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ON FOX-H FUNCTION FRACTIONAL INTEGRAL OPERATORS AND M SERIES

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In the present investigation, the fractional operators involving Fox-H function due to Saxsena-Kumbhat, are applied to the M series which is further extension of both Mittag-Leffler function and generalized hypergeometric function ${}_pF_q$. The H-function fractional operators have found essential application in the solution of kinetic equation, fractional reaction and fractional diffusion. The results are mostly derived in a closed form in the terms of the H-function suitable for numerical computation.

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1. Introduction and Preliminaries:

The Subject of fractional calculus deals with investigations of integrals and derivatives has gained importance and popularity during the last four decades or so, mainly due to its vast potential demonstrated applications in fields of science and engineering . Different extensions of various fractional integrations operators are studied by Kalla [14] , Mc Bride [5], Kilbas [1] ,Kiryakova [4A] , Purohit Kalla [13] etc.

In the present paper we introduce a fractional integral operator involving H- function for Re$(\alpha) > 0, a_i, b_j \in \mathbb{C}, \alpha_j, \beta_j > 0, i = 1 \dots p; j = 1 \dots . q$, $\rho \epsilon\ \mathbb{C}, \sigma > 0$ *as follows* :

$$\left(I_{0,+}^{m,n,p,q,\alpha,\sigma} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} H_{p,q}^{m,n}\left[(x-t)^{\sigma} \mid \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right] f(t)dt \qquad (1.1)$$

and

$$\left(I_{0,-}^{m,n,p,q,\alpha,\sigma} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} H_{p,q}^{m,n}\left[(t-x)^{\sigma} \mid \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right] f(t)dt \qquad (1.2)$$

In (1.1), (1.2) $H_{p,q}^{m,n}(.)$ denotes Fox`s H-function ( ):

The H-Function introduced & defined by Fox-H[**] in 1961 , as

$$H_{p,q}^{m,n}(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s) z^s\, ds, \qquad (1.3)$$

Where $\mathcal{L}$ is a suitable path in the complex plane $\mathbb{C}$. and

$$\mathcal{H}_{p,q}^{m,n}(s) = \frac{A(s)B(s)}{C(s)D\,(s)}, \qquad (1.4)$$

$$\text{A(s)} = \prod_{j=1}^{m} \Gamma(b_j - \beta_j s), \quad B(s) = \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s), \qquad (1.5)$$

$$C(s) = \prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s), \qquad D(s) = \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s) \tag{1.6}$$

With $0 \leq n \leq p, 1 \leq m \leq q, \{a_j, b_j\} \in \mathbb{C}, \{\alpha_j, \beta_j\} \in \mathbb{R}^+$. With all convergence conditions as given by Braaksma [*].

The Properties of these operators were studied by Saigo [ ] Mathai & Saxena [ ] following which we can easily obtain sided and right – handed sided generalized integration of type (1.1) and (1.2) for power function as follows :

$$\left(I_{0,+}^{m,n,p,q,\alpha,\sigma} x^{\rho-1}\right)(x) = \frac{\Gamma(\rho)}{\Gamma(\alpha)} x^{\rho+\alpha-1} H_{p+1,q+1}^{m,n+1}\left[x^{\sigma} \,\middle|\, \begin{matrix} (1-\alpha,\sigma)(\,a_j,\alpha_j)_{1,p} \\ (1-\alpha-\rho,\sigma)(\,b_j,\beta_j)_{1,q} \end{matrix}\right] \tag{1.7}$$

where $Re(\alpha) > 0, m, n, p, q \in N_0$ with $0 \leq n \leq p, 1 \leq m \leq q$,
$\alpha_j \, \beta_j \in R_+, a_j, b_j \in R \; or \; C, i = 1 \ldots . p \, ; j = 1, \ldots . . q$, with all convergence condition as given by A.M.Mathai [ ] .

Further Let $\alpha^* = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0$

and

$$\left(I_{0,-}^{m,n,p,q,\alpha\sigma,} x^{\rho-1}\right)(x) = \frac{1}{\Gamma(\alpha)\Gamma(1-\rho)} x^{\alpha+\rho-1} H_{p+1,q+1}^{m+1,n+1}\left[x^{\sigma} \,\middle|\, \begin{matrix} (a_j,\alpha_j)(1-\alpha,\sigma) \\ (1-\rho-\alpha,\sigma)(b_j,\beta_j) \end{matrix}\right] \tag{1.8}$$

provided $\alpha \in C, R(\alpha) > 0$ and further the constants $a_i, b_j \in C, \alpha_j, \beta_j > 0, i = 1, \ldots . p; j = 1, \ldots . . q$,

$\rho \in C, \sigma > 0$ satisfy $\sigma_{1 \leq j \leq n}^{max} \left[\frac{Re\,(a_j)-1}{\alpha_j}\right] + R(\rho) + R(\alpha) < 1$ and $1 + \gamma\sigma > R(\rho) + R(\alpha)$

