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On Hankel and Topelitz Determinants for some Special Class of Analytic Functions Involving Conical Domains Defined by Subordination

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***Abstract*--- In this paper, we derive an estimation for the Hankel and Topelitz determinant with domains bounded by conical sections involving Ruscheweygh derivative. The authors sincerely hope this article will revive and encourage the other researchers to obtain similar sort of estimates for other classes connected with conical domains.**

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1 INTRODUCTION

Let $\mathcal{A}$ denote the class of all functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \ and \ |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathcal{U}$ with Montal normalization. The analytic criteria for the familiar class of starlike and convex function are as follows.

Definition 1 *Let $f(z)$ be given by (1.1). Then $f \in \mathcal{S}^*$ if and only if*

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \ (z \in \mathcal{U}). \qquad (1.2)$$

Definition 2 *Let f be given by (1.1). Then $f \in \mathcal{C}$ if and only if* $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ (z \in \mathcal{U}). \qquad (1.3)$

It follows that $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$. Further, we recall the following definitions of the familiar classes of k-uniformly convex functions and k-starlike functions as follows:

$$k-\mathcal{ST} = \left\{ \begin{array}{l} f : f \in \mathcal{S} \ and \ \Re\left(\frac{zf'(z)}{f(z)}\right) > \\ k\left|\frac{zf'(z)}{f(z)} - 1\right| \ (z \in \mathcal{U} : k \geq 0) \end{array} \right\}, \qquad (1.4)$$

$$k-\mathcal{UCV} = \left\{ \begin{array}{l} f : f \in \mathcal{S} \ and \ \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \\ k\left|\frac{zf''(z)}{f'(z)}\right| \ (z \in \mathcal{U} : k \geq 0) \end{array} \right\}. \qquad (1.5)$$

The function classes $k-\mathcal{ST}$ and $k-\mathcal{UCV}$ were introduced and investigated by Kanas and Wisniowska [17, 18] respectively (see the work [15] also). For a fixed $k \geq 0$, the class $k-\mathcal{UCV}$ is defined purely geometrically as a subclass of univalent functions which map the intersection of $\mathcal{U}$ with any disk centered at the point $z = \zeta(|\zeta| < k)$ onto a convex domain. In the case when $k = 0$ inequality (1.4) and (1.5) reduces to the well known class of starlike [6] and convex functions respectively. When $k = 1$ the inequality (1.4) the class $\mathcal{UCV}$ introduced by Goodman [5, 6] and studied extensively by Rønning [33] and independently by Ma and Minda [26, 27]. The class $k-\mathcal{ST}$ is related to the class $k-\mathcal{UCV}$ by means of the well-known Alexander transformation between the usual classes of convex and starlike functions (see the works in [16]-[18], [26, 33]). Some more interesting developments involving the classes $k-\mathcal{UCV}$ and $k-\mathcal{ST}$ were presented by Lecko and Wisniowska [23], Kanas [10]-[14] and also other [1, 28, 29, 35] (one can also refer to [2], [37] and [38] for some more related works). Very recently, a system investigation of a class of functions with q-differential operator involving conical domain was done by Kanas and Raducanu [19].

By the familiar principle of differential subordination between analytic functions $f(z)$ and $g(z)$ in $\mathcal{U}$, we say that $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$ if there exists an analytic function $w(z)$ satisfying the following conditions: $w(0)=0$ *and* $|w(z)|<1$ $(z\in U)$, such that $f(z)=g(w(z))$ $(z\in U)$. We denote this subordination by $f(z)\prec g(z)$ $(z\in U)$. In particular, if $g(z)$ is univalent in U, then it is known that

$$f(z)\prec g(z)\ (z\in U)\Leftrightarrow f(0)=g(0)$$

$$\text{and}\ \ f(U)\subset g(U)$$

.

Kanas[10]-[18] introduced and studied the different concepts using conical region. For $0\le k<\infty$ defined over the domain Ω_k as follows:

$$\Omega_k=\{u+iv:u^2>k^2(u-1)^2+k^2v^2\} \qquad (1.6)$$

which maps U onto the conic domain Ω_k. The explicit form of the extremal function that maps U onto the conic domain $\Omega_{k,\eta}$ is given by We note that the explicit form of function $\Omega_k(z)$.

