On Soft $\pi$Gr-Continuous Functions in Soft Topological Spaces

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Abstract: The purpose of this paper is to study soft $\pi$-continuous function and soft $\pi$gr-irresolute function in soft topological spaces. Also, we introduce the concepts such as soft $\pi$gr-closure, soft $\pi$gr-interior and exhibit some results related to soft $\pi$gr-continuity, soft $\pi$gr-open map and soft $\pi$gr-closed map. Further, we study the relation between soft $\pi$gr-continuous function and other soft continuous functions.


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I. INTRODUCTION.


II. PRELIMINARIES.

Definition 2.1 ([15],[16],[22])
Let U be the initial universe and P(U) denote the power set of U. Let E denote the set of all parameters. Let A be a non-empty subset of E. A pair (F, A) is called a soft set over U, where F is a mapping given by F: A → P(U). In other words, a soft set over U is a parameterized family of subsets of the universe U. For e ∈ A, F(e) may be considered as the set E-elementary approximations of the soft set (F, A). Clearly, a soft set is not a set. For two soft sets (F, A) and (G, B) over the common universe U, we say that (F, A) is a soft subset of (G, B) if (i) A ⊆ B and (ii) for all e ∈ A, F(e) and G(e) are identical approximations. We write (F, A) ⊆ (G, B), (F, A) is said to be a soft superset of (G, B), if (G, B) is a soft subset of (F, A).

Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.2([15],[16],[22])
For a soft set (F, A) over the universe U, the relative $\pi$gr-closure of (F, A) is denoted by $\overline{\pi}F$ and is defined by $\overline{\pi}F = \pi\alpha F$ where $\pi\alpha F = U\setminus F(e)$ for all e ∈ A.

Definition 2.3([15],[16],[22])
A soft set (F, A) over X is said to be a null soft set denoted by $\varnothing$ or $\varnothing_A$ if for all e ∈ A, F(e) = φ(null set).

Definition 2.4([15],[16],[22])
A soft set (F, A) over X is said to be absolute soft set denoted by $\tilde{A}$ or $\tilde{X}_A$ if for all e ∈ A, F(e) = X. Clearly, we have $\tilde{X}_A = \varphi_A$ and $\varnothing_A = \tilde{X}_A$.

Definition 2.5 ([15],[16],[22])
The union of two soft sets (F, A) and (G, B) over the common universe U is soft set (H, C) where $C = A \cup B$ and for all $e \in C$, $H(e) = F(e)$ if $e \in A \setminus B$, $H(e) = G(e)$ if $e \in B \setminus A$ and $H(e) = F(e) \cup G(e)$ if $e \in A \cap B$. We write (F, A) $\cup$ (G, B) = (H, C).

Definition 2.6([15],[16],[22])
The intersection (H, C) of two soft sets (F, A) and (G, B) over a common universe U denoted (F, A) $\cap$ (G, B) is defined as $C = A \cap B$ and $H(e) = F(e) \cap G(e)$ for all e ∈ C.

Definition 2.7 ([2],[11],[15])
Let $\tau$ be the collection of soft sets over X, then $\tau$ is called a soft topology on X if $\tau$ satisfies the following axioms:
1) $\varnothing, \tilde{X}$ belong to $\tau$. 
2) The union of any number of soft sets in \( \tau \) belongs to \( \tau \).

3) The intersection of any two soft sets in \( \tau \) belongs to \( \tau \).

The triplet \((X, \tau, E)\) is called a soft topological space over \( X \). For simplicity, we can take the soft topological spaces \((X,\tau_1,E)\) as \( X \) and \((Y,\tau_2,E)\) as \( Y \) throughout this work respectively. Let us denote the collection of soft sets over the universe \( X \) and \( Y \) as \( SS(X) \) and \( SS(Y) \) respectively.

**Definition 2.8** ([2],[11],[15])

Let \((X,\tau,E)\) be soft topological space over \( X \). A soft set \((F,E)\) over \( X \) is said to be soft closed in \( X \), if its relative complement \((F,E)\) belongs to \( \tau \). The relative complement is a mapping \( F' : E \rightarrow P(X) \) defined by \( F(e) = X - F(e) \) for all \( e \in E \).

**Definition 2.9** ([2],[11],[15])

Let \((X,\tau,E)\) be a soft topological space over \( X \) and \( x \in X \). Then \((G,E)\) is said to be a soft neighborhood of \((F,E)\) if there exists a soft open set \((H,E)\) such that \((F,E)\) is a soft neighborhood of \((G,E)\). The neighborhood of \((F,E)\) is the family of all its neighborhoods.

