On Pathos Adjacency Line Graph Of A Regular Binary Tree

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Abstract

In this communication, the concept of Pathos Adjacency Line Graph PAL(RT) of a Regular Binary tree T is introduced. Its study is concentrated only on trees. We decompose PAL(RT) into an edge disjoint complete bipartite subgraphs and then give the reconstruction of T. We also present a characterization of those graphs whose PAL(RT) are planar, outerplanar, maximal outerplanar, minimally non-outerplanar and Eulerian.

Keywords: *Line graph; Regular Binary Tree; Planar; Outerplanar; Minimally non-outerplanar; Inner Vertex Number i(G).*

1. Introduction:

The line graph of a graph G, denoted by *L(G)*, is the graph whose vertices are the edges of G with two vertices of *L(G)* are adjacent whenever the corresponding edges of G are adjacent. The concept of pathos of a graph G was introduced by Harary [1], as a collection of minimum number of edge disjoint open paths whose union is *G*. The path number of a graph *G* is the number of paths in pathos. A Regular binary tree *T* is a binary tree in which the following conditions hold: a) There is exactly one vertex of degree two, namely the root, b) All vertices other than root have degree one or three [2]. The order (size) of *T* is the number of vertices (edges) in it. The path number of *T* is equal to β, where 2β is the number of odd degree vertices of *T*. The edge degree of an edge *uv* of *T* is the sum of the degrees of *u, v* [3]. A graph *G* is planar if it can be drawn in the plane in such a way that any intersection of two distinct edges occurs only at a vertex of the graphs. A graph *G* is called outerplanar if *G* has an embedding in the plane in such a way that each vertex bounds the infinite face. An outerplanar graph *G* is maximal outerplanar if no edge can be added without losing its outer planarity. If *G* is a planar graph, then the inner vertex number $i(G)$ of a graph *G* is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of *G* in the plane. A graph *G* is said to be minimally non-outerplanar if $i(G) = 1$[4].

The Graph Valued function is defined as follows:

The pathos adjacency line graph *PAL(RT)* of a regular binary tree *T* is defined as a graph in which;

a) *V(PAL(RT))* is the union of the set of edges and path of pathos of *T* in which two vertices are adjacent if and only if the corresponding edges of

T are adjacent and edge lies on the corresponding path P_i of pathos.

b) With reference to root r of T, the pathos vertex $P_m(v_i, v_j)$ is adjacent to $P_n(v_k, v_l)$ in $PAL(RT)$ if and only if the pathos P_m and P_n have the common vertex v_c in T such that there exists an edge between v_i and v_l through v_c.

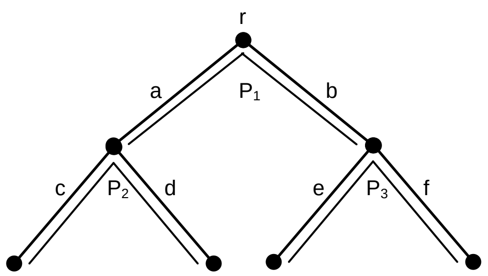

Figure 1. Regular Binary Tree T

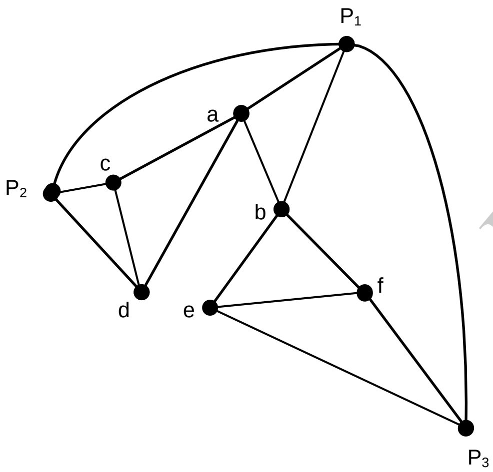

Figure 2. PAL(RT)

We need the following results to prove further results:

Theorem [A] [2]: A connected graph G is Eulerian if and only if each vertex in G has an even degree.

Theorem [B] [5]: A graph G is a non-empty path if and only if it is a connected graph with $p \geq 2$ vertices and $\sum_{i=1}^{p} d_i^2 - 4p + 6 = 0$.

Theorem [C] [6]: Every maximal outerplanar graph G with p vertices has (2p-3) edges.

Theorem [D] [6]: If G is a (p, q) graph whose vertices have degree d_i, then *L(G)* has q-vertices and q_L edges, where $q_L = -q + \frac{1}{2}\sum_{i=1}^{p} d_i^2$.

PAL(RT) Decomposition and Reconstruction of T:

We recall a complete bipartite graph is a bipartite graph (i.e., a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent) such that every pair of vertices in the two sets are adjacent.

We have the following cases.

