On $\delta g^*$-Closed Sets In Bitopological Spaces

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Abstract

The aim of this paper is to introduce the concept of $(i, j)$-$\delta g^*$-closed sets in bitopological spaces and study their properties. We prove that this class lies between the class of $(i, j)$-$\delta$-closed sets and the class of $(i, j)$-$\delta g^*$-closed sets. Also we discuss some basic properties and applications of $(i, j)$-$\delta g^*$-closed sets, which defines a new class of spaces namely $(i, j)$-$\delta \tau T_{1/2}^2$-spaces, $(i, j)$-$\delta g^* T_{1/2}^2$-spaces, $(i, j)$-$\delta g^{*} T_{\delta g}^2$-spaces and $(i, j)$-$\delta g T_{\delta g}^2$-spaces.

Keywords: $(1, 2)$-$\delta g$-closed set, $(1, 2)$-$\delta$-closed set, $(1, 2)$-$\delta g^*$-closed set.

1. Introduction

A triple $(X, \tau_1, \tau_2)$ where $X$ is a non-empty set and $\tau_1$ and $\tau_2$ are topologies on $X$ is called a bitopological space and Kelly [8] initiated the study of such spaces. Njastad[12], Velicko [20] introduced the concept of $\alpha$-open sets and $\delta$-closed sets respectively. Dontchev and Ganster [4] studied $\delta$-generalized closed set in topological spaces. Levine [10] introduced generalization of closed sets and discussed their properties. In 1985, Fukutake [5] introduced the concepts of $g$-closed sets in bitopological spaces and after that several authors turned their attention towards generalizations of various concepts of topology by considering bitopological spaces. Also M. E. Abd El-Monsef [1] et al investigated $\alpha$-closed sets in topological spaces. Sheik John et al [14] introduced $g^*$-closed sets in bitopological spaces. Sudha et al. [16] introduced the concept of $\delta g^*$-closed sets in topological spaces and investigated its relationship with the other types of closed sets. The purpose of the present paper is to define a new class of closed sets called $(i, j)$-$\delta g^*$-closed sets and we discuss some basic properties of $(i, j)$-$\delta g^*$-closed sets in bitopological spaces. Applying these sets, we obtain the new spaces called $(i, j)$-$\delta gT\tau_{1/2}$-space, $(i, j)$-$\delta gT\tau_{1/2}$-space, $(i, j)$-$\delta g\tau_{\delta g}$-space and $(i, j)$-$\delta g\tau_{\delta g}$-space.

2. Preliminaries

If $A$ is a subset of $X$ with the topology $\tau$, then the closure of $A$ is denoted by $\tau$-$\text{cl}(A)$ or $\text{cl}(A)$, the interior of $A$ is denoted by $\tau$-$\text{int}(A)$ or $\text{int}(A)$ and the complement of $A$ in $X$ is denoted by $A^c$.

2.1. Definition

A subset $A$ of a topological space $(X, \tau)$ is called a

(i) semi-open set [9] if $A \subseteq \text{cl}(\text{int}(A))$,
(ii) $\alpha$-open set [12] if $A \subseteq \text{int}(\text{cl}(A))$,
(iii) regular open set [16] if $A = \text{int}(\text{cl}(A))$,
(iv) Pre-open set [11] if $A \subseteq \text{int}(\text{cl}(A))$.

The complement of a semi open (resp. $\alpha$-open, regular open, pre-open) set is called semi-closed (resp. $\alpha$-closed, regular closed, pre-closed).

The semi-closure [3] (resp. $\alpha$-closure [12], pre-closure [11]) of a subset $A$ of $(X, \tau)$, denoted by $\text{scl}(A)$ (resp. $\text{cl}_\alpha(A)$, $\text{pcl}(A)$) is defined to be the intersection of all semi-closed (resp.$\alpha$-closed, pre-closed) sets containing $A$. It is known that $\text{scl}(A)$ (resp. $\text{cl}_\alpha(A)$, $\text{pcl}(A)$) is a semi-closed (resp.$\alpha$-closed, pre-closed) set.

2.2. Definition

The $\delta$-interior [20] of a subset $A$ of $X$ is the union of all regular open sets of $X$ contained in $A$ and is denoted by $\text{int}_\delta(A)$.

The subset $A$ is called
δ-open [20] if \( A = \text{int}_\delta(A) \). i.e., a set is δ-open if it is the union of regular open sets, the complement of a δ-open is called δ-closed. Alternatively, a set \( A \subseteq X \) is called δ-closed [20] if \( A = \text{cl}_\delta(A) \), where \( \text{cl}_\delta(A) = \{ x \in X; \text{int}(\text{cl}(\{x\})) \cap A \neq \emptyset, U \in \tau \text{ and } x \in U \} \). Every δ-closed set is closed [20].