$$p_0(z)=\frac{1+z}{1-z}=1+2z+2z^2+2z^3+2z^4+\dots\ (z\in U),\ k=0.$$

$$p_1(z)=1+\frac{2}{\pi^2}\log^2\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\ (z\in U),\ k=1.$$

$$=1+\frac{8}{\pi^2}z+\frac{16}{3\pi^2}z^2+\frac{184}{45\pi^2}z^3+\dots.$$

$$p_k(z)=1+\frac{2}{1-k^2}\sinh^2\left(A(k)\ arc\tanh\sqrt{z}\right)(z\in U)\ ,0<k<1$$

$$=\frac{1}{1-k^2}\cos\left\{A(k)i\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\}-\frac{k^2}{1-k^2}$$

$$=1+\frac{1}{1-k^2}\sum_{n=1}^{\infty}\left[\sum_{l=1}^{2n}2^l\binom{A}{l}\binom{2n-1}{2n-l}\right]z^n$$

where $A=A(k)=\frac{2}{\pi}\arccos k$.

$$=1+\frac{2A^2}{1-k^2}z+\frac{4A^2+2A^4}{3(1-k^2)}z^2$$

$$+\frac{\frac{46A^2}{15}+\frac{8A^4}{3}+\frac{4A^6}{15}}{3(1-k^2)}z^3+\dots$$

with $u(z)=\frac{z-\sqrt{\kappa}}{1-\sqrt{\kappa}z}$

where $K(\kappa)$ dentes the Legendre's complete elliptic integral of the first kind, and $K'(k)$ is the complementary integral of $K(\kappa)$. and $\kappa\in(0,1)$ is chosen such that $k=\cosh\frac{\pi K'(\kappa)}{4K(\kappa)}$.which maps the unit disk U onto the conic domains are respectively for $0<k<1$,

$$\Omega_k=\left\{u+iv:\left(\frac{u+\frac{k^2}{1-k^2}}{\frac{k}{1-k^2}}\right)^2-\left(\frac{v}{\frac{1}{\sqrt{1-k^2}}}\right)^2>1\right\},$$

for $k>1$,

$$\Omega_k=\left\{u+iv:\left(\frac{u+\frac{k^2}{k^2-1}}{\frac{k}{k^2-1}}\right)^2+\left(\frac{v}{\frac{1}{\sqrt{k^2-1}}}\right)^2<1\right\}.$$

By virtue of

$$p(z)=\frac{zf'(z)}{f(z)}\prec p_k(z) or\ p(z)=1+\frac{zf''(z)}{f'(z)}\prec p_k(z)$$

and the properties of the domains, we have

$$\Re(p(z))>\Re(p_k(z))>\frac{k}{k+1}$$

The q^{th} Hankel determinant for $q\ge1$ and $n\ge0$ is stated by Noonan and Thomas [30] as

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & . & . & . & a_{n+q+1}\\ a_{n+1} & . & . & . & . & .\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ a_{n+q-1} & . & . & . & . & a_{n+2q-2}\end{vmatrix}.$$

This determinant has also been considered by several authors, for example, Noor [31] determined the rate of growth of $H_q(n)$ as $n\to\infty$ for functions $f(z)$ given by (1.1) with bounded boundary. Ehrenborg in [4] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties was discussed by Layman [22]. Easily, one can observe that the Fekete and Szegö functional $|a_3-\mu a_2^2|$ can be represented in terms of Hankel determinant as $H_2(1)$. Fekete and Szegö the estimate $|a_3-\mu a_2^2|$ where μ is real and $f\in S$.

$H_2(2)=\begin{vmatrix} a_2 & a_3\\ a_3 & a_4\end{vmatrix}$. Janteng et al.[9] had established on the second Hankel determinant for functions $f(z)$

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belongs to S^* and C . In this paper, we will follow the same procedure or method used by them in finding $|a_2a_4 - a_3^2|$ for the domains bounded by conic sections. and define the symmetric Topelitz determinant $T_q(n)$ as follows:

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & & \vdots \\ \vdots & & & \\ a_{n+q-1} & \cdots & & a_n \end{vmatrix}.$$

That is, for example

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \qquad T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}$$

$$T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

For $f \in S$, the problem of finding the best possible bounds for $\| a_{n+1} | - | a_n \|$ has a long history [3]. It is known fact from [3], for a constant c, that $\| a_{n+1} | - | a_n \| \leq c$. However, finding exact values of the constant c for S and its subclasses has proved difficult. It is very trivial from the definition that finding estimates for $T_n(q)$ is related to finding bounds for $A(n) := | a_{n+1} - a_n |$ and the best possible upper bound obtainable for $A(n)$ is $2n+1$ which is for the function $k(z) = \dfrac{z}{(1-z)^2}$. Therefore, obtaining bounds for $A(n)$ is different to finding bounds for $\| a_{n+1} | - | a_n \|$. In a very recent investigation, some sharp estimates for $T_n(q)$ for low values of n and q involving symmetric Topelitz determinants whose entries are the coefficients a_n of starlike and close-to-convex functions are obtained by Thomas and Halim [8]. For $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=1}^{\infty} b_n z^n,$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) = z + \sum_{n=1}^{\infty} a_n b_n z^n.$$