Case 1: Here we decompose *L(T)* of *PAL(RT)* into an edge disjoint complete bipartite subgraphs as follows. Let $\{v_1, v_2, ...v_n\}$ are the vertices of subgraph *L(T)* of *PAL(RT).* Then two vertices $v_i, v_j (v_i \neq v_j)$ are adjacent in *PAL(RT)* decomposition if they are adjacent in *L(T)* such that they do not form a cycle in *L(T)* and the edge forming a cycle in *L(T)* is taken as a separate component in decomposition.

Case 2: The pathos vertex corresponding to the pathos of *T* and vertices corresponding to edges of T are adjacent in *PAL(RT)* decomposition if they are adjacent in *PAL(RT).*

Case 3: Two pathos vertices P_i, P_j $(P_i \neq P_j)$ corresponding to the pathos of *T* are adjacent in *PAL(RT)* decomposition if they are adjacent in *PAL(RT).* Hence, *PAL(RT)* consists of mutually edge disjoint complete bipartite subgraphs.

Conversely, let T' be the graph of the type which is described above. We can construct the regular binary tree *T* which has T' as its pathos adjacency line graph as follows.

We first consider the line graph *L(T)* decomposition of *PAL(RT).* Let the vertices $\{v_1', v_2', ...v_n'\}$ of *L(T)* decomposition are the edges in T. Then if two vertices $v_i', v_j' (v_i' \neq v_j')$ are adjacent in decomposition, then the corresponding edges v_i', v_j' are adjacent in *T.* We finally consider *PAL(RT)* decomposition components $L_i's$ having vertices corresponding to the pathos and edges of T. Then, draw the pathos in T along each of the pendant vertices corresponding to edges of T in decomposition components. Hence, we have the following theorem.

Theorem 1: *PAL(RT) is the pathos adjacency line graph of some regular binary tree T if and only if PAL(RT) can be partitioned into mutually edge disjoint complete bipartite subgraphs.*

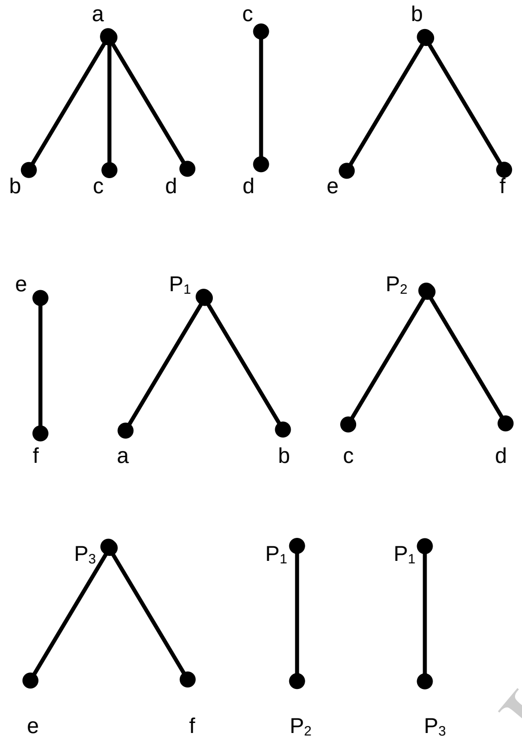

Figure 3. Decomposition of PAL(RT)

In the following theorem, we obtain the number of vertices and edges in *PAL(RT).*

Theorem 2: *If G is a (p, q) graph, where the vertices have degree d_i, then it's PAL(RT) has (q + k) vertices and* $\left|q_{PAL(RT)}\right| = \frac{1}{2}\sum_{i=1}^{p} d_i^{\,2} + (P'-1)$, *where k is being path number and* P' *is the number of pathos in T.*

Proof: By the definition, the number of vertices in $PAL(RT)$ is (q + k). The number of edges in $PAL(RT)$ is the sum of edges in *L(G),* the number of edges lies on the path P' of the pathos of G which is q and the number of adjacency of the pathos of G which is $(P'-1)$.

Hence, by Theorem [D] we have,

$$\left|q_{PAL(RT)}\right| = -q + \frac{1}{2}\sum_{i=1}^{p} d_i^{\,2} + q + (P'-1)$$

$$\Rightarrow \left|q_{PAL(RT)}\right| = \frac{1}{2}\sum_{i=1}^{p} d_i^{\,2} + (P'-1).$$

Theorem 3: *For any regular binary tree T, PAL(RT) is always planar.*

Proof: We have the following cases.

Case 1: Suppose *T* is $K_{1,2}$.i.e., one-level regular binary tree T. Then by definition, *PAL(RT)* is K_3 which is a planar.

Case 2: Consider *T* with at least $(2^n - 1)$ vertices, $n \geq 3$. i.e., level of T is at least two. Then, there exists exactly one block as K_2 and remaining $(n-1)$ number of blocks as K_3 in *L(T).* Also, T has exactly $(2^n - 1)$, $n \geq 2$ number of path of pathos. The edges joining these blocks from the pathos vertices are adjacent to at most two vertices of each block of *L(T)*. This gives a planar *PAL(RT).*

Theorem 4: *Pathos adjacency line graph PAL(RT) of a regular binary tree T is an outerplanar if and only if T is a star graph* $K_{1,2}$.