### 2.3. Definition

A subset \( A \) of \((X, \tau)\) is called

1) \( \delta \)-generalized closed (briefly \( \delta \)-g-closed) [4] if \( \text{cl}_\delta(A) \subseteq U \) whenever \( A \subseteq U \text{ and } U \) is open in \((X, \tau)\).
2) generalized closed (briefly g-closed) [10] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \text{ and } U \) is open in \((X, \tau)\).
3) \( g' \)-closed [19] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \text{ and } U \) is g-open in \((X, \tau)\).

Throughout this paper by the spaces \( X \) and \( Y \) represent non-empty bitopological spaces on which no separation axioms are assumed, unless otherwise mentioned and the integers \( i, j \in \{1, 2\} \).

For a subset \( A \) of \( X \), \( \tau_i - \text{cl}(A) \) (resp. \( \tau_i - \text{int}(A) \), \( \tau_i - \text{pcl}(A) \)) denote the closure (resp. interior, pre closure) of \( A \) with respect to the topology \( \tau_i \). We denote the family of all g-open subsets of \( X \) with respect to the topology \( \tau_i \) by \( \text{GO}(X, \tau_i) \) and the family of all \( \tau_i \)-closed sets is denoted by the symbol \( F_i \). By \( (i, j) \) we mean the pair of topologies \((\tau_i, \tau_j)\).

### 2.4. Definition

A subset \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is called

1) \( (i, j) \)-g-closed [5] if \( \tau_j - \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \text{ and } U \) is open in \( \tau_i \).
2) \( (i, j) \)-g*-closed [14] if \( \tau_j - \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \text{ and } U \) is g-open in \( \tau_i \).
3) \( (i, j) \)-rg-closed [2] if \( \tau_j - \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \text{ and } U \) is regular open in \( \tau_i \).
4) \( (i, j) \)-wg-closed [6] if \( \tau_j - \text{cl}(\tau_i - \text{int}(A)) \subseteq U \) whenever \( A \subseteq U \text{ and } U \) is open in \( \tau_i \).
5) \( (i, j) \)-gpr-closed [6] if \( \tau_j - \text{pcl}(A) \subseteq U \) whenever \( A \subseteq U \text{ and } U \) is regular open in \( \tau_i \).
6) \( (i, j) \)-ag*-closed [18] if \( \tau_j - \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \text{ and } U \) is open in \( \tau_i \).
7) \( (i, j) \)-g*p-closed [17] if \( \tau_j - \text{pcl}(A) \subseteq U \) whenever \( A \subseteq U \text{ and } U \) is g-open in \( \tau_i \).
8) \( (i, j) \)-w-closed [7] if \( \tau_j - \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \text{ and } U \) is semi-open in \( \tau_i \).
9) \( (i, j) \)-ag*-closed [13] if \( \tau_j - \text{acl}(A) \subseteq U \) whenever \( A \subseteq U \text{ and } U \) is g*-open in \( \tau_i \).

### 2.5. Definition

A bitopological space \((X, \tau_1, \tau_2)\) is called

1) \( (i, j) \)-\( T_{1/2} \)-space [5] if every \( (i, j) \)-g-closed set is \( \tau_j \)-closed.
2) \( (i, j) \)-\( T_{1/2} \)-space [14] if every \( (i, j) \)-g*-closed set is \( \tau_j \)-closed.
3) \( (i, j) \)-\( T_{1/2} \)-space [14] if every \( (i, j) \)-g-closed set is \( (i, j) \)-g*-closed.

### 3. \( (i, j) \)-\( \delta \)-g*-closed sets in bitopological spaces

In this section we introduce the concept of \( (i, j) \)-\( \delta \)-g*-closed sets in bitopological spaces and discuss the related properties.

### 3.1. Definition

A subset \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is said to be an \( (i, j) \)-\( \delta \)-g*-closed set if \( \tau_j - \text{cl}_\delta(A) \subseteq U \), whenever \( A \subseteq U \text{ and } U \in \text{GO}(X, \tau_i) \).

We denote the family of all \( (i, j) \)-\( \delta \)-g*-closed sets in \((X, \tau_1, \tau_2)\) by \( D_\delta(i, j) \).

### 3.2. Remark

By setting \( \tau_1 = \tau_2 \) in Definition 3.1., a \( (i, j) \)-\( \delta \)-g*-closed set is \( \delta \)-g*-closed.

### 3.3. Proposition

If \( A \) is \( \tau_j \)-\( \delta \)-closed subset of \((X, \tau_1, \tau_2)\), then \( A \) is \( (i, j) \)-\( \delta \)-g*-closed.
Proof: Let A be a $\tau_j$-$\delta$-closed subset of $(X, \tau_1, \tau_2)$. Then $\tau_j - \text{cl}_g(A) = A$. Let $U \in \text{GO}(X, \tau_1)$ such that $A \subseteq U$, then $\tau_j - \text{cl}_g(A) = A \subseteq U$ which implies A is (i, j) - $\delta g^*$-closed.

