On Certain Topological Properties of Double Paranormed Null Sequence Space

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***Abstract*— The aim of this paper is to introduce and study a new class $c_0\,((S_{mn}\,,\|\,.\,\|_{mn}),\,\bar{\gamma}\,,\,\bar{u}\,)$ of double sequences with their terms in a normed space S as a generalization of the familiar sequence space c_0 . We investigate the condition in terms of $\bar{\gamma}$ and $\bar{u}$ so that a class is contained in or equal to another class of same kind and thereby derive the conditions of their equality. We further explore some of the preliminary results that characterize the linear topological structures of the space $c_0\,((S_{mn}\,,\|\,.\,\|_{mn}),\,\bar{\gamma}\,,\,\bar{u}\,)$ when topologized it with suitable natural paranorm .**

Keywords: Sequence space, Double sequence, Paranormed space, GK-space.

I.INTRODUCTION

We begin with recalling some notations and basic definitions that are used in this paper.

Let S be a normed space over **C** , the field of complex numbers. Let $\omega(S)$ denotes the linear space of all sequences $\bar{s} = (s_k)$ with $s_k \in S$, $k \geq 1$ with usual coordinate wise operations .We shall denote ω (**C**) by ω . Any subspace S of ω is then called a sequence space. A normed space valued sequence space or a generalized sequence space is a linear space of sequences with their terms in a normed space. Several workers like Kamthan and Gupta [6], Khan [7], Kolk [8], Köthe [9], Maddox [11], Malkowski and Rakocevic [13], Pahari [16,17,18], Ruckle [21] etc. have introduced and studied some properties of vector and scalar valued single sequence spaces, when sequences are taken from a Banach space.

The theory of single sequence spaces has also been extended to the spaces of double sequences and studied by several workers. Boos Leiger [3], Gupta and Kamthan [4], Milovidov and Povolotzki [14], Morics [15], Rao [20] and many others have made their significant contributions and enriched the theories in this direction. In the recent years, Savas [22], Subramanian *et al* [23] and many others have introduced and studied various types of double sequence spaces using orlicz function.

The notion of convergence of a single sequence (a_n) leads to various notions of convergence for a double sequence (a_{mn}) by using many senses. The double sequence (a_{mn}) in various sequence spaces c_0, c, ℓ_∞, ℓ_p depending upon the mode of m and n tending to infinity lead to several spaces, see Maddox [12].

A paranormed space (S, G) is a linear space S with zero element θ together with a function $G : S \to \boldsymbol{R}_+$ (called a paranorm on S) which satisfies the following axioms:

PN_1: $G(\theta) = 0$; PN_2: $G(\eta) = G(-\eta)$ for all $\eta \in S$;

PN_3: $G(\eta + \nu) \leq G(\eta) + G(\nu)$ for all $\eta, \nu \in S$; and

PN_4: Scalar multiplication is continuous.

Note that the continuity of scalar multiplication is equivalent to

(i) if $G(\eta_n) \to 0$ and $\xi_n \to \xi$ as $n \to \infty$, then $G(\xi_n\, \eta_n) \to 0$ as $n \to \infty$, and

(ii) if $\xi_n \to 0$ as $n \to \infty$ and η be any element in S, then $G(\xi_n\, \eta) \to 0$, see Wilansky [24].

A paranorm is called total if $G(\eta) = 0$ implies $\eta = \theta$.

The concept of paranorm is closely related to linear metric space; see Wilansky [24] and its studies on sequence spaces were initiated by Maddox [10] and many others. Basariv and Altundag [1], Bhardwaj and Bala [2], Khan [7], Parasar and Choudhary [19], and many others further studied various types of paranormed sequence spaces .

Concerning K–property of scalar sequence spaces, see Kamthan and Gupta [6], GAK–space have been defined for vector valued sequence spaces and also they are defined for Banach space valued function space, see Gupta and Patterson [5]. We now introduce the following definition for double sequence spaces:

Let $V(S_{mn})$ be a class of sequences $\{\, \bar{s} = (s_{mn}),\, s_{mn} \in S_{mn}\,;\, m, n \geq 1\}$.The topological sequence space $(V(S_{mn}), \mathfrak{I})$ equipped with the linear topology $\mathfrak{I}$ is said to be a GK-space if the map $P_{ij} : V(S_{mn}) \to S_{ij}$, defined by $P_{ij}\,(\bar{s}) = s_{ij}$ is continuous for each $i, j \geq 1$.