Let $n \in \mathsf{N}_0 = 0,1,2,\ldots$. The Ruscheweyh derivative [34] of the n^{th} order of $f(z)$, denoted by $D^n f(z)$, is defined by

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z)$$
$$= z + \sum_{k=2}^{\infty} \frac{\Gamma(n+k)}{\Gamma(n+1)(k-1)!} a_k z^k. \qquad (1.7)$$

The Ruscheweyh derivative gave an impulse for various generalisation of well known classes of functions. The class R_n was studied by Kanas and Yaguchi [20] and Singh and Singh [36], which is given by the following definition

$$\Re\left(\frac{z(D^n f(z))'}{D^n f(z)} \right) > 0, \ z \in \mathsf{U}. \qquad (1.8)$$

We note that $R_0 = \mathsf{S}^*$ and $R_1 = \mathsf{C}$.

In this paper we derive the Hankel determinant and Topelitz matrices for the class R_n.

2 Preliminaries

The following lemmas will be required in our investigation. Let P be the family of all functions p analytic in U for which $\Re(p(z)) > 0$ for $z \in \mathsf{U}$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + \ldots. \qquad (2.1)$$

Lemma 1 *[21] Let the function w in the Schwarz function is given by*

$$w(z) = w_1 z + w_2 z^2 + \ldots, \ z \in \mathsf{U}$$

Then for every complex number s,

$$| w_2 - s w_1^2 | \leq 1 + (| s | - 1) | w_1 |^2$$

and $| w_3 - s w_2^2 | \leq max\left\{ \dfrac{1}{3}, | s - 1 | \right\}.$

Lemma 2 *[32] If $p \in P$ then $| c_k | \leq 2$ for each k.*

Lemma 3 *[7] The power series for $p(z)$ give (2.1) in U to a function in p if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & . & . & . & c_n \\ c_{-1} & 2 & c_1 & . & . & . & c_{n-1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ c_{-n} & c_{-n} & c_{-n} & . & . & . & 2 \end{vmatrix}$$

and $c_{-k} = \bar{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{n} \rho_k p_0(e^{it_k} z)$, $\rho_k > 0$, $t_k \neq t_j$, for $k \neq j$; in this case $D_n > 0$ for $n < m-1$ and $D_n = 0$ for $n \geq m$.

Lemma 4 [25] *Let the function* $p \in P$ *be given by the power series (2.1), then*

$$2c_2 = c_1^2 + x(4-c_1^2) \tag{2.2}$$

for some x, $|x| \leq 1$, and

$$4c_3 = c_1^3 + 2(4-c_1^2)c_1 x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z \tag{2.3}$$

for some z, $|z| \leq 1$.

3 Main Results

Theorem 1 *Let* $0 \leq k \leq 1$ *and if the function* $f(z)$ *given by (1.1) be in the class* R_n. *then*

$$|a_2 a_4 - a_3^2| \leq \begin{cases} \dfrac{P_1^2}{4}, & n = 0; \\ \dfrac{[8N + P_1^2(A(n) - B(n))]^2}{2P_1^2 A(n) - P_1^2 B(n) - 16(M-N)} + B(n)P_1^2, & n \geq 1, \end{cases}$$

where P_1, P_2 and P_3 are the coefficients of $p_k(z) = 1 + P_1 z + P_2 z^2 + P_3 z^3 + \ldots$.

$$M = A(n)\left(\frac{3P_1^3}{16} + \frac{P_1^4}{16} - \frac{3P_1^2}{16} + \frac{P_1 P_3}{8} + \frac{3P_1^2 P_2}{16}\right) - B(n)\left(\frac{P_2^2}{16} + \frac{P_1^4}{16} + \frac{P_1^2 P_2}{8}\right)$$

$$N = A(n)\left(\frac{P_1 P_2}{4} + \frac{3P_1^3}{16}\right) + B(n)\left(\frac{-P_1^2}{8} + \frac{P_1 P_2}{8} + \frac{P_1^3}{8}\right)$$

$$A(n) = \frac{1}{(n+1)^2(n+2)(n+3)}$$ and

$$B(n) = \frac{1}{(n+1)^2(n+2)^2}.$$

Proof. Let us consider a function $q(z)$ given by

$$q(z) = \frac{z(D^n f(z))'}{D^n f(z)} \quad (z \in \mathrm{U}).$$

Then, for $f \in R_n$, we have the following subordination:

$$q(z) \prec p_k(z) \quad (z \in \mathrm{U}). \tag{3.1}$$

where

$$p_k(z) = 1 + P_1 z + P_2 z^2 + \cdots.$$

Using the subordination relation (3.1), we see that the function $h(z)$ given by

$$p(z) = \frac{1 + p_k^{-1}(q(z))}{1 - p_k^{-1}(q(z))} = 1 + c_1 z + c_2 z^2 + \ldots$$