The converse of the above proposition is not true as seen from the following example.

3.4. Example

Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \{X, \phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$. Then the subset $\{b, c\}$ is $(1, 2) - \delta g^*$-closed but not $\tau_2 - \delta$-closed set.

3.5. Proposition

If A is both $\tau_1$-g-open and (i, j) - $\delta g^*$-closed, then A is $\tau_j - \delta$-closed.

Proof: Let A be both $\tau_1$-g-open and (i, j) - $\delta g^*$-closed. Since A is (i, j)-$\delta g^*$-closed, we have $A \subseteq U$ and $U \in \text{GO}(X, \tau_1)$ which implies $\tau_j - \text{cl}_g(A) \subseteq U$ and since A is $\tau_1$-g-open. Put $A = U$, then we have $\tau_j - \text{cl}_g(A) \subseteq A$, implies A is a $\tau_j - \delta$-closed set.

3.6. Proposition

If A is both $\tau_1$-g-open and (i, j) - $\delta g^*$-closed, then A is $\tau_j$-closed.

Proof: Since $\delta - \text{closedness} \Rightarrow \text{closedness}$, the result follows the above Proposition 3.5.

3.7. Proposition

If $A, B \in D_g^*(i, j)$, then $A \cup B \in D_g^*(i, j)$.

Proof: Let A and B be (i, j)-$\delta g^*$-closed. Let $A \cup B \subseteq U$ where $U \in \text{GO}(X, \tau_1)$. Now $A \cup B \subseteq U$ implies $A \subseteq U$ and $B \subseteq U$. Since $A, B \in D_g^*(i, j)$, implies $\tau_j - \text{cl}_g(A) \subseteq U$ and $\tau_j - \text{cl}_g(B) \subseteq U$. Then $(\tau_j - \text{cl}_g(A) \cup \tau_j - \text{cl}_g(B)) \subseteq U$. That is $\tau_j - \text{cl}_g(A \cup B) \subseteq U$. Hence $A \cup B \in D_g^*(i, j)$.

3.8. Remark

The intersection of two $(i, j)$-$\delta g^*$-closed need not be $(i, j)$-$\delta g^*$-closed as seen from the following example.

3.9. Example

Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \{X, \phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$. Then $\{a, b\}$ & $\{b, c\}$ are $(1, 2) - \delta g^*$-closed sets but $\{a, b\} \cap \{b, c\} = \{b\}$ is not $(1, 2) - \delta g^*$-closed.

3.10. Proposition

For each element $x$ of $(X, \tau_1, \tau_2)$, $\{x\}$ is $\tau_j$-g-closed or $\{x\}$ is $(i, j)$-$\delta g^*$-closed.

Proof: If $\{x\}$ is $\tau_j$-g-closed, then the proof is over. Assume $\{x\}$ is not $\tau_j$-g-closed. Then $\{x\}$ is not $\tau_j$-g-open. So the only $\tau_j$-g-open containing $\{x\}$ in $X$. Hence $\{x\}$ is $(i, j)$-$\delta g^*$-closed.

3.11. Proposition

If A is (i, j) - $\delta g^*$-closed, then $\tau_j - \text{cl}_g(A) / A$ contains no empty $\tau_j$-g-closed set.

Proof: Let A be (i, j)-$\delta g^*$-closed and F be a non empty $\tau_j$-g-closed subset of $\tau_j - \text{cl}_g(A) / A$. Now $F \subseteq \tau_j - \text{cl}_g(A) / A = \tau_j - \text{cl}_g(A) \cap A^c$ which implies $F \subseteq \tau_j - \text{cl}_g(A)$ and $F \subseteq A^c$. Therefore $A \subseteq F^c$. Since $F^c$ is $\tau_j$-g-open and A is (i, j)-$\delta g^*$-closed in X, we have $\tau_j - \text{cl}_g(A) \subseteq F^c$ which implies that $F^c \subseteq (\tau_j - \text{cl}_g(A)) \cap (\tau_j - \text{cl}_g(A))^c$. Therefore $F = \phi$. Hence $\tau_j - \text{cl}_g(A) / A$ contains no non-empty $\tau_j$-g-closed set.

The following example shows that the reverse implication of the above theorem is not true.

3.12. Example

Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, c\}\}$, $\tau_2 = \{X, \phi, \{a, b\}\}$. If $A = \{a\}$, then $\tau_j - \text{cl}_g(A) / A = \{b, c\}$ does not contain any non-
empty $\tau_1$-g-closed set. But $A$ is not $(1, 2) - \delta g^*$-closed.