**Definition 2.10** ([3])

Let \((X,\tau,E)\) be a soft topological space over \( X \), \((G,E)\) be a soft set over \( X \) and \( x \in X \). Then \((G,E)\) is said to be a soft neighborhood of \( x \), if there exists a soft open set \((F,E)\) such that \( x \in (F,E) \subsetneq (G,E) \).

**Definition 2.11** ([24])

Let \((X,\tau,E)\) be a soft topological space over \( X \), \((G,E)\) be a soft set over \( X \) and \( x \in X \). Then \((G,E)\) is said to be a soft interior point of \((G,E)\), if there exists a soft open set \((F,E)\) such that \( x \in (F,E) \subsetneq (G,E) \).

**Definition 2.12**

A soft subset \((A,E)\) of \( X \) is called
(i) a soft generalized closed (Soft g-closed)[[11]] in a soft topological space \((X,\tau,E)\) if \( s-cl(A,E) \subsetneq (U,E) \) whenever \((A,E) \subsetneq (U,E) \) and \((U,E)\) is soft open in \( X \).
(ii) a soft semi open \([3]\) if \( (A,E) \subsetneq s-int(s-cl(A,E)) \).
(iii) a soft regular open if \((A,E) = s-int(s-cl(A,E)) \).
(iv) a soft \( \alpha \)-open if \((A,E) \subsetneq s-cl(s-int(A,E)) \).
(v) a soft b-open if \((A,E) \subsetneq s-cl(s-int(A,E)) \cup s-int(s-cl(A,E)) \).
(vi) a soft pre-open if \((A,E) \subsetneq s-int(s-cl(A,E)) \).
(vii) a soft clopen if \((A,E)\) is both soft open and soft closed.

The complement of the soft semi open, soft regular open, soft \( \alpha \)-open, soft b-open, soft pre-open sets are their respective soft semi closed, soft regular closed, soft \( \alpha \)-closed, soft b-closed and soft pre-closed sets.

The finite union of soft regular open sets is called soft \( \pi \)-open set and its complement is soft \( \pi \)-closed set.

The soft regular open set of \( X \) is denoted by \( SRO(X) \) or \( SRO(X,\tau,E) \).

**Definition 2.13**([11])

A soft topological space \( X \) is called a soft \( T_{1/2} \)-space if every soft g-closed set is soft closed in \( X \).

**Definition 2.14**([9])

The soft regular closure of \((A,E)\) is the intersection of all soft regular closed sets containing \((A,E)\). The soft regular closed set \((A,E)\) is the union of all soft regular open sets contained in \((A,E)\) and is denoted by \( s-cl(A,E) \).

Similarly, we define soft \( \alpha \)-closure, soft pre-closure, soft semi closure and soft \( b \)-closure of the soft set \((A,E)\) of a topological space \( X \) and are denoted by \( s-cl(A,E) \), \( s-cl(A,E) \), \( s-cl(A,E) \) and \( s-cl(A,E) \), respectively.

**Definition 2.15** ([10],[14])

Let \((F,E)\) be a soft set over \( X \). The soft set \((F,E)\) is called a soft point denoted by \( e_\phi \) if for the element \( e_\phi \in E \), \( F(e_\phi) \neq \phi \) for all \( e \in E - \{e_\phi\} \).

**Definition 2.16** ([3])

A soft set \((G,E)\) in a soft topological space \((X,\tau,E)\) is called a soft neighborhood of the soft point \( e_\phi \) if there exists a soft open set \((H,E)\) such that \( e_\phi \in (H,E) \subsetneq (G,E) \).

A soft set \((G,E)\) in a soft topological space \((X,\tau,E)\) is called a soft neighborhood of the soft set \((F,E)\) if there exists an soft open set \((H,E)\) such that \( (F,E) \subsetneq (H,E) \subsetneq (G,E) \).

The neighborhood system of the soft point \( e_\phi \) denoted by \( N_\phi \), is the family of all its neighborhoods.

**Definition 2.17**([3])

Let \((X,\tau,E)\) be a soft topological space over \( X \), \((G,E)\) be a soft set over \( X \) and \( x \in X \). Then \((G,E)\) is said to be a soft interior point of \((G,E)\), if there exists a soft open set \((F,E)\) such that \( x \in (F,E) \subsetneq (G,E) \).

**Definition 2.18** ([10],[14])

Let \( SS(X)_\lambda \) and \( SS(Y)_\beta \) be two soft classes. Then \( u : X \rightarrow Y \) and \( p : A \rightarrow B \) be two functions. Then the function \( f_{pu} : SS(X)_\lambda \rightarrow SS(Y)_\beta \) and its inverse are defined as

(i) Let \((F,A)\) be a soft set in \( SS(X)_\lambda \). The image of \((F,A)\) under \( f_{pu} \) written as \( (f_{pu}(F),p(A)) \) is a soft set in \( SS(Y)_\beta \) such that \( f_{pu}(F)(y) = \begin{cases} u(F(x)) \cup p^{-1}(y) \cap A \neq \phi \\ \phi \text{, otherwise.} \end{cases} \) for all \( y \in B \).