is analytic and has positive real part in the open unit disk U. We also have

$$q(z) = p_k\left(\frac{p(z)-1}{p(z)+1}\right) \quad (z \in \mathrm{U}).$$

$$z(D^n f(z))' = D^n f(z) \quad p_k\left(\frac{p(z)-1}{p(z)+1}\right) \tag{3.2}$$

$$z + 2(n+1)a_2 z^2 + 3\frac{(n+1)(n+2)}{2!}a_3 z^3 + \ldots$$
$$= 1 + \frac{P_1 c_1}{2} z + \left(\frac{P_1 c_2}{2} - \frac{P_1 c_1^2}{4} + \frac{P_2 c_1^2}{4}\right) z^2 + \left(\frac{P_1 c_1^3}{8} - \frac{P_1 c_1 c_2}{2} + \frac{P_1 c_3}{2} + \frac{P_2 c_1 c_2}{2} - \frac{P_2 c_1^3}{4} + \frac{P_3 c_1^3}{8}\right) z^3 + \ldots \tag{3.3}$$

Equating the like terms of (3.3), we get

$$a_2 = \frac{P_1 c_1}{2(n+1)}, \tag{3.4}$$

$$a_3 = \frac{1}{(n+1)(n+2)}\left[\frac{P_1 c_2}{2} + \left(-\frac{P_1}{4} + \frac{P_2}{4} + \frac{P_1^2}{4}\right)c_1^2\right], \tag{3.5}$$

$$a_4 = \frac{P_1 c_1}{(n+1)(n+2)(n+3)}\left[P_1 c_3 + \left(P_2 - P_1 + \frac{3P_1^2}{4}\right)c_1 c_2 + \left(\frac{P_3}{4} + \frac{P_1}{4} - \frac{P_2}{2} - \frac{3P_1^2}{4} + \frac{3P_1 P_2}{8} + \frac{P_1^3}{8}\right)c_1^3\right]$$

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$$
\begin{aligned}
a_2a_4-a_3^2 &= A(n)\frac{P_1^2c_1c_3}{2} \\
&+\left[\begin{array}{l} A(n)\left(\frac{P_1P_2}{2}-\frac{P_1^2}{2}+\frac{3P_1^3}{8}\right)- \\ B(n)\left(\frac{-P_1^2}{4}+\frac{P_1P_2}{4}+\frac{P_1^3}{4}\right)\end{array}\right]c_1^2c_2 \\
&-B(n)\frac{P_1^2}{4}c_2^2+ \\
&\left[A(n)\left(\begin{array}{l}\frac{P_1P_3}{8}+\frac{P_1^2}{8}-\frac{P_1P_2}{4} \\ -\frac{3P_1^3}{16}+\frac{3P_1^2P_2}{16}+\frac{P_1^4}{16}\end{array}\right)\right. \\
&\left. -B(n)\left(\begin{array}{l}\frac{P_1^2}{16}+\frac{P_2^2}{16}-\frac{P_1^3}{8} \\ +\frac{P_1^4}{16}-\frac{P_1P_2}{8}+\frac{P_1^2P_2}{8}\end{array}\right)\right]c_1^4
\end{aligned}
\tag{3.6}
$$

where $A(n)=\dfrac{1}{(n+1)^2(n+2)(n+3)}$ and $B(n)=\dfrac{1}{(n+1)^2(n+2)^2}$.

Making use of Lemma 4 to express c_2 in terms of c_1 and for simplicity, we have taken $Y=4-c_1^2$ and $Z=(1-|x|^2)z$. Without loss of generality, let us assume that $c=c_1$, where $0\le c\le 2$. Applying triangle inequality, we get,

$$
\begin{aligned}
|a_2a_4-a_3^2| &= \left|\left\{A(n)\left(\frac{P_1^4}{16}+\frac{P_1P_3}{8}+\frac{3P_1^2P_2}{16}\right)\right.\right. \\
&\left. -B(n)\left(\frac{P_2^2}{16}+\frac{P_1^4}{16}+\frac{P_1^2P_2}{8}\right)\right\}c^4 \\
&+Y\left\{\begin{array}{l} A(n)\left(\frac{P_1P_2}{4}+\frac{3P_1^3}{16}\right)- \\ B(n)\left(\frac{P_1P_2}{8}+\frac{P_1^3}{8}\right)\end{array}\right\}c^2x \\
&-A(n)\frac{P_1^2}{8}c^2Yx^2-B(n)\frac{P_1^2}{16}Y^2x^2 \\
&\left.+A(n)\frac{P_1^2}{4}cYZ\right|
\end{aligned}
\tag{3.7}
$$