3.13. Corollary

If $A$ is $(i, j)$-$\delta g^*$-closed in $(X, \tau_1, \tau_2)$, then $A$ is $\tau_j$-\(\delta\)-closed if and only if $\tau_j - cl_{\delta}(A)$ is $\tau_j$-g-closed.

Proof: (Necessity) Let $A \subseteq D^i_\delta(i, j)$ and let $A$ be $\tau_j$-$\delta$-closed. Then $\tau_j - cl_{\delta}(A) = A$, i.e., $\tau_j - cl_{\delta}(A)/A = \phi$ and hence $\tau_j - cl_{\delta}(A)/A$ is $\tau_j$-g-closed.

(Sufficiency) If $\tau_j - cl_{\delta}(A)/A$ is $\tau_j$-g-closed, then by Proposition 3.11, $\tau_j - cl_{\delta}(A)/A = \phi$, since $A$ is $(i, j)$-$\delta g^*$-closed. Hence $\tau_j - cl_{\delta}(A) = A$. Therefore $A$ is $\tau_j$-$\delta$-closed.

3.14. Proposition

If $A$ is an $(i, j)$-$\delta g^*$-closed set, then $\tau_i - cl_{\delta}(x) \cap A \neq \phi$ holds for each $x \in \tau_i - cl_{\delta}(A)$.

Proof: Let $A$ be $(i, j)$-$\delta g^*$-closed and we know $\tau_i \subseteq GO(X, \tau_i)$. Suppose $\tau_i - cl_{\delta}(x) \cap A \neq \phi$, for some $x \in \tau_i - cl_{\delta}(A)$, then $A \subseteq X - \tau_i - cl_{\delta}(x) = B$, say. Then $B$ is a $\tau_i$-$\delta$-open set. Since a $\delta$-open set is an open set and a open set is g-open, $B$ is g-open in $X$. Since $A$ is $(i, j)$-$\delta g^*$-closed, we get $\tau_i - cl_{\delta}(A) \subseteq B = X - \tau_j - cl_{\delta}(x)$. Then $\tau_j - cl_{\delta}(A) \cap \tau_j - cl_{\delta}(x) = \phi$ which implies that $\tau_j - cl_{\delta}(A) \cap (x) = \phi$. Hence $x \neq \tau_j - cl_{\delta}(A)$, which is a contradiction.

The converse of the above proposition is not true as seen in the following example.

3.15. Example

Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \{X, \phi, \{a\}, \{b, c\}\}$. The subset $A = \{b\}$ in $(X, \tau_1, \tau_2)$ is not $(1, 2) - \delta g^*$-closed. However $\tau_1 - cl_{\delta}(x) \cap A \neq \phi$ holds for each $x \in \tau_2 - cl_{\delta}(a)$.

3.16. Proposition

If $A$ is an $(i, j)$-$\delta g^*$-closed set of $(X, \tau_1, \tau_j)$, such that $A \subseteq B \subseteq \tau_j - cl_{\delta}(A)$, then $B$ is also an $(i, j)$-$\delta g^*$-closed set of $(X, \tau_1, \tau_j)$.

Proof: Let $U$ be a $\tau_j$-g-open set in $(X, \tau_1, \tau_j)$ such that $B \subseteq U$ and hence $A \subseteq U$. Since $A$ is $(i, j)$-$\delta g^*$-closed, $\tau_j - cl_{\delta}(A) \subseteq U$. Hence $\tau_j - cl_{\delta}(B) \subseteq U$ which implies that $B$ is a $(i, j)$-$\delta g^*$-closed set of $(X, \tau_1, \tau_j)$.

3.17. Proposition

Let $A \subseteq Y \subseteq X$ and suppose that $A$ is $(i, j)$-$\delta g^*$-closed in $X$. Then $A$ is $(i, j)$-$\delta g^*$-closed relative to $Y$.

Proof: Let $A \in D^i_\delta(i, j) & A \subseteq Y \cap U$, U is g-open in $X$. A $\subseteq Y \cap U$ implies $A \subseteq U$ and since $A \in D^i_\delta(i, j)$, $\tau_j - cl_{\delta}(A) \subseteq U$. Then $\tau_j - cl_{\delta}(A) \cap Y \subseteq U \cap Y$. Hence $\tau_j - cl_{\delta}(A) \cap Y \subseteq U \cap Y$. Therefore $A$ is $(i, j)$-$\delta g^*$-closed relative to $Y$.

3.18. Theorem

In a bitopological space $(X, \tau_1, \tau_2)$, $GO(X, \tau_i) \subseteq F_{\delta}$ if and only if every subset of $X$ is an $(i, j)$-$\delta g^*$-closed set, where $F_{\delta}$ is the collection of $\delta$-closed sets with respect to $\tau_j$.