II.THE CLASS $c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{u})$ OF DOUBLE SEQUENCES

Let $\bar{u} = (u_{mn})$ and $\bar{v} = (v_{mn})$ be any double sequences of strictly positive real numbers and $\bar{\gamma} = (\gamma_{mn})$ and $\bar{\mu} = (\mu_{mn})$ be double sequences of non-zero complex numbers. Let $(S_{mn}, \|.\|_{mn})$,$m, n \geq 1$ be normed space over the field ***C*** of complex numbers with zero element θ.We now introduce and study the following class of Banach space valued double sequences:

$$c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{u}) = \{\bar{s} = (s_{mn}): s_{mn} \in S_{mn}, m, n \geq 1, \text{ and } \|\gamma_{mn} s_{mn}\|_{mn}^{u_{mn}} \to 0 \text{ as } m+n \to \infty\}.$$

Further, by $\bar{u} = (u_{mn}) \in \ell_\infty$, we mean $\sup u_{mn} < \infty$. We denote $A(\xi) = \max(1, |\xi|)$ and the zero element of this class by $\bar{\theta} = (\theta_{mn})$ for all m,n .

III. SOME CONTAINMENT RELATIONS

In this section we investigate the conditions in terms of $\bar{u}$ and $\bar{\gamma}$, so that a class $c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{u})$ is contained in or equal to another class of same kind and thereby derive the conditions of their equality.

Theorem 3.1: For any $\bar{\gamma} = (\gamma_{mn})$, $c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{u}) \subset c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{v})$ if and only if $\liminf_{m+n\to\infty} \frac{v_{mn}}{u_{mn}} > 0$.

Proof: For the sufficiency of the condition, suppose that the condition hold, and $\bar{s} = (s_{mn}) \in c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{u})$. Then there exists a constant $K > 0$ such that $v_{mn} > K\, u_{mn}$ for all sufficiently large values of m,n. Further $\|\gamma_{mn} s_{mn}\|_{mn}^{u_{mn}} < 1$ for all sufficiently large values of m,n and so

$\|\gamma_{mn} s_{mn}\|_{mn}^{u_{mn}} \leq (\|\gamma_{mn} s_{mn}\|_{mn}^{u_{mn}})^K$ for all sufficiently large values of m,n which implies that $\bar{s} \in c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{v})$. Hence

$$c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{u}) \subset c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{v}).$$

For the necessity of the condition suppose the inclusion holds, but $\liminf_{m+n\to\infty} \frac{v_{mn}}{u_{mn}} = 0$. Then there exist subsequences $(m(k))$ of (m) and $(n(k))$ of (n) respectively such that

$$k\, v_{m(k)\, n(k)} < u_{m(k)\, n(k)},\ k \geq 1.$$

Now taking $z_{mn} \in S_{mn}$ with $\|z_{mn}\|_{mn} = 1$, we define a sequence $\bar{s} = (s_{mn})$ by

$$s_{mn} = \begin{cases} \gamma_{mn}^{-1} k^{-1/u_{mn}} z_{mn}, & m = m(k),\ n = n(k),\ k \geq 1 \text{ and} \\ \theta, & \text{otherwise.} \end{cases}$$

Then we see that $\bar{s} \in c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{u})$. Since $\|\gamma_{m(k)\, n(k)} s_{m(k)\, n(k)}\|_{m(k)n(k)}^{u_{m(k)n(k)}} = \frac{1}{k}$, $k \geq 1$ and $|\gamma_{mn} s_{mn}|_{mn}^{v_{mn}} = \theta$, otherwise. But for each $k \geq 1$,

$$\|\gamma_{m(k)\, n(k)} s_{m(k)\, n(k)}\|_{m(k)n(k)}^{u_{m(k)n(k)}} = k^{-v_{m(k)\, n(k)}/u_{m(k)\, n(k)}} > k^{-1/k} > e^{-1/2}$$

shows that $\bar{s} \notin c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{u})$, a contradiction. This cmpletes the proof.

Theorem 3.2: For any $\bar{\gamma} = (\gamma_{mn})$, $c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{v}) \subset c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{u})$ if and only if $\limsup_{m+n\to\infty} \frac{v_{mn}}{u_{mn}} < \infty$.

