On Domination and Energy of Zero Divisor Graphs

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***Abstract*— The energy E (G) of a graph G is the sum of the absolute values of the eigenvalues of G. In this paper, we study the characterization of eigenvalues and energy of adjacency matrix corresponding to zero-divisor graphs of finite commutative ring. Also, we study the eigenvalue and energy of $\Gamma(Z_n)$ where n=2p, 3p, 5p, pq, p^2, 4p where p, q are distinct prime numbers. Finally, we give the relationship between the domination number, energy, rank and eigenvalues of complete zero-divisor graphs.**

Keywords* — *Commutative ring, Zero-divisor graph, Energy graph.

I. INTRODUCTION

The energy E(G), of a graph G is defined to be the sum of the absolute values of its eigenvalues. Hence, if A(G) is the adjacency matrix of G, and $\{\lambda_1,\lambda_2,\lambda_3...\lambda_n\}$ are the eigenvalues of A(G), then E(G)= $\sum_{i=1}^{n}|\lambda_i|$. The set $\{\lambda_1,\lambda_2,...,\lambda_n\}$ is the spectrum of G and denoted by Spec G. The totally disconnected graph K_n^c has zero energy while the complete graph K_n with the maximum possible number of edges (among graphs on n vertices) has energy 2(n-1).Graphs for which the energy is greater than 2(n-1) are called hyperenergetic graphs. If E(G) 2(n-1), then G is called non-hyperenergetic.

The energy of G was first defined by I. Gutman in 1978 as the sum of the absolute values of the eigenvalues of A(G) in [2]. Energy of graph originated from theoretical chemistry. Huckel molecular orbital theory is a field of theoretical chemistry where graph eigen values occur. The carbon atoms of a hydrocarbon system correspond to vertices of a graph associated with the molecule. From Huckel theory, the energy of a molecular graph is equal to the total - electron energy of a conjugated hydrocarbon [3].

The concept of energy has been generalized in two different directions. Let A be an m×n matrix and A^* denote its adjoint(conjugate transpose of A).The singular values $s_1(A)\geq s_2(A)\geq\ldots\geq s_m(A)$ of a matrix A are the square roots of the eigenvalues of AA^*.Note that if A is an n×nHermitian matrix(i.e , $A = A^*$) , then the singular values of A are the absolute values of the eigenvalues of A.For any $A\in M_{m,n}$define the energy of A , E(A) = $\sum_{i=1}^{m} s_i(A)$ .From the above, we note that the usual energy of a graphG , E(G) = E(A(G)).

Another generalization of energy is defined as follows:Let Mbe a matrix associated with G. Suppose $\mu_1, \mu_2,\ldots\mu_n$ are the eigenvalues of M and $\overline{\mu}$ is the average of $\mu_1, \mu_2,\ldots\mu_n$.The M – energy of G is then defined as the absolute deviation $E_{M(G)}$= $\sum_{i=1}^{n}|\mu_i-\overline{\mu}|$.If M is the adjacency matrix A(G), then $\overline{\mu}$ =0.Hence, the usual energy E(G) = E_A(G).For notation and zero-divisor graph terminology, we in general follow [4,5,6,7].

Let R be a commutative ring and let Z(R) be its set of zero-divisors. The zero-divisor graph of a ring is the graph (simple) whose vertex set is the set of non-zero zero-divisors, and an edge is drawn between two distinct vertices if their product is zero. Throughout this paperff, we consider the commutative ring by R and zero-divisor graph $\Gamma(R)$by $\Gamma(Z_n)$. The idea of a zero-divisor graph of a commutative ring was introduced by I.Beck in [1], where he was mainly interested in colourings. The zero-divisor graph is very useful to find the algebraic structures and properties of rings.

Claude Berge introduced the theory of Domination in 1958. The inspiration for this concept was drawn from the classical problem of covering chessboard with minimum number of chess pieces.

The most common definition given of a dominating set is that it is a set of vertices $D \subset V$ in a graph G = (V, E) having the property that every vertex v V-D is adjacent to atleast one vertex in D . The domination number (G) is the cardinality of a smallest dominating set of G.