(ii) Let \((G,B)\) be a soft set in \( SS(Y)_\beta \). The inverse image of \((G,B)\) under \( f_{pu} \) written as \( f_{pu}^{-1}(G,B) \) is a soft set in \( SS(X)_\lambda \) such that \( f_{pu}^{-1}(G)(x) = \begin{cases} f_{pu}^{-1}(G)(p(x)) \cup \phi \text{, otherwise.} \end{cases} \) for all \( x \in A \).
Definition: 2.19([10],[14])
Let \((X,\tau,A)\) and \((Y,\tau^*,B)\) be soft topological spaces and \(f_{pu} : SS(X)_\lambda \to SS(Y)_\mu\) be a function. Then function \(f_{pu}\) is called soft continuous if \(f_{pu}^{-1}(G,B) \in \tau\) for all \((G,B) \in \tau^*\).

Definition: 2.20 [10]
Let \((X,\tau,A)\) and \((Y,\tau^*,B)\) be soft topological spaces and \(f_{pu} : SS(X)_\lambda \to SS(Y)_\mu\) be a function. Then the function \(f_{pu}\) is called soft open mapping if \(f_{pu}((G,A)) \in \tau^*\) for all \((G,A) \in \tau\).

Proof: Let \((F,A)\) and \((G,A)\) be soft topological spaces and \(f_{pu} : SS(X)_\lambda \to SS(Y)_\mu\) be a function. Then the function \(f_{pu}\) is called soft open mapping if \(f_{pu}((G,A)) \in \tau^*\) for all \((G,A) \in \tau\).

Similarly, a function \(f_{pu} : SS(X)_\lambda \to SS(Y)_\mu\) is called a soft closed map if for a closed set \((F,A)\) in \(\tau\), the image \(f_{pu}((G,B))\) is soft closed in \(\tau^*\).

Definition: 2.21([9])
A soft subset \((G,A)\) of a soft topological space \(X\) is called a soft \(\pi\)GR-closed set in \(X\) if \(s\pi gr \text{-cl}(G,A) \subset (X,A)\) whenever \((G,A) \subset (X,A)\), where \((X,A)\) is soft \(\pi\)-open in \(X\). We denote the soft \(\pi\)GR-closed set of \(X\) by \(S\pi GRO(X)\).

III. SOFT \(\pi\)GR-CLOSURE AND SOFT \(\pi\)GR-INTERIOR.

Let us introduce the following definitions.

Definition: 3.1
Let \((X,\tau,A)\) be a soft topological space over \(X\), \((G,A)\) be a soft set over \(X\) and \(x \in X\). Then \((G,A)\) is said to be a soft \(\pi\)GR-neighborhood of \(x\) if there exists a soft \(\pi\)-open set \((F,A)\) such that \(x \in (F,A) \subset (G,A)\). The soft \(\pi\)GR-neighborhood of a point \(x\) is denoted by \(s\pi gr\text{-nbhd}(x)\).

Definition: 3.2
The soft \(\pi\)GR-Closure of a soft set \((G,A)\) is defined to be the intersection of all soft \(\pi\)GR-closed sets containing the soft set \((G,A)\) and is denoted by \(s\pi gr\text{-cl}(G,A)\).

The soft \(\pi\)GR-Interior of a soft set \((G,A)\) is defined to be the union of all soft \(\pi\)-open sets containing the soft set \((G,A)\) and is denoted by \(s\pi gr\text{-int}(G,A)\).

Lemma: 3.3
Let \((F_1,A)\) and \((F_2,A)\) be subsets of \((X,\tau,A)\). Then
(i) \(s\pi gr\text{-cl}(\varphi_A) = \varphi_A\) and \(s\pi gr\text{-cl}(X) = X\).
(ii) \(s\pi gr\text{-cl}(F_1,A) \supseteq \text{Int}(F_1,A)\).
(iii) \(s\pi gr\text{-cl}(F_1,A) \supseteq \text{Int}(F_1,A)\).
(iv) \(s\pi gr\text{-cl}(F_1,A) \cap \text{Int}(F_2,B) \subseteq \text{Int}(F_1,A) \cap \text{Int}(F_2,B)\).

Proof: Obvious.