Proof:

Let the condition hold, and $\bar{s} = (s_{mn}) \in c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{v})$. Then there exists a constant $L > 0$ such that $v_{mn} < L\, u_{mn}$ for all sufficiently large values of m,n. Further $\|\gamma_{mn} s_{mn}\|_{mn}^{u_{mn}} \to 0$ as $m+n \to \infty$ together with

$\|\gamma_{mn} s_{mn}\|_{mn}^{u_{mn}} \leq (\|\gamma_{mn} s_{mn}\|_{mn}^{v_{mn}})^{1/L}$for all sufficiently large values of m, n implies that $\bar{s} \in c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{u})$ and hence

$$c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{v}) \subset c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{u}).$$

Conversely let the inclusion hold but $\limsup_{m+n\to\infty} \frac{v_{mn}}{u_{mn}} = \infty$.Then there exists subsequences $(m(k))$ of (m) and $(n(k))$ of (n) respectively for each $k \geq 1$;

$v_{m(k)\, n(k)} < k\, u_{m(k)\, n(k)}$.

Thus for $z_{mn} \in S_{mn}$ with $\|z_{mn}\|_{mn} = 1$ the sequence

$\bar{s} = (s_{mn})$ defined as

$$s_{mn} = \begin{cases} \gamma_{mn}^{-1} k^{-1/v_{mn}} z_{mn}, & m = m(k),\ n = n(k),\ k \geq 1 \text{ and} \\ \theta, & \text{otherwise.} \end{cases}$$

is in $c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{u})$ but for each $k \geq 1$,

$$\|\gamma_{m(k)\, n(k)} s_{m(k)\, n(k)}\|_{m(k)n(k)}^{u_{m(k)n(k)}} = k^{-v_{m(k)\, n(k)}/u_{m(k)\, n(k)}} > k^{-1/k} > e^{-1/2}.$$

This shows that $\bar{s} \notin c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{u})$, a contradiction. This completes the proof.

Theorem 3.3: For any $\bar{\gamma} = (\gamma_{mn})$, $c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u}) = c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{v})$ if and only if

$$0 < \liminf_{m+n\to\infty} \frac{v_{mn}}{u_{mn}} \leq \limsup_{m+n\to\infty} \frac{v_{mn}}{u_{mn}} < \infty.$$

Theorem 3.4: For any $\bar{u} = (u_{mn})$, $c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u}) \subset c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\mu}, \bar{u})$

if and only if $\liminf_{m+n\to\infty} \left|\frac{\gamma_{mn}}{\mu_{mn}}\right|^{u_{mn}} > 0.$

Proof:

For the sufficiency of the condition, suppose that $\liminf_{m+n\to\infty} \left|\frac{\gamma_{mn}}{\mu_{mn}}\right|^{u_{mn}} > 0$ and $\bar{s} = (s_{mn}) \in c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$. Then there exists a constant $K > 0$ such that $K < \left|\frac{\gamma_{mn}}{\mu_{mn}}\right|^{u_{mn}}$ for sufficiently large values of m, n. Thus

$K \|\mu_{mn} s_{mn}\|_{mn}^{u_{mn}} < \|\gamma_{mn} s_{mn}\|_{mn}^{u_{mn}}$ for all sufficiently large

values of m, n and so $\|\gamma_{mn} s_{mn}\|_{mn}^{u_{mn}} \to 0$ implies that

$\|\mu_{mn} s_{mn}\|_{mn}^{u_{mn}} \to 0$ and hence $\bar{s} \in c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\mu}, \bar{u})$.

This proves that

$c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u}) \subset c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\mu}, \bar{u})$.

For the necessity, suppose that $\liminf_{m+n\to\infty} \left|\frac{\gamma_{mn}}{\mu_{mn}}\right|^{u_{mn}} = 0$. Then there exist subsequences $(m(k))$ of (m) and $(n(k))$ of (n) respectively such that for each $k \geq 1$, $k\,|\gamma_{m(k)\,n(k)}|^{u_{m(k)n(k)}} < |\mu_{m(k)\,n(k)}|^{u_{m(k)n(k)}}$.

Now for $z_{mn} \in S_{mn}$ with $\|z_{mn}\|_{mn} = 1$, define the sequence $\bar{s} = (s_{mn})$ by

$$s_{mn} = \begin{cases} \gamma_{mn}^{-1}\, k^{-1/u_{mn}}\, z_{mn}, & m = m(k),\ n = n(k),\ k \geq 1 \text{ and} \\ \theta, & \text{otherwise.} \end{cases}$$

Now, $\|\gamma_{m(k)\,n(k)}\, s_{m(k)\,n(k)}\|_{m(k)n(k)}^{u_{m(k)n(k)}} = \frac{1}{k}$, $k \geq 1$

and $\|\gamma_{mn} s_{mn}\|_{mn}^{u_{mn}} = 0$, otherwise.