Throughout this paper G will denote a simple (no loops or multiple edges), undirected graph with n vertices and m edges. If $\{v_1, v_2, \ldots v_n\}$ is the set of vertices of G , then the adjacency matrix of G, A(G) = A = $[a_{ij}]$ is an n × n matrix, where a_{ij}= 1 if v_i and v_j are adjacent and a_{ij}= 0 otherwise. Thus, A is a symmetric (0, 1) – matrix with real eigenvalues and zeros on the diagonal. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of A, then we have $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ *and* $\lambda_1 + \lambda_2 + ..\lambda_n = 0$.

II EIGENVALUES & ENERGY OF ZERO DIVISOR GRAPHS

In this paper, we evaluate the eigenvalues of zero-divisor graph and find the energy of some generalized zero divisor graphs.

Theorem1: For any graph $\Gamma(Z_{2p})$, where p is any prime number, then the eigenvalues are $\sqrt{p-1}, -\sqrt{p-1}$ and $E(\Gamma(Z_{2p})) = 2\sqrt{p-1}$.

Proof: The vertex set of $\Gamma(Z_{2p})$is {2, 4, 6… 2(p-1), p}. Let u=4 and v=p then 2p must divides uv. That is 2p divides 4p. Clearly, u and v are adjacent vertices. Similarly, any vertices u in $V(\Gamma(Z_{2p}))$and v=p then 2p must divides uv. It seems that p is adjacent to all the vertices in $V(\Gamma(Z_{2p}))$.Let u=4≠p and w=6≠p in $V(\Gamma(Z_{2p}))$ such that uw ≠0. It means that 2p does not divide uv=24. Clearly, no two vertices in $\Gamma(Z_{2p})$ are adjacent, except p.

Case(i): Let p=3.The eigenvalues of $\Gamma(Z_6)$are $\sqrt{2}$,-$\sqrt{2}$. Then the energy of $\Gamma(Z_6)$=Sum of the absolute values of the eigenvalues=$\left|\sqrt{2}\right| + \left|-\sqrt{2}\right| = 2\sqrt{2}.$

Case(ii): Let p=5.The eigenvalues of $\Gamma(Z_{10})$are $\sqrt{4}$,-$\sqrt{4}$. Then the energy of $\Gamma(Z_{10})$ =Sum of the absolute values of the eigenvalues=$\left|\sqrt{4}\right| + \left|-\sqrt{4}\right| = 2\sqrt{4}.$

Case (iii): Let p>5, is a prime number.

In general, the eigenvalues of $\Gamma(Z_{2p})$are $\sqrt{p-1}, -\sqrt{p-1}$. Then the energy of $\Gamma(Z_{2p})$=Sum of the absolute values of the eigenvalues=$\left|\sqrt{p-1}\right| + \left|-\sqrt{p-1}\right| = 2\sqrt{p-1}$.

Theorem 2: For any graph $\Gamma(Z_{3p})$, where p is any prime number, then the Eigenvalues are $\sqrt{2(p-1)}, -\sqrt{2(p-1)}$ and $E(\Gamma(Z_{3p})) = 2\sqrt{2(p-1)}$.

Proof: The vertex set of $\Gamma(Z_{3p})$is {3,6,9,…3(p-1),p,2p}. Let u and v be two vertices in $\Gamma(Z_{3p})$with maximum degree. Let u=p and v=2p then there exist any other vertex w≠p≠2p in $\Gamma(Z_{3p})$such that,is adjacent to both u and v. That is, uw =vw= 0. But uv=$2p^2$ which is not divided by 3p. Therefore u and v are non-adjacent vertices. Then the vertex set V can be partitioned into two parts V1 and V2 such that V_1={u,v}={p,2p} and V_2={3,6,9,…,3(p-1)}. Clearly $|V_1|$=2 and $|V_2|$=p-1, then
$|V|=|V_1|+|V_2|$=p+1.

Case(i): Let p=5.The eigenvalues of $\Gamma(Z_{15})$ are $\sqrt{8}$, $-\sqrt{8}$ .Then the energy of $\Gamma(Z_{15})$= Sum of the absolute values of the eigenvalues=$\left|\sqrt{8}\right| + \left|-\sqrt{8}\right| = 2\sqrt{8}$.

Case (ii): Let p=7.The Eigenvalues of $\Gamma(Z_{21})$ are $\sqrt{12}, -\sqrt{12}$. Then the energy of $\Gamma(Z_{21})$= Sum of the absolute values of the Eigenvalues=$\left|\sqrt{12}\right| + \left|-\sqrt{12}\right| = 2\sqrt{12}$.