Sharma and Jain [7A] introduced the generalized M-series as the function defined by means of the power series:

$$ {}_{p}^{\alpha}M_{q}^{\beta}(a_1, a_2, \ldots, a_p;\, b_1, b_2, \ldots b_q; z) = {}_{p}^{\alpha}M_{q}^{\beta}(z) = {}_{p}^{\alpha}M_{q}^{\beta}\left((a_j)_1^p \,;\, (b_j)_1^q; z\right)$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z, \alpha, \beta \in C, R(\alpha) > 0 \tag{1.9}$$

where, $(a_j)_n, (b_j)_n$ are the known Pochammer symbols. The series (1.7) is defined when none of the parameters $b_j's$, $j = 1, 2, \ldots, q$, is a negative integer or zero; if any numerator parameter a_j is a negative integer or zero, then the series terminates to a polynomial in z. The series in

(1.7) is convergent for all z if $p \leq q$, it is convergent for $|z| < \delta$ if $p = q + 1$ and divergent, if $p > q + 1$. When $p = q + 1$ and $|z| = \delta$, the series can converge on conditions depending on the parameters. Properties of M-series are further studied by Saxena [8A], Chouhan and Sarswat [9A] etc.

The generalized Mittag- Leffler function [10A], is obtained from (1.7) for $p = q = 1$; $a = \gamma \in C$; b = 1, as

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{m=0}^{\infty} \frac{(\gamma)_m}{\Gamma(\alpha m + \beta)} \frac{z^m}{m!} = \sum_{m=0}^{\infty} \frac{(\gamma)_m}{(1)_m} \frac{z^m}{\Gamma(\alpha m + \beta)} = {}_1^{\alpha}M_1^{\beta}(\gamma\, ;\, 1; z). \qquad (1.10)$$

The generalized M-series (1.7) can be represented as a special case of the Wright generalized hypergeometric function (1.6), as

$${}_p^{\alpha}M_q^{\beta}\left(\left(a_j\right)_1^p ;\, \left(b_j\right)_1^q ; z\right) = k\ {}_{p+1}\psi_{q+1}\left[\begin{matrix}(a_1, 1), \dots, (a_p, 1), (1,1); \\ (b_1, 1), \dots, (b_q, 1), (\beta, \alpha);\end{matrix}\ z\right], \qquad (1.11)$$

where $k = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}$.

1. Main Results:

In this section, the image formulas for the M-series involving Fox-H Function fractional integral operators (1.1) and (1.2) are established:

Theorem 2.1 Let m, n, p, q ∈ N_0 with $0 \leq n \leq p, 1 \leq m \leq q, \alpha_j, \beta_j \in R_+, a_j, b_j \in R \text{ or } C, i = 1, \dots \dots p, j = 1, \dots \dots q, Re(\alpha) > 0, \alpha \in C$.

$\alpha^* = 0, \gamma\mu + R(\delta) < -1;\ \sigma_{1 \leq j \leq m}^{min}\left[\frac{R(b_j)}{\beta_j}\right] + R(\rho) > 0$ and $\gamma\sigma < R(\rho)$.

Then there holds the formula

$$\left(I_{0,+}^{m,n,p,q,\alpha,\sigma} t^{\rho-1}\ {}_{p_1}^{\upsilon}M_{q_1}^{\delta}(a_1 t^{\upsilon})\right)(x)$$

$$= \sum_{n_1=0}^{\infty} \frac{(a_1^{`})_{n_1} \cdots (a_{p_1}^{`})_{n_1}}{(b_1^{`})_{n_1} \cdots (b_{q_1}^{`})_{n_1}} \frac{a^{n_1}\Gamma(\upsilon n_1 + \rho)}{\Gamma(\upsilon n_1 + \delta)\Gamma(\alpha)} x^{\upsilon n_1 + \rho + \alpha - 1} . H_{p+1,q+1}^{m,n+1}\left[x^{\sigma}\, \Big|\, \begin{matrix}(1-\alpha, \sigma)(a_j, \alpha_j)_{1,p} \\ (1-\alpha-\upsilon n_1 - \rho, \sigma)(b_j, \beta_j)_{1,q}\end{matrix}\right]$$

$$\dots \dots \dots . (2.1)$$

Proof: Using (1.1) and (1.9), and then changing the order of integration and summation, we get

$$\left(I_{0,+}^{m,n,p,q,\alpha,\sigma} t^{\rho-1} {}_{p_1}^{\ v}M_{q_1}^{\delta}(a_1 t^{v})\right)(x) = \sum_{n_1=0}^{\infty} \frac{(\grave{a}_1)_{n_1} \cdots (\grave{a}_{p_1})_{n_1}}{(\grave{b}_1)_{n_1} \cdots (\grave{b}_{q_1})_{n_1}} \frac{a^{n_1}}{\Gamma(vn_1 + \delta)} I_{0,+}^{m,n,p,q,\alpha,\sigma} t^{vn_1+\rho-1}$$

Interpreting the right hand side of above equation, in view of the definition (1.7), we arrive at result (2.1).