Lemma: 3.4
Let \((F,A)\) and \((G,A)\) be soft subsets of the soft topological space \(X\). Then \(s\pi gr\text{-cl}((F,A) \cap (G,A)) \subseteq s\pi gr\text{-cl}(F,A) \cap s\pi gr\text{-cl}(G,A)\).

Proof: Since \((F,A) \cap (G,A) \subseteq (F,A), (G,A)\)
\(s\pi gr\text{-cl}((F,A) \cap (G,A)) \subseteq s\pi gr\text{-cl}(F,A)\) and \(s\pi gr\text{-cl}((F,A) \cap (G,A)) \subseteq s\pi gr\text{-cl}(F,A)\)
\(s\pi gr\text{-cl}((F,A) \cap (G,A)) \subseteq s\pi gr\text{-cl}(F,A)\cap s\pi gr\text{-cl}(G,A)\).

Lemma: 3.5
Let \((F,A)\) and \((G,A)\) be soft subsets of a soft topological space \(X\). Then
(i) \(s\pi gr\text{-int}(X) = X\), \(s\pi gr\text{-int}(\varphi_A) = \varphi_A\).
(ii) \(s\pi gr\text{-int}((F,A)) \subseteq (F,A)\).
(iii) If \((F,A)\) is any soft \(\pi\)-open set contained in \((G,A)\), then \((F,A) \subseteq s\pi gr\text{-int}(G,A)\).
(iv) If \((F,A) \subseteq (G,A)\), then \(s\pi gr\text{-int}(F,A) \subseteq s\pi gr\text{-int}(G,A)\).

Proof: Straight Forward.

Theorem: 3.6
If a subset \((F,A)\) of a soft topological space \(X\) is soft \(\pi\)-open, then \(s\pi gr\text{-int}(F,A) = (F,A)\).

Proof: Obvious.

Theorem: 3.7
If \((F,A)\) and \((G,A)\) are soft subsets of a soft topological space \(X\), then \(s\pi gr\text{-int}(F,A) \cup s\pi gr\text{-int}(G,A) \subseteq s\pi gr\text{-int}((F,A) \cup (G,A))\).

Proof: We know that \((F,A) \subseteq (F,A) \cup (G,A)\).

Theorem: 3.8
If \((F,A)\) and \((G,A)\) are soft subsets of a space \(X\), then \(s\pi gr\text{-int}(F,A) \cap s\pi gr\text{-int}(G,A) \subseteq s\pi gr\text{-int}((F,A) \cap (G,A))\).

Proof: We know that \((F,A) \cap (G,A) \subseteq (F,A) \cap (G,A)\).

Lemma: 3.9
If \((F,A)\) be a soft subset of a soft topological space \(X\), then \((X- s\pi gr\text{-int}(F,A)) = s\pi gr\text{-cl}(X-(F,A))\).

Proof: Let \(x \in X- (s\pi gr\text{-int}(F,A))\). Then \(x \notin s\pi gr\text{-int}(F,A)\).
That is every soft \(\pi\)-open set \((G,A)\) containing \(x\) is such that \((G,A) \supseteq (F,A)\).
Every soft \(\pi\)-open set \((H,A)\) containing \(x\) intersects \((F,A)\).

Hence \((X- s\pi gr\text{-int}(F,A)) \subseteq s\pi gr\text{-cl}(X-(F,A))\).

Conversely, let \(x \in s\pi gr\text{-cl}(X-(F,A))\). Thus every soft \(\pi\)-open set \((H,A)\) containing \(x\) intersects \((F,A)\) (i.e.) every
soft $\pi gr$-open set $(H,A)$ containing $x$ is such that $(H,A) \not\subseteq (F,A)$.

$\Rightarrow x \notin \pi gr$-int$(F,A))$. Thus $\pi gr$-cl$(X-(F,A)) \subsetneq (X-\pi gr$-int$(F,A)))$ and hence $(X-\pi gr$-int$(F,A)) = \pi gr$-cl$(X-(F,A))$.

**Remark:** 3.10

If $(F,A)$ be a soft subset of a soft topological space $X$, then $(X-\pi gr$-cl$(F,A))) = \pi gr$-cl$(X-(F,A))$.

IV. **SOFT $\pi GR$-CONTINUOUS FUNCTIONS.**

**Definition:** 4.1

Let $(X,\tau,A)$ and $(Y,\tau^*,B)$ be soft topological spaces and $f_{pu}: SS(X)_A \rightarrow SS(Y)_B$ be a function. Then the function $f_{pu}$ is called

1. soft $\pi gr$-continuous if $f_{pu}^{-1}((G,B))$ is a soft $\pi gr$-closed set for every soft closed set $(G,B)$ in $(Y,\tau^*,B)$.