Case(iii):Let p>7, is a prime number.

In general, the eigenvalues of $\Gamma(Z_{3p})$are $\sqrt{2(p-1)}, -\sqrt{2(p-1)}$. Then the energy of $\Gamma(Z_{3p})$= Sum of the absolute values of the eigenvalues=$\left|\sqrt{2(p-1)}\right| + \left|-\sqrt{2(p-1)}\right| = 2\sqrt{2(p-1)}.$

Theorem 3: For any graph $\Gamma(Z_{5p})$,where p is any prime number, then the eigenvalues are $\sqrt{4(p-1)}, -\sqrt{4(p-1)}$ and $E(\Gamma(Z_{5p})) = 2\sqrt{4(p-1)}$.

Proof: The vertex set of $\Gamma(Z_{5p})$ is {5,10, . . .5(p-1), p, 2p,3p,4p}. Clearly $|V(\Gamma(Z_{5p}))| = p+3$.Let u and v be any two vertices in $\Gamma(Z_{5p})$ with maximum and minimum degree, respectively. Let u = p and v=10 then 5p must divide uv which implies that u and v are adjacent.

Let u=p and v=2p then 5p does not divide uv=$2p^2$, which implies that u and v are non adjacent vertices. Then the vertex set V can be partitioned into two parts V_1 and V_2, where V_1={p,2p,3p,4p} and V_2= {5,10,…,5(p-1)}.Clearly any vertices u and v in $V_1$are non-adjacent as same as in V_2. Let u=p in V_1 and v=10 are in V_2 then 5p divides uv=10p. Finally, we note that, every vertex in V_1 is adjacent to all the vertices in V_2.Moreover $V(\Gamma(Z_{5p}))=V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$.

Case(i):Let p=7.The eigenvalues of $\Gamma(Z_{35})$ are $\sqrt{24}, -\sqrt{24}$ .Then the energy of $\Gamma(Z_{35})$ = Sum of the absolute values of the eigenvalues= $\left|\sqrt{24}\right| + \left|-\sqrt{24}\right| = 2\sqrt{24}$.

Case(ii):Let p=11.The eigenvalues of $\Gamma(Z_{55})$ are $\sqrt{40}, -\sqrt{40}$ .Then the energy of $\Gamma(Z_{55})$ = Sum of the absolute values of the eigenvalues= $\left|\sqrt{40}\right| + \left|-\sqrt{40}\right| = 2\sqrt{40}$.

Case (iii): Let p>11 , is a prime number.In general, the eigenvalues of $\Gamma(Z_{5p})$ are $\sqrt{4(p-1)}, -\sqrt{4(p-1)}$. Then the energy of $\Gamma(Z_{5p})$=Sum of the absolute values of the Eigenvalues= $\left|\sqrt{4(p-1)}\right| + \left|-\sqrt{4(p-1)}\right| = 2\sqrt{4(p-1)}$.

Theorem 4: For any graph $\Gamma(Z_{p^2})$, where p>2 is any prime number, then the eigenvalues are $\underbrace{-1,-1,\ldots-1}_{(p-2)\,times}, (p-2)$ and $E\left(\Gamma\left(Z_{p^2}\right)\right) = 2(p-2)$.

Proof: If p is any prime, then $V\left(\Gamma\left(Z_{p^2}\right)\right)$={p,2p,3p,4p,…,(p-1)p}. Clearly p is adjacent to all the vertices in $\Gamma(Z_{p^2})$. Also note that, any two vertices in $\Gamma(Z_{p^2})$ is adjacent and hence $\Gamma(Z_{p^2})$ is a complete graph, namely K_{p-1} .

Case(i): Let p=3.The eigenvalues of $\Gamma(Z_{3^2})$ are -1,1. Then the energy of $\Gamma(Z_{3^2})$=Sum of the absolute values of the eigenvalues=2.

Case(ii): Let p=5.The eigenvalues of $\Gamma(Z_{5^2})$ are -1,-1,-1,3. Then the energy of $\Gamma(Z_{5^2})$=Sum of the absolute values of the eigenvalues=6.

Case(iii): Let p=7.The eigenvalues of $\Gamma(Z_{7^2})$ are -1,-1,-1,-1-1,5. Then the energy of $\Gamma(Z_{7^2})$=Sum of the absolute values of the eigenvalues=10.