But for each $k \geq 1$,

$$\|\mu_{m(k)\,n(k)} s_{m(k)\,n(k)}\|_{m(k)n(k)}^{u_{m(k)\,n(k)}}$$

$$= \|\mu_{m(k)\,n(k)}\, \gamma_{m(k)n(k)}^{-1}\, k^{-1/u_{m(k)n(k)}}\, u_{m(k)\,n(k)}\, z_{m(k)\,n(k)}\|_{m(k)n(k)}^{u_{m(k)\,n(k)}}$$

$$= \left|\frac{\mu_{m(k)\,n(k)}}{\gamma_{m(k)\,n(k)}}\right|^{u_{m(k)\,n(k)}} . \frac{1}{k} > 1$$

which shows that $\bar{s} \in c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$ but $\bar{s} \notin c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\mu}, \bar{u})$, a contradiction. This completes the proof.

Theorem 3.5: For any $\bar{u} = (u_{mn})$, $c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\mu}, \bar{u}) \subset c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$

if and only if $\limsup_{m+n\to\infty} \left|\frac{\gamma_{mn}}{\mu_{mn}}\right|^{u_{mn}} < \infty.$

Proof:

Let the condition hold, and $\bar{s} = (s_{mn}) \in c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\mu}, \bar{u})$. Then there exists $0 < L < \infty$ such that $|\gamma_{mn}|^{u_{mn}} < L|\mu_{mn}|^{u_{mn}}$, for all sufficiently large values of m,n and so $\|\gamma_{mn} s_{mn}\|_{mn}^{u_{mn}} \leq \|\mu_{mn} s_{mn}\|_{mn}^{u_{mn}}$

implies that $\bar{s} \in c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$. This shows that $c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\mu}, \bar{u}) \subset c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$.

Conversely let

$c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\mu}, \bar{u}) \subset c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$

but $\limsup_{m+n\to\infty} \left|\frac{\gamma_{mn}}{\mu_{mn}}\right|^{u_{mn}} = \infty$. Then there exist subsequences $(m(k))$ of (m) and $n(k))$ of (n) respectively such that for each $k \geq 1$

$$|\gamma_{m(k)\,n(k)}|^{u_{m(k)n(k)}} > k\,|\mu_{m(k)\,n(k)}|^{u_{m(k)n(k)}}.$$

Now taking $z_{mn} \in S_{mn}$, such that $\|z_{mn}\|_{mn} = 1$, we define the sequence $\bar{s} = (s_{mn})$ by

$$s_{mn} = \begin{cases} \mu_{mn}^{-1}\, k^{-1/u_{mn}}\, z_{mn}, & m = m(k),\ n = n(k),\ k \geq 1 \text{ and} \\ \theta, & \text{otherwise.} \end{cases}$$

Then, $\|\mu_{m(k)\,n(k)}\, s_{m(k)\,n(k)}\|_{m(k)n(k)}^{u_{m(k)n(k)}} = \frac{1}{k}$, $k \geq 1$ and

$\|\mu_{mn} s_{mn}\|_{mn}^{u_{mn}} = 0$, otherwise

shows that $\bar{s} \in c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\mu}, \bar{u})$. But on the other hand

$\|\gamma_{m(k)\,n(k)}\, s_{m(k)\,n(k)}\|_{m(k)n(k)}^{u_{m(k)n(k)}} > 1$, for all $k = 1$

implies that $\bar{s} \notin c_0((S_{mn}, \|.\|_{mn}), \bar{\gamma}, \bar{u})$, a contradiction. This completes the proof.