$$
\left|a_2a_4-a_3^2\right|\le\left|M\right|c^4+c^2Y|x|N
$$

$$
+c^2Y|x|^2\ A(n)\frac{P_1^2}{8}+B(n)\frac{P_1^2}{16}Y^2|x|^2+A(n)\frac{P_1^2}{4}c^2Y(1-|x|^2):=\Phi(c,|x|).
$$

where

$$
M=A(n)\left(\frac{3P_1^3}{16}+\frac{P_1^4}{16}-\frac{3P_1^2}{16}+\frac{P_1P_3}{8}+\frac{3P_1^2P_2}{16}\right)-B(n)\left(\frac{P_2^2}{16}+\frac{P_1^4}{16}+\frac{P_1^2P_2}{8}\right)
$$

$$
N=A(n)\left(\frac{P_1P_2}{4}+\frac{3P_1^3}{16}\right)+B(n)\left(\frac{P_1P_2}{8}+\frac{P_1^3}{8}\right)
$$

Trivially, $\Phi'(|x|)>0$ on $[0,1]$, and so $\Phi(|x|)\le\Phi(1)$. Hence

$$
\begin{aligned}
\left|a_2a_4-a_3^2\right| &\le\left|M\right|c^4+c^2(4-c^2)N \\
&+c^2(4-c^2)A(n)\frac{P_1^2}{8}+ \\
&B(n)\frac{P_1^2}{16}(4-c^2)^2:=G(c).
\end{aligned}
$$

$$
G(c)=\left[M-N-\frac{P_1^2}{8}A(n)+B(n)\frac{P_1^2}{16}\right]c^4+\left[4N+\frac{P_1^2}{2}A(n)-\frac{P_1^2}{2}B(n)\right]c^2+B(n)P_1^2
$$

For optimum value of $G(c)$, consider $G'(c)=0$ this implies that $c=0$ or

$$c^2 = 4\left[\frac{P_1^2(A(n)-B(n))+8N}{2P_1^2A(n)-P_1^2B(n)-16(M-N)}\right].$$

Since each of these coefficients p_k's are positive, applying the properties of $p_k(z)$, this show that the following result.

For $n=0$, G has a maximum attained at $c=0$. The upperbound for (3.7) corresponds to $|x|=1$ and $c=0$, in which case

$$|a_2a_4-a_3^2| \le B(n)P_1^2 = \frac{P_1^2}{4}.$$

For $n\ge 1$, G has a maximum attained at

$$c = 2\sqrt{\frac{8N+P_1^2(A(n)-B(n))}{2P_1^2A(n)-P_1^2B(n)-16(M-N)}}.$$

The upperbound for (3.7) corresponds to $|x|=1$ and

$$c = 2\sqrt{\frac{8N+P_1^2(A(n)-B(n))}{2P_1^2A(n)-P_1^2B(n)-16(M-N)}},$$

in which case

$$|a_2a_4-a_3^2| \le \frac{[8N+P_1^2(A(n)-B(n))]^2}{2P_1^2A(n)-P_1^2B(n)-16(M-N)} + B(n)P_1^2.$$

which completes the proof.

For the choices of $k=0$, and $n=0$, Theorem 1 reduces to a result in [9] and [24]:

Remark 1 *If* $f\in S^*$, *then* $|a_2a_4-a_3^2|\le 1$.

If $k=1$, and $n=0$, then $P_1=\frac{8}{\pi^2}$, $P_2=\frac{16}{3\pi^2}$ and $P_3=\frac{184}{45\pi^2}$ and then we get the following result:

Corollary 1 *If* $f\in SP$, *then* $|a_2a_4-a_3^2|\le\frac{16}{\pi^4}$.

If $0<k<1$, and $n=0$, then $P_1=\frac{2A^2(k)}{1-k^2}$, $P_2=\frac{4A^2(k)+2A^4(k)}{3(1-k^2)}$, and $P_3=\frac{46A^2(k)+40A^4(k)+4A^6(k)}{45(1-k^2)}$ and then we get the following result:

Corollary 2 *If* $f\in k-ST$, *then*

$$|a_2a_4-a_3^2| \le \frac{A^4(k)}{(1-k^2)^2}.$$

If $k=0$, and $n=1$, then $P_1=P_2=P_3=2$. Theorem 1 reduces to a result in [9]:

Theorem 2 *Let* $0\le k\le 1$ *and if the function* $f(z)$ *given by (1.1) be in the class* R_n. *then*

$$|a_3^2-a_2^2| \le \begin{cases} \frac{P_1^2}{4}, & n=0; \\ B(n)P_1^2-\frac{S^2}{4R}, & n\ge 1, \end{cases}$$

where P_1, P_2 and P_3 are the coefficients of $P_k(z)=1+P_1z+P_2z^2+P_3z^3+\ldots$.

$$R = B(n)\left(\frac{-P_1P_2}{8}+\frac{P_1^2}{16}+\frac{P_2^2}{16}-\frac{P_1^3}{8}+\frac{P_1^4}{16}+\frac{P_1^2P_2}{8}\right)$$

$$S = B(n)\left(\frac{-P_1^2}{2}+\frac{P_1P_2}{2}+\frac{P_1^3}{2}-(n+2)^2\frac{P_1^2}{4}\right)$$

and $B(n)=\frac{1}{(n+1)^2(n+2)^2}$.