Case(iv): Let p>7, is a prime number.The eigenvalues of $\Gamma(Z_{p^2})$ are $\underbrace{-1,-1,\ldots,-1}_{(p-2)\,times}, (p-2)$.

Then the energy of $\Gamma(Z_{p^2})$= Sum of the absolute values of the eigenvalues= $\left|(-1,\ldots-1)(p-2)\text{times}\right| + \left|p-2\right| = 2(p-2)$.

Theorem5: In $\Gamma(Z_{pq})$, if p and q are distinct prime numbers with p<q, then the eigenvalues are $\sqrt{(p-1)(q-1)}, -\sqrt{(p-1)(q-1)}$ and $E\left(\Gamma\left(Z_{pq}\right)\right) = 2\sqrt{(p-1)(q-1)}$.

Proof: The proof is by the method of induction on p and q. The vertex set of $\Gamma(Z_{pq})$ is {p,2p,3p,…,p(q-1),q,2q,3q,…,(p-1)q}.

Case(i): Let p=2,q is any prime>2,Using theorem (2.1), for any graph $\Gamma(Z_{2q})$, where q>2 is any prime number, then the eigenvalues are $\sqrt{q-1}, -\sqrt{q-1}$ .Without loss of generality, the eigenvalues can be written as $\sqrt{(p-1)(q-1)}$, $-\sqrt{(p-1)(q-1)}$, where p=2. Then, $E\left(\Gamma\left(Z_{2q}\right)\right) = 2\sqrt{(q-1)} = 2\sqrt{(2-1)(q-1)} = 2\sqrt{(p-1)(q-1)}$, where p=2.

Case(ii): Let p=3,q is any prime>3,Using theorem (2.2), for any graph $\Gamma(Z_{3q})$, where q>3 is any prime number, then the eigenvalues are $\sqrt{2(q-1)}, -\sqrt{2(q-1)}$. Without loss of generality, the eigenvalues can be written as $\sqrt{(p-1)(q-1)}$, $-\sqrt{(p-1)(q-1)}$,where p=3. Then, $E\left(\Gamma\left(Z_{3q}\right)\right) = 2\sqrt{2(q-1)} = 2\sqrt{(3-1)(q-1)} = 2\sqrt{(p-1)(q-1)}$, where p=3.

Case(iii):Let p =5,q is any prime>5,Using theorem (2.3), for any graph $\Gamma(Z_{5q})$, where q>5 is any prime number, then the eigenvalues are $\sqrt{4(q-1)}, -\sqrt{4(q-1)}$. Without loss of generality, the eigenvalues can be written as $\sqrt{(p-1)(q-1)}, -\sqrt{(p-1)(q-1)}$,where p=4. Then, $E\left(\Gamma\left(Z_{5q}\right)\right) = 2\sqrt{4(q-1)} = 2\sqrt{(5-1)(q-1)} = 2\sqrt{(p-1)(q-1)}$, where p=5.

Case(iv): Let p<q,The vertex set of $\Gamma(Z_{pq})$, is {p,2p,3p,…,p(q-1),q,2q,3q,…,(p-1)q}.Using theabove cases, the eigenvalues of $\Gamma(Z_{pq})$ are $\sqrt{(p-1)(q-1)}, -\sqrt{(p-1)(q-1)}$ where p and q are distinct primes with p<q. Then the energy of $\Gamma(Z_{pq})$ is $E\left(\Gamma\left(Z_{pq}\right)\right) = \left|\sqrt{(p-1)(q-1)}\right| + \left|-\sqrt{(p-1)(q-1)}\right| = 2\sqrt{(p-1)(q-1)}$.

Theorem6:If p > 4 is any prime, thenthe eigenvalues are in the form of $\pm$ x, $\pm$ 2.4142x,where x and 2.4142x are less than p and $E(\Gamma(z_{4p})) = 2(x + 2.4142x)$.

Proof: The vertex of $\Gamma(Z_{4p})$ is {2,4, …, 2(2p-1),p,2p,3p} with $\left|V(\Gamma(Z_{4P}))\right| = 2p+1$.Let v = 2p be a vertex and let w be any vertex except p and 3p such that 4p divides vw.

Clearly, v is adjacent to all the vertices in $V(\Gamma(Z_{4P}))$ expect p and 3p. Let P, S,N(S) be the pendant set, minimum degree set, neighborhood of S, respectively. Since v has maximum degree then $v \in N(S)$.