Proof: Suppose that $GO(X, \tau_i) \subseteq F_{\delta}$. Let $A$ be a subset of $(X, \tau_1, \tau_2)$ such that $A \subseteq U$ where $U \in GO(X, \tau_i)$. Then $\tau_j - cl_{\delta}(A) \subseteq \tau_j - cl_{\delta}(U) = U$. Therefore $A$ is $(i, j)$-$\delta g^*$-closed set.

Conversely, suppose that every subset of $X$ is $(i, j)$-$\delta g^*$-closed. Let $U \in GO(X, \tau_i)$. Since $U$ is $(i, j)$-$\delta g^*$-closed, we have $\tau_j - cl_{\delta}(U) \subseteq U$. Therefore $U \in F_{\delta}$ and hence $GO(X, \tau_i) \subseteq F_{\delta}$.

3.19. Proposition

Every $(i, j)$-$\delta g^*$-closed set is $(i, j)$-g-closed.

Proof: Let $A$ be $(i, j)$-$\delta g^*$-closed. Let $A \subseteq U$ and $U$ be a open set in $\tau_j$. Since every open set is g-
open, U is a g-open set. Then \( \tau_j - cl_\delta(A) \subseteq U \), we know that \( \tau_j - cl(U) \subseteq \tau_j - cl_\delta(U) \subseteq U \). Hence A is (i, j)-g-closed.

3.20. Remark

A (i, j)-g-closed need not be (i, j)-\( \delta g^* \)-closed as shown in the following example.

3.21. Example

Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}\} \), \( \tau_2 = \{X, \phi, \{a, b\}\} \). Then the set {b} is (1, 2)-g-closed but not (1, 2)-\( \delta g^* \)-closed.

3.22. Proposition

Every (i, j)-\( \delta g^* \)-closed set is (i, j)-g*-closed.

**Proof:** Let A be (i, j)-\( \delta g^* \)-closed. Let \( A \subseteq U \) and U be a g-open set in \( \tau_i \). Then \( \tau_j - cl_\delta(A) \subseteq U \), we know that \( \tau_j - cl(U) \subseteq \tau_j - cl_\delta(U) \subseteq U \). Hence A is (i, j)-g*-closed.

3.23. Remark

A (i, j)-g*-closed need not be (i, j)-\( \delta g^* \)-closed as shown in the following example.

3.24. Example

Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}\} \), \( \tau_2 = \{X, \phi, \{b, c\}\} \). Then the set \( \{a\} \) is (1, 2)-g*-closed but not (1, 2)-\( \delta g^* \)-closed.

3.25. Proposition

Every (i, j)-\( \delta g^* \)-closed set is (i, j)-rg-closed.

**Proof:** The proof follows from every regular open set is g-open.

3.26. Remark

A (i, j)-rg-closed need not be (i, j)-\( \delta g^* \)-closed as shown in the following example.

3.27. Example

Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}, \{a, b\}\} \), \( \tau_2 = \{X, \phi, \{a, b\}\} \). Then the set \( \{a, b\} \) is (1, 2)-rg-closed but not (1, 2)-\( \delta g^* \)-closed.

3.28. Proposition

Every (i, j)-\( \delta g^* \)-closed set is (i, j)-wg-closed.

**Proof:** Let A be (i, j)-\( \delta g^* \)-closed. Let \( A \subseteq U \) and U be a open set in \( \tau_i \). Since every open set is g-open, U is g-open in \( \tau_i \) Now \( \tau_i - int(A) \subseteq \tau_i \), implies \( \tau_j - cl(\tau_i - int(A)) \subseteq \tau_j - cl_\delta(A) \subseteq \tau_j - cl_\delta(A) \). Since A is (i, j)-\( \delta g^* \)-closed, \( \tau_j - cl_\delta(A) \subseteq U \). Therefore \( \tau_j - cl(\tau_i - int(A)) \subseteq U \). Hence A is (i, j)-wg-closed.

3.29. Remark

A (i, j)-wg-closed need not be (i, j)-\( \delta g^* \)-closed as shown in the following example.

3.30. Example

Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}\} \), \( \tau_2 = \{X, \phi, \{b, c\}\} \). Then the set \( \{b\} \) is (1, 2)-wg-closed but not (1, 2)-\( \delta g^* \)-closed.

3.31. Proposition

Every (i,j)-\( \delta g^* \)-closed set is (i,j)-ag*-closed.

**Proof:** Let A be (i, j)-\( \delta g^* \)-closed. Let \( A \subseteq U \subseteq \text{GO}(X, \tau_i) \), since \( \tau_i \subseteq \text{GO}(X, \tau_i) \). Then \( \tau_j - cl_\delta(A) \subseteq U \). We know \( \tau_j - acl(A) \subseteq \tau_j - cl_\delta(A) \) which implies \( \tau_j - acl(A) \subseteq U \). Therefore A is (i, j)-ag*-closed.

