B_i^{(1)},\dots,B_i^{(2)})_{1\,q}:,(d_i^{(1)},D_i^{(1)})_{1.q1}:\dots\ (d_i^{(2)},CD_i^{(2)})_{1.q2} \end{matrix} \right.$$

$$= \prod_{i=1}^{2} \frac{1}{\Gamma(\nu_i-\beta_i)} \int_0^1 \dots\dots \int_0^1 \prod_{i=1}^{2} (1-t_i)^{\nu_i-\beta_i-1} (t_i)^{\beta_i-1}$$

$$H^{0,n:m_1,n_1+1:\dots:m_2,n_2+1}_{p,q:p_1+1,q_1:\dots:p_2+1,q_2} \begin{bmatrix} x_1 t_1 \\ \vdots \\ x_2 t_2 \end{bmatrix} \left| \begin{matrix} (a_i;A_i^{(1)},\dots,A_i^{(2)})_{1\,p}:(1-\nu_1,1),(c_i^{(1)},C_i^{(1)})_{1.p1}:\dots(1-\nu_2,1),\ (c_i^{(2)},C_i^{(2)})_{1.p2} \\ (b_i;B_i^{(1)},\dots,B_i^{(2)})_{1\,q}:,(d_i^{(1)},D_i^{(1)})_{1.q1}:\dots\ (d_i^{(2)},CD_i^{(2)})_{1.q2} \end{matrix} \right. \Bigg]$$

$$dt_1 \dots . dt_2 \qquad (9)$$

where | arg x_1 | < ½ Mπ, | arg x_2 | < ½ Nπ,

$$M = -\sum^{p}(A_i)^{(1)}$$

$$-\sum^{q}(B_i)^{(1)} - \sum^{m_1}(d_i)^{(1)} - \sum^{q_1}(D_i)^{(1)} + \sum^{n_1}(c_i)^{(1)} - \sum^{p_1}(C_i)^{(1)} > 0,$$

$$N = -\sum^{p}(A_i)^{(2)}$$

$$-\sum^{q}(B_i)^{(2)} - \sum^{m_2}(d_i)^{(2)} - \sum^{q_2}(D_i)^{(2)} + \sum^{n_2}(c_i)^{(2)} - \sum^{p_2}(C_i)^{(2)} > 0,$$

(ii) Again if we take n = p = q = 0, r = 1 in (8), we easily arrive at the following integral representation of the H-function:

$$H^{m_1,n_1+1}_{p_1+1,q_1}\left[x_1\left|\begin{matrix}(1-\beta_1,1),\left(c_i{}^{(1)},C_i{}^{(1)}\right)_{1,p1}\\ \left(d_i{}^{(1)},D_i{}^{(1)}\right)_{1,q1}\end{matrix}\right.\right]$$

$$= \frac{1}{\Gamma(\nu_1-\beta_1)}\int_0^1 (1-t_1)^{\nu_1-\beta_1-1}(t_1)^{\beta_1-1} \times H^{m_1,n_1+1}_{p_1+1,q_1}\left[x_1\left|\begin{matrix}(1-\beta_1,1),\left(c_i{}^{(1)},C_i{}^{(1)}\right)_{1,p1}\\ \left(d_i{}^{(1)},D_i{}^{(1)}\right)_{1,q1}\end{matrix}\right.\right] dt_1 \quad (10)$$

where

$$\sum_{j=1}^{n_1} C_i{}^{(1)} - \sum_{j=n_1+1}^{p_1} C_i{}^{(1)} + \sum_{j=1}^{m_1} D_i{}^{(1)} - \sum_{j=m_2+1}^{q_1} D_i{}^{(1)} \equiv A > 0,$$

$$|\arg x_1| < ½ A\pi.$$

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