Proof: Suppose *PAL(RT)* is an outerplanar. Assume that *T* has a vertex of degree 3. Then, *T* consists of exactly four vertices of odd degree. By definition, each block of *L(T)* is either K_2 or K_3. The number of path of pathos in T is two. The edges joining these blocks from the pathos vertices are adjacent to at most two vertices of each block of *L(T)*. The adjacency of pathos vertices gives *PAL(RT)* in which $i(PAL(RT)) > 0$, a contradiction.

Conversely, suppose *T* is a star graph $K_{1,2}$ .Then, by definition it forms K_3 in *PAL(RT)*, which is an outerplanar.

Theorem 5: *Pathos adjacency line graph PAL(RT) of a regular binary tree T is maximal outerplanar if and only T is one- level regular binary tree*.

Proof: Suppose *PAL(RT)* is maximal outerplanar. Then, *PAL(RT)* is connected. Hence, *T* is connected. Let *T* be connected regular binary tree with *p = 3* vertices, *q = 2 edges* and the path number *k = 1*. Then, *PAL(RT)* has *(q + k)* vertices. A regular binary tree with three vertices is a path P_3. Also, the number of path of pathos in T is exactly one, $P'-1=0$. Hence, by Theorem [2], $|q_{PAL(RT)}| = \frac{1}{2}\sum_{i=1}^{p} d_i^{\,2}$.

Since, *PAL(RT)* is maximal outerplanar, by Theorem[C], it has $[2(q+k)-3]$edges.

$$\therefore \frac{1}{2}\sum_{i=1}^{p} d_i^{\,2} = 2(q+k)-3$$

$$\Rightarrow \frac{1}{2}\sum_{i=1}^{p} d_i^{\,2} = 2(p-1+k)-3, \text{ for a tree, } q = p - 1$$

$$\Rightarrow \sum_{i=1}^{p} d_i^{\,2} = 4(p-1+k)-6$$

$$\Rightarrow \sum_{i=1}^{p} d_i^{\,2} = 4p+4k-10.$$

For a tree which is a path, path number $k = 1$.

$$\therefore \sum_{i=1}^{p} d_i^{\,2} = 4p+4(1)-10$$

$\Rightarrow \sum_{i=1}^{p} d_i^{\,2} - 4p + 6 = 0$. Hence, by Theorem [B], *G* is a non-empty path.

Conversely, suppose T is one-level regular binary tree. Then, by definition, it forms K_3 in *PAL(RT)*, which is a maximal outerplanar. Hence, *PAL(RT)* is maximal outerplanar.

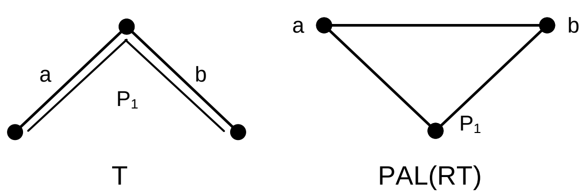

Figure 4.

Theorem 6: *For any regular binary tree T, pathos adjacency line graph PAL(RT) is not minimally non-outerplanar*.

Proof: Suppose *T* is $K_{1,2}$. Then by Theorem [4], *PAL(RT)* is an outerplanar, a contradiction. Since the maximum degree of a vertex in *T* is three, *T* has $2^n - 1$ vertices, $n \geq 3$. Then, by definition of *L(T)*, there exists exactly one block as K_2 and remaining $(n-1)$ number of blocks as K_3. Also, T has exactly $2^n - 1$, $n \geq 2$ number of path of pathos. The edges joining these blocks from the pathos vertices are adjacent to at most two vertices of each block of *L(T)*. The adjacency of pathos vertices gives *PAL(RT)* such that $i(PAL(RT)) \geq 2$, a contradiction. Hence *PAL(RT)* is not minimally non-outerplanar.

Theorem 7: *Pathos adjacency line graph PAL(RT) of a regular binary tree T is Eulerian if and only if the order of T is exactly three*.

Proof: Suppose *PAL(RT)* is Eulerian. Assume that the order of *T* is at least 5. Then, there exists exactly one block as K_2 and at least one block as K_3 in *L(T)*. Also, T has at least two path of pathos. The edges joining these blocks from the pathos vertices are adjacent to at most two vertices of each block of *L(T)*. Hence the even degree vertices of K_3 in *PAL(RT)* will be incremented by one. By Theorem [A], *PAL(RT)* is non-Eulerian.

Conversely, suppose that the order of *T* is exactly three. Then, by definition it forms K_2 in *L(T)*. The number of path of pathos in T is exactly one. The edges joining K_2 from the pathos vertex forms K_3 in *PAL(RT)*. By Theorem [A], *PAL(RT)* is Eulerian.

Conclusion: The pathos adjacency line graph is presented to and characterized for a regular binary tree T. Since the pathos vertices are adjacent in *PAL(RT)*, it is always a block. It is interesting to apply this concept to certain classes of trees.

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