2. soft $\pi gr$-irresolute if $f_{pu}^{-1}((G,B))$ is a soft $\pi gr$-closed set for every soft $\pi gr$-closed set $(G,B)$ in $(Y,\tau^*,B)$.

3. soft regular continuous if $f_{pu}^{-1}((G,B))$ is soft regular closed in $(X,\tau,A)$ for every soft closed set $(G,B)$ in $(Y,\tau^*,B)$.

4. soft $R$-map if for a soft regular closed set $(G,B)$ in $(X,\tau,A)$, $f_{pu}^{-1}((G,B))$ is soft regular closed in $\tau$.

**Definition:** 4.2

Let $(X,\tau,A)$ and $(Y,\tau^*,B)$ be soft topological spaces and $f_{pu}: SS(X)_A \rightarrow SS(Y)_B$ be a function. Then

1. The function $f_{pu}$ is called soft $\pi gr$-open if $f_{pu}(G,A))$ is soft $\pi gr$-open set for every soft open set $(G,A)$ in $(X,\tau,A)$.

2. The function $f_{pu}$ is called soft $\pi gr$-closed if $f_{pu}(G,A))$ is soft $\pi gr$-closed set for every soft closed set $(G,A)$ in $(X,\tau,A)$.

**Remark:** 4.3

Soft $\pi gr$-continuity and soft continuity are independent concepts.

**Example:** 4.4

(i) Let $X=\{h_1, h_2, h_3\}$, $Y=\{a,b\}$, $E=\{e_1,e_2\}$. Let $F_1$, $F_2$, $F_3$ are functions defined from $E$ to $P(X)$ as follows:

$F_1(e_1) = \{h_1,h_2\}$, $F_2(e_2) = \{h_2\}$

$F_3(e_1) = \{h_3\}$, $F_3(e_2) = \{h_2\}$

Then $\tau_1 = \{\phi, X, (F_1,E),(F_2,E),(F_3,E)\}$ is soft topology on $X$.

Let $G_1, G_2$ are functions from $E$ to $P(Y)$ and are defined as follows:

$G_1(e_1) = \{a\}$, $G_2(e_2) = \{a\}$

Then $\tau_2 = \{\phi, Y, (G_1,E), (G_2,E), (G_3,E)\}$ be a soft topology on $Y$. Let $f: X \rightarrow Y$ be a function defined as $f(h_1) = f(h_2) = \{a\}$ and $f(h_3) = \{b\}$.

The inverse image of the soft open sets $(G_1,E)$ and $(G_2,E)$ in $Y$ are soft $\pi gr$-open in $X$ but not soft open in $X$. Hence soft $\pi gr$-continuity need not be soft continuity.

(ii) Let $X=\{a,b,c,d\}$, $E=\{e_1,e_2\}$. Let $F_1, F_2, \ldots, F_6$ are functions from $E$ to $P(X)$ and are defined as follows:

$F_1(e_1) = \{c\}$, $F_1(e_2) = \{a\}$

$F_2(e_1) = \{d\}$, $F_2(e_2) = \{a\}$

$F_3(e_1) = \{c,d\}$, $F_3(e_2) = \{a,b\}$

$F_4(e_1) = \{a,d\}$, $F_4(e_2) = \{b,d\}$

$F_5(e_1) = \{b,c,d\}$, $F_5(e_2) = \{a,b,c\}$

$F_6(e_1) = \{a,c,d\}$, $F_6(e_2) = \{a,b,d\}$

Then $\tau_1 = \{\phi, X, (F_1,E), \ldots, (F_6,E)\}$ is a soft topology and elements in $\tau$ are soft open sets.

Let $G_1, G_2, G_3$ are functions from $E$ to $P(Y)$ and are defined as follows:

$G_1(e_1) = \{a\}$, $G_1(e_2) = \{d\}$

$G_2(e_1) = \{b\}$, $G_2(e_2) = \{c\}$

$G_3(e_1) = \{a,b\}$, $G_3(e_2) = \{c,d\}$

$G_4(e_1) = \{b,c,d\}$, $G_4(e_2) = \{a,b,c\}$

Then $\tau_2 = \{\phi, Y, (G_1,E), \ldots, (G_4,E)\}$ be a soft topology on $Y$. Here the inverse image of the soft open set $(G_1,E)$ in $Y$ is not soft $\pi gr$-open in $X$ but soft open in $X$. Hence soft $\pi gr$-continuity need not be soft $\pi gr$-continuous.

**Theorem:** 4.5

Every soft regular continuous function is soft $\pi gr$-continuous but not conversely.