Proof. From the equations (3.4) and (3.5), we get

$$\begin{aligned} a_3^2-a_2^2 &= B(n)\frac{P_1^2}{4}c_2^2-\frac{P_1^2c_1^2}{4(n+1)^2} \\ &+B(n)\left[\frac{-P_1^2}{4}+\frac{P_1P_2}{4}+\frac{P_1^3}{4}\right]c_1^2c_2 \\ &+B(n)\left(\frac{P_1^2}{16}+\frac{P_2^2}{16}-\frac{P_1^3}{8}-\frac{P_1P_2}{8}+\frac{P_1^4}{16}+\frac{P_1^2P_2}{8}\right)c_1^4 \end{aligned} \tag{3.8}$$

where $B(n)=\frac{1}{(n+1)^2(n+2)^2}$.

Making use of Lemma 4 to express c_2 in terms of c_1 and for simplicity, we have taken $Y=4-c_1^2$ and $Z=(1-|x|^2)z$. Without loss of generality, let us assume that $c=c_1$, where $0\le c\le 2$. Applying triangle inequality, we get,

$$\begin{aligned}|a_3^2-a_2^2| &= \Bigg| B(n)\left[\frac{P_1^2P_2}{8}+\frac{P_2^2}{16}+\frac{P_1^4}{16}\right]c^4 \\ &+B(n)Y\left[\frac{P_1P_2}{8}+\frac{P_1^3}{8}\right]c^2x \\ &+Y^2B(n)\frac{P_1^2}{16}x^2-\frac{P_1^2c^2}{4(n+1)^2}\Bigg|\end{aligned} \quad (3.9)$$

$$\begin{aligned}|a_3^2-a_2^2| &\le \Bigg| B(n)\left[\frac{P_1^2P_2}{8}+\frac{P_2^2}{16}+\frac{P_1^4}{16}\right]c^4-\frac{P_1^2c^2}{4(n+1)^2}\Bigg| \\ &+B(n)Y\left[\frac{P_1P_2}{8}+\frac{P_1^3}{8}\right]c^2|x| \\ &+Y^2B(n)\frac{P_1^2}{16}|x|^2\end{aligned}$$

Trivially, $\Phi'(|x|)>0$ on $[0,1]$, and so $\Phi(|x|)\le\Phi(1)$. Hence

$$\begin{aligned}|a_3^2-a_2^2| &\le \Bigg| B(n)\left[\frac{P_1^2P_2}{8}+\frac{P_2^2}{16}+\frac{P_1^4}{16}\right]c^4-\frac{P_1^2c^2}{4(n+1)^2}\Bigg| \\ &+B(n)Y\left[\frac{P_1P_2}{8}+\frac{P_1^3}{8}\right]c^2 \\ &+Y^2B(n)\frac{P_1^2}{16}:=G(c)\end{aligned}$$

$$G(c)=B(n)\left(\frac{P_1^2P_2}{8}+\frac{P_2^2}{16}+\frac{P_1^4}{16}-\frac{P_1P_2}{8}-\frac{P_1^3}{8}+\frac{P_1^2}{16}\right)c^4+$$
$$B(n)\left(\frac{P_1P_2}{2}+\frac{P_1^3}{2}-\frac{P_1^2}{2}-(n+1)^2\frac{P_1^2}{2}\right)c^2+B(n)P_1^2$$

$$G(c)=Rc^4+Sc^2+B(n)P_1^2.$$
$$G'(c)=4Rc^3+2Sc$$

where

$$R=B(n)\left[\frac{P_1^2P_2}{8}+\frac{P_2^2}{16}+\frac{P_1^4}{16}-\frac{P_1P_2}{8}-\frac{P_1^3}{8}+\frac{P_1^2}{16}\right]$$

$$S=B(n)\left[\frac{P_1P_2}{2}+\frac{P_1^3}{2}-\frac{P_1^2}{2}-(n+2)^2\frac{P_1^2}{4}\right].$$

Now $G'(c)=0$ implies $c=0$ or $c^2=\frac{-S}{2R}$

Since each of these coefficients p_k's are positive, applying the properties of $p_k(z)$, this show that the following result.