3.32. Remark

A (i, j)-ag*-closed need not be (i, j)-\( \delta g^* \)-closed as shown in the following example.

3.33. Example

Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a, b\}\} \), \( \tau_2 = \{X, \phi, \{a\},\{b\}\} \). Then the set \( \{a\} \) is (1, 2)-ag*-closed but not (1, 2)-\( \delta g^* \)-closed.

3.34. Proposition

Every (i,j)-\( \delta g^* \)-closed set is(i,j)-gpr-closed.

**Proof:** Let A be (i, j)-\( \delta g^* \)-closed. Let \( A \subseteq U \subseteq \text{open} \) and U be a regular open set. Since every regular open set is g-open, U is g-open. Since A is (i, j)-\( \delta g^* \)-closed, \( \tau_j - cl_\delta(A) \subseteq U \). We
know that $\tau_j - pcl(A) \subseteq \tau_j - cl_\delta(A)$. That is, $\tau_j - pcl(A) \subseteq \tau_j - cl_\delta(A) \subseteq U$ Therefore $A$ is $(i, j)$-gpr-closed.

3.35. Remark

A $(i, j)$-gpr-closed need not be $(i, j)$- $\delta g^*$-closed as shown in the following example.

3.36. Example

Let $X = \{a, b, c\}, \tau_1 = \{X, \phi, \{a, b\}\}, \tau_2 = \{X, \phi, \{a\}, \{b, c\}\}$. Then the set $\{b\}$ is $(1, 2)$-gpr-closed but not $(1, 2)$- $\delta g^*$-closed.

3.37. Proposition

Every $(i, j)$- $\delta g^*$-closed set is $(i, j)$-g*-$p$-closed.

Proof: Let $A$ be $(i, j)$- $\delta g^*$-closed. Let $A \subseteq U$ and $U$ is g-open in $\tau_i$. Then $\tau_j - cl_\delta(A) \subseteq U$. We know $\tau_j - pcl(A) \subseteq \tau_j - cl_\delta(A)$. Therefore $\tau_j - pcl(A) \subseteq U$. Hence $A$ is $(i, j)$-g*-$p$-closed.

3.38. Remark

A $(i, j)$-g*-$p$-closed need not be $(i, j)$- $\delta g^*$-closed as shown in the following example.

3.39. Example

Let $X = \{a, b, c\}, \tau_1 = \{X, \phi, \{a, b\}\}, \tau_2 = \{X, \phi, \{a\}, \{b, c\}\}$. Then the set $\{b\}$ is $(1, 2)$-g*-$p$-closed but not $(1, 2)$- $\delta g^*$-closed.

3.40. Proposition

Every $(i, j)$- $\delta g^*$-closed set is $(i, j)$-sag*-closed.

Proof: Let $A$ be $(i, j)$- $\delta g^*$-closed. Let $A \subseteq U$ and $U$ is g*-$p$-open in $\tau_i$. Since every g*-$p$-open set is g-open, $U$ is g-open. Then $\tau_j - acl(A) \subseteq U$. We know $\tau_j - acl(A) \subseteq \tau_j - cl_\delta(A)$, which implies $\tau_j - acl(A) \subseteq U$. Therefore $A$ is $(i, j)$-sag*-closed.

3.41. Remark

A $(i, j)$-sag*-closed need not be $(i, j)$- $\delta g^*$-closed as shown in the following example.

3.42. Example

Let $X = \{a, b, c\}, \tau_1 = \{X, \phi, \{a, b\}\}, \tau_2 = \{X, \phi, \{a\}, \{b, a\}\}$. Then the set $\{b\}$ is $(1, 2)$-sag*-closed but not $(1, 2)$- $\delta g^*$-closed.

3.43. Proposition

Every $(i, j)$- $\delta g^*$-closed set is $(i, j)$-g-p* closed.

Proof: The proof follows from the fact that every open set is g-open.

3.44. Remark

A $(i, j)$- $\delta g^*$-closed need not be $(i, j)$-g*-$p$-closed as shown in the following example.

3.45. Example

Let $X = \{a, b, c\}, \tau_1 = \{X, \phi, \{a\}\}, \tau_2 = \{X, \phi, \{a\}, \{b, a\}\}$. Then the set $\{b\}$ is $(1, 2)$- $\delta g^*$-closed but not $(1, 2)$-g*-$p$-closed.

3.46. Remark

The following examples show that $(i, j)$-w closed and $(i, j)$- $\delta g^*$-closed are independent to each other.

3.47. Example

Let $X = \{a, b, c\}, \tau_1 = \{X, \phi, \{a\}\}, \tau_2 = \{X, \phi, \{a\}, \{b, a\}\}$. Then the set $\{a\}$ is $(1, 2)$-w-closed but not $(1, 2)$- $\delta g^*$-closed.