**Proof:** Straight forward.

**Example:** 4.6

In example 4.4(i), The inverse image of the soft open sets $(G_1,E)$ and $(G_2,E)$ in $Y$ are soft $\pi gr$-open in $X$ but not soft regular open in $X$. Hence soft $\pi gr$-continuity need not be soft regular continuous.
Theorem: 4.7
Every soft $\pi gr$-continuous function is soft rg-continuous, soft $\pi g$-continuous, soft $\pi g^*$-continuous, soft $\pi ga$-continuous, soft $\pi gp$-continuous, soft $\pi gs$-continuous and soft $\pi gb$-continuous.

Proof: Follows from the definitions.

Remark: 4.8
The converse of the above need not be true and is shown in the following example.

Example: 4.9
In example 4.4(ii), the inverse image the soft open set $\tau_4 \subseteq G_4$ in $Y$ is soft not $\pi gr$-open in $X$ but soft rg-open, soft $\pi  g$-open, soft $\pi g^*$-open, soft $\pi ga$-open, soft $\pi gp$-open, soft $\pi gs$-open, soft $\pi gb$-open and soft $\pi gb$-open in $X$. Hence soft rg-continuous, soft $\pi ga$-continuous, soft $\pi gp$-continuous, soft $\pi gs$-continuous, soft $\pi gb$-continuous, soft $\pi g^*$-continuous, soft $\pi ga$-continuous and soft $\pi gb$-continuous but not soft $\pi gr$-continuous.

Remark: 4.10
Soft $\pi gr$-continuity and soft $\pi gr$-irresolute are independent concepts.

Example: 4.11
(i) Let $X = \{a, b, c, d\}$, $Y = \{e_1, e_2\}$. Let $F_1, F_2, \ldots, F_6$ be functions from $E$ to $P(Y)$ and are defined as follows:

$F_1(e_1) = \{c\}$, $F_2(e_2) = \{a\}$,

$F_3(e_1) = \{d\}$, $F_2(e_2) = \{b\}$,

$F_3(e_1) = \{c, d\}$, $F_3(e_2) = \{a, b\}$,

$F_4(e_1) = \{a, d\}$, $F_4(e_2) = \{b, d\}$,

$F_5(e_1) = \{b, c, d\}$, $F_5(e_2) = \{a, b, c\}$,

$F_6(e_1) = \{a, c, d\}$, $F_6(e_2) = \{a, b, d\}$.

Then $\tau_1 = \{\phi, \tilde{X}, (F_1, E), \ldots, (F_6, E)\}$ is a soft topology and elements in $\tau$ are soft open sets.

Let $G_1, G_2, G_3$ are functions from $E$ to $P(Y)$ and are defined as follows:

$G_1(e_1) = \{a\}$, $G_1(e_2) = \{d\}$,

$G_2(e_1) = \{b, c, d\}$, $G_2(e_2) = \{a, b, c\}$.

Then $\tau_2 = \{\phi, Y, (G_1, E), (G_2, E)\}$ is a soft topology on $Y$. Let $f: X \to Y$ be an identity map. Here the inverse image of the soft open set in $Y$ is soft $\pi gr$-open in $X$, but the inverse image of soft $\pi gr$-open sets in $Y$ are not soft $\pi gr$-open in $X$. Hence soft $\pi gr$-continuous function need not be soft $\pi gr$-irresolute.

(ii) Let $X = \{a, b, c\} = Y$, $E = \{e_1, e_2\}$. Let $F_1, F_2, F_3$ are functions from $E$ to $P(X)$ and are defined as follows:

$F_1(e_1) = \{a, c\}$, $F_1(e_2) = \{\phi\}$,

$F_2(e_1) = \{b\}$, $F_2(e_2) = \{a\}$,

$F_3(e_1) = X$, $F_3(e_2) = \{a\}$.

Then $\tau_3 = \{\phi, \tilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$ is a soft topology on $X$ and elements in $\tau$ are soft open sets of $X$.

Let $G_1, G_2, G_3, G_4, G_5, G_6$ are functions from $E$ to $P(Y)$ and are defined as follows:

$G_1(e_1) = \{a\}$, $G_1(e_2) = \{b\}$,

$G_2(e_1) = \{b, c\}$, $G_2(e_2) = \{a\}$,

$G_3(e_1) = X$, $G_3(e_2) = \{a, b\}$,

$G_4(e_1) = \{a, b\}$, $G_4(e_2) = \{a, b, c\}$,

$G_5(e_1) = \{b\}$, $G_5(e_2) = \{\phi\}$,

$G_6(e_1) = \{a, b\}$, $G_6(e_2) = \{b\}$.