For $n=0$, G has a maximum attained at $c=0$. The upperbound for (3.9) corresponds to $|x|=1$ and $c=0$, in which case

$$|a_3^2-a_2^2|\le B(n)P_1^2=\frac{P_1^2}{4}.$$

For $n\ge1$, G has a maximum attained at $c^2=\frac{-S}{2R}$. The upperbound for (3.9) corresponds to $|x|=1$ and $c^2=\frac{-S}{2R}$, in which case

$$|a_3^2-a_2^2|\le B(n)P_1^2-\frac{S^2}{4R}$$

which completes the proof.

If $k=0$, and $n=0$, then we get the following result.

Corollary 3 *If* $f\in S^*$, *then* $|a_3^2-a_2^2|\le1.$

If $k=1$, and $n=0$, then $P_1=\frac{8}{\pi^2}$, $P_2=\frac{16}{3\pi^2}$ and $P_3=\frac{184}{45\pi^2}$ and then we get the following result:

Corollary 4 *If* $f\in SP$, *then* $|a_3^2-a_2^2|\le\frac{16}{\pi^4}$.

Theorem 3 *Let* $0\le k\le1$ *and if the function* $f(z)$ *given by (1.1) be in the class* R_n. *then*

$$|1+2a_2^2(a_3-1)-a_3^2|\le\begin{cases}1+\frac{P_1^2}{4}, & n=0;\\ 1+B(n)P_1^2-\frac{W^2}{4V}, & n\ge1,\end{cases}$$

where P_1, P_2 and P_3 are the coefficients of $P_k(z)=1+P_1z+P_2z^2+P_3z^3+\dots$.

$$V=C(n)\left(\frac{P_1^2P_2}{8}+\frac{P_1^4}{8}\right)-B(n)\left(\frac{P_2^2}{16}+\frac{P_1^4}{16}+\frac{P_1^2P_2}{8}-\frac{P_1^2}{16}+\frac{P_1P_2}{8}+\frac{P_1^3}{8}\right)$$
$$-(n+2)\frac{P_1^3}{8}$$

$$W=B(n)\left(\frac{-P_1^2}{2}+\frac{P_1P_2}{2}+\frac{P_1^3}{2}+(n+2)\frac{P_1^2}{2}-(n+2)^2\frac{P_1^2}{2}\right)$$

$$C(n)=\frac{1}{(n+1)^2(n+2)}$$ and

$$B(n)=\frac{1}{(n+1)^2(n+2)^2}$$

Proof. From the equations (3.4) and (3.5), we get

$$1+2a_2^2(a_3-1)-a_3^2=1+C(n)\left[\left(\frac{-P_1^3}{8}+\frac{P_2P_1^2}{8}+\frac{P_1^4}{8}\right.\right.$$
$$\left.\left.-\left(\frac{n+1}{n+2}\right)\left(\frac{P_1^2}{16}+\frac{P_2^2}{16}+\frac{P_1^4}{16}-\frac{P_1P_2}{8}-\frac{P_1^3}{8}+\frac{P_2P_1^2}{8}\right)\right)c_1^4\right]$$
$$+C(n)\left[\frac{P_1^3}{4}-\left(\frac{n+1}{n+2}\right)\left(\frac{-P_1^2}{4}+\frac{P_1P_2}{4}+\frac{P_1^3}{4}\right)\right]c_1^2c_2$$
$$-B(n)\frac{P_1^2c_2^2}{4}-\frac{P_1^2c_1^2}{2(n+1)^2},$$

where $C(n)=\dfrac{1}{(n+1)^3(n+2)}$.

Making use of Lemma 4 to express c_2 in terms of c_1 and for simplicity, we have taken $Y=4-c_1^2$. Without loss of generality, let us assume that $c=c_1$, where $0\le c\le 2$. Applying triangle inequality, we get,

$$|1+2a_2^2(a_3-1)-a_3^2|=$$
$$\left|1+C(n)\left(\frac{P_1^2P_2}{8}+\frac{P_1^4}{8}-\left(\frac{n+1}{n+2}\right)\left(\frac{P_2^2}{16}+\frac{P_1^4}{16}+\frac{P_1^2P_2}{8}\right)\right)c^4\right.$$
$$+C(n)\left[\frac{P_1^3}{8}-\left(\frac{n+1}{n+2}\right)\left(\frac{P_1^3}{8}+\frac{P_1P_2}{8}\right)\right]Yc^2x$$
$$\left.-B(n)\frac{P_1^2}{16}x^2Y^2-\frac{P_1^2}{2(n+1)^2}c^2\right| \tag{3.10}$$

$$|1+2a_2^2(a_3-1)-a_3^2|=\left|1+C(n)\left(\frac{P_1^2P_2}{8}+\frac{P_1^4}{8}\right.\right.$$
$$\left.\left.-\left(\frac{n+1}{n+2}\right)\left(\frac{P_2^2}{16}+\frac{P_1^4}{16}+\frac{P_1^2P_2}{8}\right)\right)c^4-\frac{P_1^2}{2(n+1)^2}c^2\right|$$
$$+B(n)\frac{P_1^2}{16}|x|^2Y^2+$$
$$C(n)Y\left(\frac{P_1^3}{8}+\left(\frac{n+1}{n+2}\right)\left(\frac{P_1P_2}{8}+\frac{P_1^3}{8}\right)\right)c^2|x|$$