On Setting $p_1 = q_1 = 1;\ a = \eta \in \mathbb{C};\ b = 1$ in (2.1), we obtained the following result.

Corollary 2.1. With the conditions on parameters mentioned in theorem (2.1), there holds the formula

$$\left(I_{0,+}^{m,n,p,q,\alpha,\sigma} t^{\rho-1} E_{v,\delta}^{\eta}(a_1 t^{v})\right)(x)$$

$$= \sum_{m_1=0}^{\infty} \frac{(n)_{m_1}}{\Gamma(vm_1 + \delta)} \frac{{a_1}^{m_1}}{\Gamma(m_1 + 1)} \frac{\Gamma(vm_1 + \rho)}{\Gamma(\alpha)} x^{vm_1+\rho+\alpha-1} . H_{p+1,q+1}^{m,n+1}\left[x^{\sigma} \middle| \begin{matrix} (1-\alpha,\sigma)(a_j,\alpha_j)_{1,p} \\ (1-\alpha-vm_1-\rho,\sigma)(b_j,\beta_j)_{1,q} \end{matrix}\right] \quad (2.2)$$

Theorem 2.2 Let $\alpha \in C$, $Re(\alpha) > 0$, $a_j, b_j \in \mathbb{C}$, $\alpha_j,\ \beta_j > 0$, $i = 1, \ldots .p; j = 1, \ldots .q$, $\sigma > 0, \rho \in \mathbb{C}$ satisfy $\sigma_{1\le j\le n}^{max}\left[\frac{Re\,(a_j)-1}{\alpha_j}\right] + R(\rho) + R(\alpha) < 1$

and $1+\gamma\sigma > R(\rho) + R(\alpha)$.

then there holds the formula

$$\left(I_{0,-}^{m,n,p,q,\alpha,\sigma} t^{-\beta_1-\delta} {}_{p_1}^{\ v}M_{q_1}^{\delta}(a_1 t^{-v})\right)(x)$$

$$= \sum_{n_1=0}^{\infty} \frac{(\grave{a}_1)_{n_1} \cdots (\grave{a}_{p_1})_{n_1}}{(\grave{b}_1)_{n_1} \cdots (\grave{b}_{q_1})_{n_1}} \frac{{a_1}^{n_1} x^{\alpha-vn_1-\beta_1-\delta}}{\Gamma(vn_1+\delta)\Gamma(\alpha)\Gamma(vn_1+\beta_1+\delta)} . H_{p+1,q+1}^{m+1,n+1}\left[x^{\sigma} \middle| \begin{matrix} (a_j,\alpha_j)(1-\alpha,\sigma) \\ (vn_1+\beta_1+\delta-\alpha,\sigma)(b_j,\beta_j) \end{matrix}\right] \quad (2.3)$$

Proof: Using (1.2) and (1.9), and then changing the order of integration and summation, we get

$$\left(I_{0,-}^{m,n,p,q,\alpha,\sigma} t^{-\beta_1-\delta} {}_{p_1}^{\ v}M_{q_1}^{\delta}(a_1 t^{-v})\right)(x)$$

$$= \sum_{n_1=0}^{\infty} \frac{(\grave{a}_1)_{n_1} \cdots (\grave{a}_{p_1})_{n_1}}{(\grave{b}_1)_{n_1} \cdots (\grave{b}_{q_1})_{n_1}} \frac{{a_1}^{n_1}}{\Gamma(vn_1 + \delta)} \left(I_{0,-}^{m,n,p,q,\alpha,\sigma} t^{-vn_1-\beta_1-\delta}\right)$$

Interpreting the right hand side of above equation, in view of the definition (1.8), we arrive at result (2.3).

On Setting $p_1 = q_1 = 1;\ a = \eta \in \mathbb{C};\ b = 1$ in (2.3), we obtained the following result.

Corollary 2.2. With the conditions on parameters given in theorem (2.3) there holds the formula

$$\left(I_{0,-}^{m,n,p,q,\alpha,\sigma} t^{-\beta_1-\delta} E_{v,\delta}^{\eta}(a_1 t^{-v})\right)(x)$$

$$= \sum_{m_1=0}^{\infty} \frac{(n)_{m_1}}{\Gamma(vm_1+\delta)} \frac{{a_1}^{m_1}}{\Gamma(m_1+1)} . \frac{x^{\alpha-vm_1-\beta_1-\delta}}{\Gamma(\alpha)\Gamma vm_1+\beta_1+\delta} . H_{p+1,q+1}^{m+1,n+1}\left[x^{\sigma} \Big|_{(vm_1+\beta_1+\delta-\alpha,\sigma)(b_j,\beta_j)}^{(a_j,\alpha_j)(1-\alpha,\sigma)}\right]$$

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