On combining Theorems 3.4 and 3.5 we get :

Theorem 3.6: For any $\bar{u} = (u_{mn})$, $c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u}) = c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\mu}, \bar{u})$ if and only if

$$0 < \liminf_{m+n\to\infty} \left|\frac{\gamma_{mn}}{\mu_{mn}}\right|^{u_{mn}} \leq \limsup_{m+n\to\infty} \left|\frac{\gamma_{mn}}{\mu_{mn}}\right|^{u_{mn}} < \infty.$$

IV. LINEAR TOPOLOGICAL STRUCTURES

OF $c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$

In this section, we shall investigate some results that characterize the linear topological structure of

$c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$ when topologized it with suitable natural paranorm . As far as the linear space structure of

$c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$ over the field C of complex numbers is concerned, we throughout take the

coordinatewise operations i.e., for $\bar{s} = (s_{mn})$, $\bar{t} = (t_{mn})$ and scalar ξ,

$$\bar{s} + \bar{t} = (s_{mn} + t_{mn}) \text{ and } \xi\,\bar{s} = (\xi s_{mn})$$

and we see below that $\bar{u} \in \ell_\infty$ is necessary and sufficient

condition for linearity of $c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$.

Theorem 4.1: $c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$ forms a linear space

over the set of complex number C if and only if $\bar{u} = (u_{mn}) \in \ell_\infty$.

Proof:

For the sufficiency of the condition, assume that

$\bar{u} = (u_{mn}) \in \ell_\infty$ and $\bar{s} = (s_{mn})$, $\bar{t} = (t_{mn}) \in c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$, $m, n \geq 1$.So that $\| \gamma_{mn} s_{mn} \|_{mn}^{u_{mn}} \to 0$ and $\|\gamma_{mn} t_{mn}\|_{mn}^{u_{mn}} \to 0$ as $m + n \to \infty$.

Then we have

$$\| \gamma_{mn}(s_{mn} + t_{mn}) \|_{mn}^{u_{mn}} \leq \| \gamma_{mn} s_{mn} \|_{mn}^{u_{mn}} + \| \gamma_{mn} t_{mn} \|_{mn}^{u_{mn}}$$

$$\to 0 \text{, as } m + n \to \infty.$$

Hence $\bar{s} + \bar{t} \in c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$.

Also it is clear that for any scalar ξ, $\xi\bar{s} \in c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$, since

$$\|\xi\, \gamma_{mn} s_{mn} \|_{mn}^{u_{mn}} = |\xi|^{u_{mn}} \| \gamma_{mn} s_{mn} \|_{mn}^{u_{mn}}$$

$$\leq A(\xi) \| \gamma_{mn} s_{mn} \|_{mn}^{u_{mn}}$$

$$\to 0 \text{ as } m + n \to \infty.$$

Conversely if $\bar{u} = (u_{mn}) \notin \ell_\infty$ then there exist subsequences $(m(k))$ of (m) and $(n(k))$ of (n) respectively such that $u_{m(k)\,n(k)} > k$ for each $k \geq 1$.

Now taking $z_{mn} \in S_{mn}$ with $\|z_{mn}\|_{mn} = 1$, we a define

sequence $\bar{s} = (s_{mn})$ by

$$s_{mn} = \begin{cases} \gamma_{mn}^{-1} k^{-1/u_{mn}} z_{mn}, & m = m(k),\ n = n(k),\ k \geq 1 \text{ and} \\ \theta, & \text{otherwise.} \end{cases}$$

Then for each $m = m(k)$, $n = n(k)$, $k \geq 1$, we have

$$\| \gamma_{mn} s_{mn} \|_{mn}^{u_{mn}} = \| \gamma_{m(k)\,n(k)}\ s_{m(k)\,n(k)} \|_{m(k)n(k)}^{u_{m(k)n(k)}}$$

$$= \frac{1}{k},\ k \geq 1$$

and $\| \gamma_{mn} s_{mn} \|_{mn}^{u_{mn}} = 0$, otherwise,

shows that $\bar{s} \in c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$ .But on the otherhand, for each $m = m(k)$, $n = n(k)$, $k \geq 1$and for scalar $\xi = 2$ we have

$$\| \mu_{mn}(\xi\ s_{mn}) \|_{mn}^{u_{mn}} = \| \mu_{m(k)\,n(k)}\, 2\, s_{m(k)\,n(k)} \|_{m(k)n(k)}^{u_{m(k)\,n(k)}}$$

$$> \frac{2^k}{k} > 1$$

showing that $\xi\bar{s} \notin c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$, a contradiction. This completes the proof. Hence c_0 $((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$ is a linear space if and only if $\bar{u} = (u_{mn}) \in \ell_\infty$.