Then $\tau_4 = \{\phi, Y, (G_1, E), (G_2, E), (G_3, E), (G_4, E), (G_5, E), (G_6, E)\}$ be a soft topology on $Y$. Let $f: X \to Y$ be an identity map. Here the inverse image of the soft $\pi gr$-open sets in $Y$ are soft $\pi gr$-open in $X$, but the inverse image of soft open set $(G_4, E) = \{\{a, b\}, \{b, c\}\}$ is not soft $\pi gr$-open in $X$. Hence soft $\pi gr$-irresoluteness need not be soft $\pi gr$-continuous.

Remark: 4.12
The above discussions are represented diagrammatically as follows:

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1-Soft continuous
2-soft regular continuous
3-soft $\pi gr$-irresolute
4-soft $\pi gr$-continuous
5-soft $\pi g$-continuous
6-soft $\pi g^*$-continuous
7-soft $\pi ga$-continuous
8-soft $\pi gp$-continuous
9-soft $\pi gs$-continuous
10-soft $\pi gb$-continuous
11-soft $\pi gs$-continuous.
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Theorem: 4.13

A soft function $f_{pu}: SS(X) \rightarrow SS(Y)$ is soft $\pi gr$-continuous when $f_{pu}(sgn-cl(F,A)) \subseteq s-cl(f_{pu}(F,A)))$ for every soft set $(F,A)$ of a soft topological space $(X, \pi)$.

Proof: Let $f_{pu}: SS(X) \rightarrow SS(Y)$ be a soft $\pi gr$-continuous function. Now, $s-cl(f_{pu}(F,A)))$ is a soft closed set of $(Y, \pi, B)$. By the soft $\pi gr$-continuity of $f_{pu}$, $f^{-1}_{pu}(s-cl(f_{pu}(F,A)))$ is soft $\pi gr$-closed and $(F,A) \subseteq f^{-1}_{pu}(s-cl(f_{pu}(F,A)))$. But $sgn-cl (F,A)$ is the smallest soft $\pi gr$-closed set containing $(F,A)$, hence $sgn-cl(F,A) \subseteq f^{-1}_{pu}(s-cl(f_{pu}(F,A)))$. The above implies $f_{pu}(sgn-cl(F,A)) \subseteq s-cl(f_{pu}(F,A)))$.

Theorem: 4.14

A soft function $f_{pu}: SS(X) \rightarrow SS(Y)$ is soft $\pi gr$-continuous when $f^{-1}_{pu}(s-int(G,B)) \subseteq sgr-int(f^{-1}_{pu}(G,B))$ for every soft set $(G,B)$ of a soft topological space $(X, \pi, \tau)$.

Proof: Let $f_{pu}: SS(X) \rightarrow SS(Y)$ be a soft $\pi gr$-continuous function. Now, $s-int(f_{pu}(G,B)))$ is a soft open set of $(Y, \pi, \tau)$, so by soft $\pi gr$-continuity of $f_{pu}$, $f^{-1}_{pu}(s-int(f_{pu}(G,B)))$ is soft $\pi gr$-open in $(X, \pi)$ and $f^{-1}_{pu}(s-int(G,B))) \subseteq (G,B)$. As $sgn-int(G,B)$ is the largest soft $\pi gr$-open set contained in $(G,B)$, $f^{-1}_{pu}(s-int(G,B))) \subseteq sgr-int(f^{-1}_{pu}(G,B))$ for every soft set $(G,B)$ of a soft topological space $(Y, \pi, \tau)$.

Theorem: 4.15

A soft function $f_{pu}: SS(X) \rightarrow SS(Y)$ is soft $\pi gr$-closed if $sgn-cl(f_{pu}(F,A))) \subseteq f_{pu}(s-cl(f_{pu}(F,A)))$ for every soft set $(F,A)$ of a soft topological space $(X, \pi, \tau)$.

Proof: Suppose that $f$ is soft $\pi gr$-closed and $(F,A)$ is any soft set of $(X, \pi, \tau)$. We have, $f_{pu}(F,A) \subseteq f_{pu}(s-cl(f_{pu}(F,A)))$. Now, $sgn-cl(f_{pu}(F,A))) \subseteq sgn-cl(f_{pu}(s-cl(f_{pu}(F,A))))$.

Since $f_{pu}(s-cl(f_{pu}(F,A)))$ is soft $\pi gr$-closed in $(Y, \pi, \tau)$, $sgn-cl(f_{pu}(s-cl(f_{pu}(F,A))))=f_{pu}(s-cl(f_{pu}(F,A)))$ for every soft set $(F,A)$ of $(X, \pi, \tau)$. Hence $sgn-cl(f_{pu}(F,A))) \subseteq f_{pu}(s-cl(f_{pu}(F,A)))$ for every soft set $(F,A)$ of a soft topological space $(X, \pi, \tau)$.