Trivially, $\Phi'(|x|)>0$ on $[0,1]$, and so $\Phi(|x|)\le\Phi(1)$. Hence

$$|1+2a_2^2(a_3-1)-a_3^2|=\left|1+C(n)\left(\frac{P_1^2P_2}{8}+\frac{P_1^4}{8}\right.\right.$$
$$\left.\left.-\left(\frac{n+1}{n+2}\right)\left(\frac{P_2^2}{16}+\frac{P_1^4}{16}+\frac{P_1^2P_2}{8}\right)\right)c^4-\frac{P_1^2}{2(n+1)^2}c^2\right|$$
$$+B(n)\frac{P_1^2}{16}Y^2+$$
$$C(n)Y\left(\frac{P_1^3}{8}+\left(\frac{n+1}{n+2}\right)\left(\frac{P_1P_2}{8}+\frac{P_1^3}{8}\right)\right)c^2$$

$$G(c)=\;1+B(n)P_1^2+\left[C(n)\left(\frac{P_1^2P_2}{8}-\frac{P_1^3}{8}+\frac{P_1^4}{8}\right)\right.$$
$$\left.-\left(\frac{n+1}{n+2}\right)\left(\frac{P_2^2}{16}+\frac{P_1^4}{16}+\frac{P_1^2P_2}{8}-\frac{P_1^3}{8}-\frac{P_1P_2}{8}-\frac{P_1^2}{16}\right)\right]c^4$$
$$+C(n)\left(\frac{P_1^3}{2}-\left(\frac{n+1}{n+2}\right)\left(\frac{P_1P_2}{2}+\frac{P_1^3}{2}+\frac{P_1^2}{2}\right)\right)c^2$$

$$G(c)=1+Vc^4+Wc^2+B(n)P_1^2$$

$$G'(c)=4Vc^3+2Wc$$

where

$$\mathrm{V}=\mathrm{C(n)}\left(\frac{P_1^2P_2}{8}-\frac{P_1^3}{8}+\frac{P_1^4}{8}\right)-$$
$$\frac{(\mathrm{n}+1)}{(\mathrm{n}+2)}\left(\frac{P_2^2}{16}+\frac{P_1^4}{16}+\frac{P_1^2P_2}{8}-\frac{P_1^3}{8}-\frac{P_1P_2}{8}-\frac{P_1^2}{8}\right)$$

$$W=C(n)\left(\frac{P_1^3}{2}-\left(\frac{n+1}{n+2}\right)\left(\frac{P_1P_2}{2}+\frac{P_1^3}{2}+\frac{P_1^2}{2}\right)\right)$$

Now $G'(c)=0$ implies $c=0$ or $c^2=\dfrac{-W}{2V}$

Since each of these coefficients p_k's are positive, applying the properties of $p_k(z)$, this show that the following result.

For $n=0$, G has a maximum attained at $c=0$. The upperbound for (3.10) corresponds to $|x|=1$ and $c=0$, in which case

$$|1+2a_2^2(a_3-1)-a_3^2|\le 1+B(n)P_1^2=1+\frac{P_1^2}{4}.$$

For $n\ge 1$, G has a maximum attained at $c^2=\dfrac{-W}{2V}$.

The upperbound for (3.10) corresponds to $|x|=1$ and

$c^2 = \frac{-S}{2R}$, in which case

$$|1+2a_2^2(a_3-1)-a_3^2| \le 1+B(n)P_1^2 - \frac{W^2}{4V}$$

which completes the proof.

If $k=0$, and $n=0$, then we get the following result.

Corollary 5 *If* $f \in S^*$, *then* $|1+2a_2^2(a_3-1)-a_3^2| \le 2.$

If $k=1$, and $n=0$, then $P_1 = \frac{8}{\pi^2}$, $P_2 = \frac{16}{3\pi^2}$ and $P_3 = \frac{184}{45\pi^2}$ and then we get the following result:

Corollary 6 *If* $f \in SP$, *then*

$$|1+2a_2^2(a_3-1)-a_3^2| \le 1+\frac{16}{\pi^4}.$$

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