Consider $\bar{u} = (u_{mn}) \in \ell_\infty$ and $\bar{s} \in c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$,we define

$$G_{\gamma,u}(\bar{s}) = \sup_{mn} \| \gamma_{mn} s_{mn} \|_{mn}^{u_{mn}} \qquad \text{...(4.1)}$$

Theorem 4.2 : Let $\bar{u} = (u_{mn}) \in \ell_\infty$ and S_{mn} be a normed space for each $m,n \geq 1$. Then $(c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u}), G_{\gamma,u})$ forms

a total paranormed space.

Proof:

It can be easily verified that $G_{\gamma,u}$ defined by (4.1) satisfy following properties of paranormed space.

PN1: $G_{\gamma,u}(\bar{s}) \geq 0$ and $G_{\gamma,u}(\bar{s}) = 0$ if and only if $\bar{s} = \bar{\theta}$

PN2: $G_{\gamma,u}(\bar{s} + \bar{t}) \leq G_{\gamma,u}(\bar{s}) + G_{\gamma,u}(\bar{t})$

PN3: $G_{\gamma,u}(\xi\bar{s}) \leq A(\xi)\, G_{\gamma,u}(\bar{s})$ where $\xi \in$ C.

Throughout the proof of the theorem, $G_{\gamma,u}$ will be denoted by G. Here we prove the continuity of scalar multiplication i.e., PN4. Further for continuity of scalar multiplication, it is sufficient to show that

(a) if $\bar{s}^{(k)} \to \bar{\theta}$ in G and $\xi_k \to \xi$ as $k \to \infty$

(b) if $\xi_k \to 0$ as $k \to \infty$ and $\bar{s} \in c_0((S_{mn}, \|\,.\,\|_{mn}), \bar{\gamma}, \bar{u})$ then $\xi_k\,\bar{s} \to \bar{\theta}$ in G as $k \to \infty$.

Now (a) is easily proved if we suppose that $|\xi_k| \leq L$ for all $k \geq 1$ and consider,

$$G(\xi_k\, \bar{s}^{(k)}) \leq \sup_{m,n} |\xi_k|^{u_{mn}} \sup_{m,n} \| \gamma_{mn} s_{mn} \|_{mn}^{u_{mn}}$$

$\leq A(L)\, G\,(\bar{s}^{\,(k)})$.

Now let $\bar{s} \in c_0\,((S_{mn}\,,\, \|\,.\,\|_{mn}),\, \bar{\gamma}\,,\, \bar{u}\,)$, $|\xi_k| \leq 1$ for all $k \geq N$ and $\varepsilon > 0$. Then there exists K such that

$\|\,\gamma_{mn}\, s_{mn}\,\|_{mn}^{u_{mn}} < \varepsilon$, for all $m + n \geq K$

and hence

$\|\xi_k\, \gamma_{mn}\, s_{mn}\,\|_{mn}^{u_{mn}} < \varepsilon$, for all $m + n \geq K$ and $k \geq N$.

Now choose N_1 so that $|\xi_k|^{u_{mn}}\, \|\,\gamma_{mn}\, s_{mn}\,\|_{mn}^{u_{mn}} < \varepsilon$, for all $k \geq N_1$, and $2 \leq m + n \leq K$.

Thus $G\,(\xi_k\, \bar{s}) \leq \varepsilon$ for all $k \geq \max(N, N_1)$ which proves (b). Hence G forms a total paranorm on

$c_0\,((S_{mn}),\, \bar{\gamma}\,,\, \bar{u},\, \|\,.\,\|_{mn})$.

Theorem 4.3 : Let $\bar{u} = (u_{mn}) \in \ell_\infty$ and S_{mn} be a normed space for each $m, n \geq 1$. Then

$(c_0\,((S_{mn}\,,\, \|\,.\,\|_{mn}),\, \bar{\gamma}\,,\, \bar{u}\,)\,,\, G_{\gamma,u})$ is a GK-space.

Proof:

For each $m,n \geq 1$, the continuity of linear map

$P_{ij} : c_0\,((S_{mn}\,,\, \|\,.\,\|_{mn}),\, \bar{\gamma}\,,\, \bar{u}\,) \rightarrow S_{ij}$ where $P_{ij}\,(\bar{s}) = s_{ij}$

follows from $\|\,P_{ij}\,(\bar{s})\| \leq |\gamma_{ij}|^{-1}\,[G(\bar{s})]^{1/u_{ij}}$ and so

$c_0\,((S_{mn}\,,\, \|\,.\,\|_{mn}),\, \bar{\gamma}\,,\, \bar{u}\,)$ is a GK-space.

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