Theorem: 4.16

A soft function $f_{pu}: SS(X) \rightarrow SS(Y)$ is soft $\pi gr$-open if $f_{pu}(s-int((F,A))) \subseteq sgr-int(f_{pu}(F,A)))$ for every soft set $(F,A)$ of a soft topological space $(X, \pi, \tau)$.

Proof: Let $f_{pu}: SS(X) \rightarrow SS(Y)$ be soft $\pi gr$-open map and $(F,A)$ be any soft set of $(X, \pi, \tau)$. Then $s-int((F,A))$ is soft open in $(X, \pi, \tau)$. Since $f$ is soft $\pi gr$-open map, $f_{pu}(s-int((F,A)))$ is soft $\pi gr$-open in $(Y, \pi, \tau)$. We have $f_{pu}(s-int((F,A))) \subseteq f_{pu}(F,A))$ for every soft set $(F,A)$ of a soft topological space $(X, \pi, \tau)$. Also, $f_{pu}(s-int((F,A))) = sgr-int(f_{pu}(s-int((F,A))))$ Hence $f_{pu}(s-int((F,A))) \subseteq sgr-int(f_{pu}(F,A)))$.

Theorem: 4.17

Let $(X, \pi)$ be a soft $\pi gr$-$T_{1/2}$-space $f_{pu}: SS(X) \rightarrow SS(Y)$ be a soft function. Then $f_{pu}$ is soft $\pi gr$-continuous iff $f_{pu}$ is soft regular continuous.

Proof: Let $f_{pu}$ be a soft $\pi gr$-continuous function. Then $f^{-1}_{pu}(G,B))$ is soft $\pi gr$-closed in $(X, \pi)$ for every soft closed set $(G,B)$ of $(Y, \pi, \tau)$. Since $(X, \pi, \tau)$ is a soft $\pi gr$-$T_{1/2}$-space, every soft $\pi gr$-closed set is soft regular closed. Hence $f^{-1}_{pu}(G,B)$ is soft regular closed in $(X, \pi)$ for every soft closed set $(G,B)$ of $(Y, \pi, \tau)$ and hence $f_{pu}$ is soft regular continuous.

Let $f_{pu}$ be a soft regular continuous function in $(X, \pi)$. Then $f^{-1}_{pu}(G,B)$ is soft regular closed in $(X, \pi)$ for every soft closed set $(G,B)$ in $(Y, \pi, \tau)$. Since every soft regular closed set is soft $\pi gr$-closed. Then $f^{-1}_{pu}(G,B)$ is soft $\pi gr$-closed in $(X, \pi)$ for every soft closed set $(G,B)$ of $(Y, \pi, \tau)$ and hence $f_{pu}$ is soft $\pi gr$-continuous.

Theorem: 4.18

A soft function $f_{pu}: SS(X) \rightarrow SS(Y)$ is soft $\pi gr$-irresolute, then

(i) $f_{pu}(sgn-cl(F,A)) \subseteq sgr-cl(f_{pu}(F,A)))$ for every soft set $(F,A)$ of $(X, \pi, \tau)$.  

(ii) $sgn-cl(f_{pu}(G,B))) \subseteq f^{-1}_{pu}(sgr-cl(G,B))$ for every soft set $(G,B)$ of $(Y, \pi, \tau)$.

Proof: (i) For every soft set $(F,A)$ of $(X, \pi, \tau)$, $sgn-cl(f_{pu}(F,A)))$ is soft $\pi gr$-closed in $(Y, \pi, \tau)$. By hypothesis , $f^{-1}_{pu}(sgn-cl(f_{pu}(F,A)))$ is soft $\pi gr$-closed in $(X, \pi, \tau)$. Also , $(F,A) = f^{-1}_{pu}(f_{pu}(F,A))) \subseteq f^{-1}_{pu}(sgr-cl(f_{pu}(F,A))))$. By the definition of soft $\pi gr$-closure, we have $sgn-cl(F,A) \subseteq f^{-1}_{pu}(sgr-cl(F,A))$. Hence, we get $f_{pu}(sgn-cl(F,A)) \subseteq sgr-cl(f_{pu}(F,A)))$.

(ii) Since $sgn-cl(G,B)$ is soft $\pi gr$-closed in $Y$ and so by hypothesis , $f^{-1}_{pu}(sgn-cl(G,B))$ is soft $\pi gr$-closed in $(X, \pi, \tau)$. Since $f^{-1}_{pu}(G,B)) \subseteq f^{-1}_{pu}(sgr-cl(G,B)))$, it follows that $sgn-cl(f^{-1}_{pu}(G,B)) \subseteq f^{-1}_{pu}(sgr-cl(G,B))$.  


REFERENCES