3.48. Example

Let $X = \{a, b, c\}, \tau_1 = \{X, \phi, \{a\}, \{a, b\}\}, \tau_2 = \{X, \phi, \{a\}\}$. Then the set $\{a\}$ is $(1, 2)$- $\delta g^*$-closed but not $(1, 2)$- w-closed.

3.49. Remark

The following diagram has shown the relationship of $(i, j)$- $\delta g^*$-closed sets with other known existing sets. $A \rightarrow B$ represents $A$ implies $B$ but not conversely and $A \leftrightarrow B$ represents $A$ and $B$ are independent to each other.
1. (i, j)- \( \delta g^* \)-closed set, 2. (i, j)- wg-closed set, 3. (i, j)- g*-closed set, 4. (i, j)- w-closed set, 5. (i, j)- g-closed set, 6. (i, j)- gog*-closed set, 7. (i, j)- rg-closed set, 8. (i, j)- gp-closed set, 9. (i, j)- gpr-closed set, 10. (i, j)- \( \delta g \)-closed set, 11. (i, j)- \( \alpha g^* \)-closed set.

4. Applications

In this section we introduce the new closed spaces namely (i, j)- \( \delta g_T^{1/2} \)-space, (i, j)- \( \delta g^*_T \)-space, (i, j)- \( \delta g_T^{3/2} \)-space, and (i, j)- \( \delta g_T^1 \)-space in bitopological spaces.

4.1. Definition

A bitopological space \((X, \tau_1, \tau_2)\) is said to be

1) (i, j)- \( \delta g_T^{1/2} \)-space if every (i, j)- g-closed set is (i, j)-\( \delta g \)-closed.

2) (i, j)- \( \delta g_T^{3/2} \)-space if every (i, j)-g-closed set is (i, j)- \( \delta g^* \)-closed.

3) (i, j)- \( \delta g_T^1 \)-space if every (i, j)-g*-closed set is (i, j)- \( \delta g^* \)-closed.

4) (i, j)- \( \delta g_T^{3/2} \)-space if every (i, j)- \( \delta g \)-closed set is (i, j)- \( \delta g^* \)-closed.

4.2. Proposition

Every (i, j)- \( \delta g_T^{1/2} \)-space is a (i, j)- \( \delta g^*_T \)-space.

Proof: Let X be a (i, j)- \( \delta g_T^{1/2} \)-space and A be (i, j)- g*-closed. Since every (i, j)- g*-closed set is (i, j)- g-closed.

Assumption, we get A is (i, j)- \( \delta g^* \)-closed. Hence X is a (i, j)- \( \delta g_T^{1/2} \)-space.

The converse of the above proposition is not true as seen by the following example.

4.3. Example

Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a, b\}\} \), \( \tau_2 = \{X, \phi, \{a\}\} \). Then \((X, \tau_1, \tau_2)\) is (i, j)- \( \delta g_T^{1/2} \)-space. But \( \{a, b\} \) is (i, j)- g-closed but not (i, j)- \( \delta g^* \)-closed. Hence \( (X, \tau_1, \tau_2) \) is not \( \delta g_T^{1/2} \)-space.

4.4. Proposition

Every (i, j) - \( \delta g_T^{1/2} \)-space is a (i, j) - \( \delta g_T^{3/2} \)-space.

Proof: Let X be a (i, j) - \( \delta g_T^{1/2} \)-space and A be (i, j)- \( \delta g \)-closed. Since every (i, j)- \( \delta g \)-closed set is (i, j)- g-closed. Then A is (i, j)- g-closed. By assumption, we get A is (i, j)- \( \delta g^* \)-closed. Hence X is a (i, j)- \( \delta g_T^{3/2} \)-space.

The converse of the above proposition is not true as seen by the following example.

4.5. Example

Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, b\}\} \), \( \tau_2 = \{X, \phi, \{a\}\} \). Then \((X, \tau_1, \tau_2)\) is (i, j)- \( \delta g_T^{3/2} \)-space not (i, j)- \( \delta g_T^{1/2} \)-space. Since \( \{b, c\} \) is (i, j)- g-closed but not (i, j)- \( \delta g^* \)-closed. Hence \( (X, \tau_1, \tau_2) \) is not \( \delta g_T^{1/2} \)-space.

4.6. Proposition

Every (i, j) - \( \delta g_T^{1/2} \)-space is a (i, j) - \( \delta g_T^{3/2} \)-space.

Proof: Let X be a (i, j) - \( \delta g_T^{1/2} \)-space and A be (i, j)- g-closed. Then A is (i, j)- \( \delta g^* \)-closed. Since every (i, j)- \( \delta g^* \)-closed set is (i, j)- \( \delta g \)-closed. We get A is (i, j)- g-closed. Hence X is a (i, j)- \( \delta g_T^{3/2} \)-space.