Case(i): Let p = 5.The vertex set of $\Gamma(Z_{20})$ is {2,4,…2(10-1),5,10,15} with $|V(\Gamma(Z_{20}))| = 11$. Let v =2p = 10 be a vertex and letw be any vertex except 5 and 15 such that 20 divides vw. Clearly, v = 10 is adjacent to all the vertices in $V(\Gamma(Z_{20}))$ except 5 and 15 then $10 \in N(S)$.Let x = 2 and y = 14 then 28 is not divisible by 20 which implies x and y are non adjacent vertices. But xv = yv = 0.

The eigenvalues of $\Gamma(Z_{20})$ are -3.6955, -1.5307, 0, 1.5307, 3.6955. Then the energy of $\Gamma(Z_{20})$ = Sum of the absolute values of the eigenvalues = 2(1.5307+3.6955) = 2(1.5307+2.4142(1.5307)) = 10.4524.

Case(ii): Let p = 7.The vertex set of $\Gamma(Z_{28})$ is {2,4,…2(14-1), 7,14,21} with $|V(\Gamma(Z_{28}))| = 2p+1 = 15$.Let v = 2p =14 be a vertex and let w be any vertex except 7 and 21 such that 28 divides vw. Clearly v =14 is adjacent to all the vertices in $V(\Gamma(Z_{28}))$ except 7 and 21 then $14 \in$ N(S). The eigenvalues of $V(\Gamma(Z_{28}))$ are -4.5261, -1.8748, 0, 1.8748, 4.5261. Then the energy of $V(\Gamma(Z_{28}))$ = Sum of the absolute values of the eigenvalues = 2(1.8748+4.5261) = 2(1.8748+2.4142(1.8748)) = 12.8018

Case(iii): Let p>7is a prime numberThe vertex set of $V(\Gamma(Z_{4p}))$ is{2,4,…2(2p-1),p,2p,3p}with $|V(\Gamma(Z_{4p}))| = 2p+1$.Let v =2p be a vertex and let w be any other vertex except p and 3p such that 4p divides vw. Clearly , v is adjacent to all the vertices $V(\Gamma(Z_{4P}))$ except p and 3p and v = 2p $\in$ N(S).

The eigenvalues of $V(\Gamma(Z_{4P}))$ are -x, -2.4142x, 0, 2.4142x, x. Then the energy of $V(\Gamma(Z_{28}))$ = Sum of the absolute values of the eigenvalues =2(x+2.4142x)=6.8284x.

III Relationship between domination, energy, rank and eigenvalues of $\Gamma(Z_n)$ In this section, we find the bounds which relate the domination number of $\Gamma(Z_n)$, energy of $\Gamma(Z_n)$, rank of $\Gamma(Z_n)$ and eigenvalues of $\Gamma(Z_n)$,for n = p^2, where p >2 is any prime number.

G	$\gamma(\Gamma(Z_n))$	$E(\Gamma(Z_n))$	$\rho(\Gamma(Z_n))$	Eigenvalues
$\Gamma(Z_6)$	1	2	2	-1,1
$\Gamma(Z_{25})$	1	6	4	-1,-1,-1,3
$\Gamma(Z_{49})$	1	10	6	-1,-1,-1,-1,-1,5
$\Gamma(Z_{121})$	1	18	10	-1,-1,-1,-1,-1-1,-1,-1,-1,9,
…	…	…	…	…
…	…	…	..	…
$\Gamma(Z_{p^2})$	1	2p -4	p-1	$\underbrace{-1,-1,\ldots -1}_{(p-2)-times},(P-2)$

III CONCLUSION

In this paper, we study the eigenvalues and energy of zero divisor graph over finite commutative rings. Graphs are the most ubiquitous models of both natural and human made structures. In computer science, zero divisor graphs are used to represent networks of communication, network flow, clique problems. For any graph $\Gamma(Z_{4P})$, where p > 3 is any prime number and 4 <x< 4+x, then the eigenvalues are x , 2.4142x, approximately, where x is a position of consecutive prime number from 5 to so on.

If an Eigen value of a zero divisor graph is a rational number, then it is an integer. Also we note that the energy of a zero divisor graph cannot be an odd integer. The energy of a zero divisor graph cannot be the square root of an oddinteger.The energy of a zero divisor graph cannot be the square root of the double of an odd integer.

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