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On exponential Diophantine equation $2^x + 41^y = z^2$

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***Abstract*. In this paper, we used different method and find all solutions of exponential Diophantine equations $2^x + 41^y = z^2$,where x,y,z are non negative integers.**

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1. INTRODUCTION

Diophantine analysis deals with various techniques of solving Diophantine equations .If a Diophantine equation has variables as exponents, it is said to be exponential Diophantine equation. In this note, we will find all solutions of exponential Diophantine equation 2^x $+41^y = z^2$,where x,y,z are non negative integers.

2. MAIN RESULTS

Theorem 1. [4] .The only solutions to the Diophantine equation 8^x $+41^y = z^2$ in non negative integers are given by $(x,y,z)\in\{(1,0,3),(1,1,7)\}$.

Theorem 2.[1]. The equation $2^x + b^y = z^2$, $b,x,y,z\in N$, $\gcd(b,z)=1, b>1, x>1, y\geq 3$ has only solution (b,x,y,z)=(17,7,3,71).

Theorem 3.[5] The equation $2^x + b^y = c^z$ admits the solution for $x>1$, $y=1$, $2\mid z$ and $2^x < b^{\frac{50}{13}}$.

Theorem 4. The only solutions $(x,y,z)\in N^3$ to the equation $2^x + 41^y = z^2$ are (3,1,7) ,(7,1,13).
Proof. Let x, y, $z\in N$. First, we determine the parity of x,y and z. Since $x\geq 1$, then, obviously z is odd. Morever ,its clear that the equation 2^x $+41^y = z^2$ may admit a solution $(x,y.z)\in N^3$ provided y is odd, otherwise, we will have a contradiction([2]). Suppose now that 2^x $+41^y = z^2$ holds true for some tuples (x,y,z) where x, y and z are all odd.We consider two cases, case 1 ,where $3\mid x$ and case 2 is where 3 is not divisible by x.

Case 1. If x=3k, where k is natural, and 2 is not divisible by k, then we have $2^{3k} + 41^y = z^2$ or equivalently $8^k + 41^y = z^2$. In view of Theorem 1, the only solution (k,y,z) $(x,y.z)\in N^3$ is (1,1,7).Hence we get (3,1,7) as only solution to $2^x + 41^y = z^2$ in N, for x divisible by 3.

Case 2. Now, suppose that 3 is not divisible by x. First, we suppose that x=1. Then we have $2 + 41^y = z^2$.Note that $41^y \equiv 1 \pmod 4$.Hence we get $2 + 41^y \equiv 3 \pmod 4$ while $z^2 \equiv 1 \pmod 4$, a contradiction. Hence x>1.Apparently, 41 is not divisible by z because of congruence $2^x \equiv 0 \pmod{41}$. So for $y\geq 3$,the only solution we get is (x,y,z)=(7,3,71)(Theorem 2). Now consider the case for y=1.Since z has quadratic exponent, we know that the equation $2^x + 41^y = z^2$ may admit a solution in N such that $x \leq \frac{50 \log 41}{13 \log 2} < 41$. (Theorem 3).

Since the bound for x is small, one can effectively use a simple mathematical program to find whether there is any integer x on the interval (1,41) that makes the quantity $\sqrt{2^x + 41}$ an integer.Nevertheles,the values of x that could satisy the equation may be obtained theoretically,and this we show as follows:

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Rewriting $2^x+41^y=z^2$ as $3((2^x+5)/3)=(z+6)(z-6)$,we see that z must be atleast 7.we know that z is odd so z=2c+3 for c≥2.It follows that $2^x+41=(2c+3)^2=4c^2+12c+9$,or equivalently, $4c^2+12c-2^x=32$.Suppose that c is even, say $c=2^s m$ for some s,m∈N where m is odd.Then, $2^x(2^{2s+2-x}m^2+2^{s+2-x}3m-1)=2^5$.......(1).Recall that x>1, 3 is not divisible by x and x is odd, so x must be atleast 7.Thus from (1), we get a contradiction. Hence c cannot be an even integer.Hence, c is odd.For x≥7 and c is odd, we have $2^x+41=4c^2+12c+9$.

Hence $2^x+32=4c^2+12c$,which gives $2^{x-2}+8=c^2+3c$.

So $2^3(2^{x-5}+1)=c(c+3)$. Now, c being odd gives c+3=8 or c=5. So $2^{x-5}=2^2$ and so x=7. Finally this gives the solution (x,y,z)=(7,1,13).

Hence the final solution for $(x,y,z)\in N^3$ to $2^x+41^y=z^2$ are (3,1,7) ,(7,1,13).

Remark 1. Further, if x,y,z are non negative, then ,in this equation, x and z can never be 0. Suppose if x=0, the equation becomes $41^y=z^2-1$ will lead to a contradiction. Indeed, for y and z in N, we have $41^\alpha(41^{\beta-\alpha}-1)=41^\beta-41^\alpha=(z+1)-(z-1)=2$,where $\alpha+\beta=y$. Evidently, α=0 gives $41^\beta=3$, which is impossible. If z=0, then $2^x+41^y=0$, again impossible for non zero x and y. If y=0, then equation becomes $2^x+1=z^2$ which have a solution (3,0,3)[3].

3. CONCLUSION

The all solutions of Diophantine equations $2^x+41^y=z^2$,where x,y,z are non negative integers is{(3,0,3), (3,1,7) ,(7,1,13)}.

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