The converse of the above proposition is not true as seen by the following example.
4.7. Example

Let \( X = \{a, b, c\} \). \( \tau_1 = \{X, \phi, \{a\}\}, \tau_2 = \{X, \phi, \{a, b\}\} \). Then \( (X, \tau_1, \tau_2) \) is (i, j)-\( \delta_g T_{\frac{1}{2}} \)-space not (i, j)-\( \delta_g T_{\frac{1}{2}} \)-space. Since \( \{b\} \) is (i, j)-\( g \)-closed but not (i, j)-\( g^* \)-closed.

4.8. Proposition

\( (X, \tau_1, \tau_2) \) is both (i, j)-\( \delta_g T_{\frac{1}{2}} \)-space and (i, j)-\( \delta_g T_{\frac{1}{2}} \)-space.

**Proof:** (Necessity): Let \( (X, \tau_1, \tau_2) \) be (i, j)-\( \delta_g T_{\frac{1}{2}} \)-space and (i, j)-\( \delta_g T_{\frac{1}{2}} \)-space. Consider \( A \) is (i, j)-\( g^* \)-closed. Then \( A \) is (i, j)-\( g \)-closed. Since \( (X, \tau_1, \tau_2) \) be (i, j)-\( \delta_g T_{\frac{1}{2}} \)-space, \( A \) is (i, j)-\( g \)-closed. Therefore \( (X, \tau_1, \tau_2) \) is a (i, j)-\( \delta_g T_{\frac{1}{2}} \)-space. (Sufficiency): It satisfies by Proposition 4.4 and Proposition 4.6.

4.9. Remark

The following examples show that (i, j)-\( \delta_g T_{\frac{1}{2}} \) and (i, j)-\( \delta_g T_{\frac{1}{2}} \) are independent to each other.

4.10. Example

Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{b\}\}, \{c\}, \{b, c\}, \{a, b\}\} \). \( \tau_2 = \{X, \phi, \{a\}\} \). Then \( (X, \tau_1, \tau_2) \) is (i, j)-\( \delta_g T_{\frac{1}{2}} \)-space. But \( \{b, c\} \) is (i, j)-\( g \)-closed but not (i, j)-\( g^* \)-closed.

4.11. Example

Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}\}, \{b, c\}\), \( \tau_2 = \{X, \phi\} \). Then \( (X, \tau_1, \tau_2) \) is (i, j)-\( \delta_g T_{\frac{1}{2}} \)-space. But \( \{b\} \) is (i, j)-\( \delta_g \)-closed but not (i, j)-\( \delta g^* \)-closed.

4.12. Remark

The following examples show that (i, j)-\( \delta_g T_{\frac{1}{2}} \) and (i, j)-\( \delta g T_{\frac{1}{2}} \) are independent to each other.

4.13. Example

Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}\}, \tau_2 = \{X, \phi, \{a, b\}\} \). Then \( (X, \tau_1, \tau_2) \) is (i, j)-\( \delta_g T_{\frac{1}{2}} \)-space. But \( \{b\} \) is (i, j)-\( \delta g \)-closed but not (i, j)-\( \delta g^* \)-closed.

4.14. Example

Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}\}, \tau_2 = \{X, \phi, \{a, b\}\} \). Then \( (X, \tau_1, \tau_2) \) is (i, j)-\( \delta_g T_{\frac{1}{2}} \)-space. But \( \{a, b\} \) is (i, j)-\( g \)-closed but not (i, j)-\( \delta g^* \)-closed.

4.15. Remark

The following examples shows that (i, j)-\( \delta_g T_{\frac{1}{2}} \) and (i, j)-\( \delta g T_{\frac{1}{2}} \) are independent to each other.

4.16. Example

Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}\}, \tau_2 = \{X, \phi, \{a, b\}\} \). Then \( (X, \tau_1, \tau_2) \) is (i, j)-\( \delta g T_{\frac{1}{2}} \)-space. But \( \{c\} \) is (i, j)-\( g^* \)-closed but not (i, j)-\( \delta g^* \)-closed.

4.17. Example

Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}\}, \tau_2 = \{X, \phi, \{a, b\}\} \). Then \( (X, \tau_1, \tau_2) \) is (i, j)-\( \delta g T_{\frac{1}{2}} \)-space. But \( \{a, b\} \) is (i, j)-\( g \)-closed but not (i, j)-\( \delta g \)-closed.

4.18. Remark

The following diagram has shown the relationship of (i, j)-\( g^* \)-closed spaces with other known existing space. \( A \longrightarrow B \) represents A implies B but not conversely and \( A \Longleftrightarrow B \) represents A and B are independent to each other.
5. References


