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NURBS based Finite Element Approach for One Dimensional Problems

A. Suresh [A]
Assistant Professor,
Department of Mechanical Engineering,
JNT University College of Engineering Jagityal,
Nachupally P.O. Kodimyal Mandal,
Karimnagar Dist., Telangana., India, - 505 501

Ch. Yella Reddy [B]
M. Tech,
Department of Mechanical Engineering,
JNT University College of Engineering Jagityal,
Nachupally P.O. Kodimyal Mandal,
Karimnagar Dist., Telangana., India, - 505 501

***Abstract* -In this work, an attempt is made to use the Non Uniform Rational B-Spline(NURBS) basis functions as the shape functions in the finite element method. These basis functions are employed in Collocation Method approximation for the spatial descretization. It uses recursive formula of NURBS basis functions for solving second order differential equations with Neumann's boundary conditions. A test case is considered to study the efficiency of this method. When the degree of the basis increased the stability and efficiency is improved. The result obtained by present method is compared and found to be in good agreement with analytical solution and finite element method.**

***Keywords:* NURBS,B-Splines, Isogeometric Method, Collocation Method, Concrete Pier.**

1. INTRODUCTION:

Mathematical models in the form of differential and partial differential equations are used to represent various engineering problems in the fields, such as Structural mechanics, Solid mechanics, Fluid flow, Heat transfer, Vibration analyses, Contact mechanics etc. The solutions to these mathematical models can be Exact, Analytical or Approximate depending on the nature of these equations. When the Exact solution is not possible, numerical methods are needed to obtain approximate solutions. Many numerical techniques are evolved and has been used increasingly in last few years. Those numerical techniques include Finite Difference Method [R.K.Panday *et al*, 2004], B-spline Collocation Method [Moshen.A *et al*, 2008], Predictor and Corrector Method [Abdalkaleg Hamad *et al*, 2014], Finite Element Method [Ch.Sridhar Reddy *et al*, 2014], and many more. In these methods, the approximating function provides higher order of continuity and is capable of providing accurate solutions with continuous gradients throughout the domain.

The basis functions for B-spline and NURBS are derived using knot space and for a particular degree. A recursive formulation was given Carl.De boor [C.de Boor *et al*, 1982] for deriving these basis functions. If we use this formulation the evaluation of basis function can be generalized up to any degree. This basis function can be used in collocation method .

In the present work, an attempt is made to use an approximating function for the field variable based on the NURBS basis function to solve the boundary value problem. A non-uniform knot vector for a particular weight vector is used to obtain the second and third degree NURBS basis functions. For the spatial descretization, Collocation Method approximation is employed.

In this paper the recursive formulation of B-spline and NURBS basis functions [Hughes, T.J.R *et al*.2005, David F. Rogers *et al*, 2002] are discussed initially then the NURBS collocation method is discussed and formulated. The effectiveness and accuracy of this method is tested using the governing equation of one dimensional structural problem. The structural problem considered is, a loaded typical concrete pier of a bridge with varying cross section, to study the variation of displacement along the pier.

Considering second order linear differential equations with variable coefficients

$$\frac{d^2U}{dx^2} + k_1 P(x)\frac{dU}{dx} + k_2 Q(x)U = F(x) \qquad (1)$$

With the boundary conditions $U(a)=d_1$, $U(b)=d_2$. Where a,b,d_1,d_2, k_1 and k_2 are variables, $P(x)$, $Q(x)$ and $F(x)$ are functions of x. Let the approximation solution be

$$U^h(x) = \sum_{i=-2}^{n-1} C_i R_{i,p}(x) \qquad (2)$$

Where C_i are constants to be determined and $R_{i,p}(x)$ are NURBS basis functions.

$U^h(x)$ is the approximate global solution to the exact solution U(x) of the considered second order singular differential equation(1).

2. A BRIEF INTRODUCTION TO B-PLINES/NURBS:

B-Splines:

A spline is the mathematical representation of real world geometries. Schoenberg[David F. Rogers *et al*,2002] was given first reference to the word B-spline and described it as a smooth piecewise polynomial curve. From mathematical point of view, a curve generated by using the

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vertices of a defining polygon and the curve is dependent on some interpolation scheme between the curve and polygon. This scheme is provided by the choice of B-spline basis functions. B-spline basis is generally has non global behaviour due to the property that each vertex of B-spline B_i is associated with a unique basis function. Thus, each vertex affects the shape of a curve only over a range of parameter values where its associated basis function and hence the degree of the resulting curve to be changed without changing the number of defining polygon vertices.
Any curve can be represented as a parametric curve, i.e., the coordinate *x* is represented as a function of a parameter t. A parametric B-spline function can be defined by

$$p(x) = \sum_{i=1}^{n+1} B_i N_{i.p}(t) \tag{3}$$

Where, $x_{min} \le x \le x_{max}$, $2 \le p < n+1$
Where the B_i are the position vectors of the n+1 defining polygon vertices, *p* is the order and the $N_{i,p}(x)$ are the normalized B-spline basis functions. B-spline curve defined as polynomial spline function of order p and degree p-1.

i. If p=1

$$N_{i,p}(x) = \begin{cases} 1 & \text{if} \quad x_i \le x < x_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

ii. If p>1

$$N_{i,p}(x) = \frac{(x - x_i) N_{i,p-1}(x)}{x_{i+p-1} - x_i} + \frac{(x_{i+p} - x) N_{i+1,p-1}(x)}{x_{i+p} - x_{i+1}} \tag{5}$$

The values of x_i are elements of knot vector satisfying the relation $x_i \le x_{i+1}$. The parameter x varies from x_{min} to x_{max} along the curve p(x).

The sum of the B-spline basis functions is 1 for any parameter value x. Positivity property guarantees that the curve segment lies completely within the convex hull of P_i. The partition of unity property ensures that the relationship between the curve and its defining control points is invariant under affine transformations. Local support property indicates that each segment of B-spline curve is influenced by only p control points.

NURBS basis functions:
Non-Uniform Rational B-Splines (NURBS) were introduced by K. Versprille [K. J. Versprille *et al*, 1975] as significant improvement that can accurately handle both analytic and modeled curves. NURBS are used in most computer graphics applications, significantly in CAE and renowed industry standards such as IGES (Initial Graphics Exchange Specification), STEP (Standard for the Exchange of Product model data).

A Rational B-spline curve is the projection of a non-rational B-spline curve defined in four-dimensional (4D) homogeneous coordinate space back into three-dimensional (3D) physical space. Specifically,

$$p(x) = \sum_{i=1}^{n+1} B^h{}_i N_{i.p}(x) \tag{6}$$

Where the B_i^h 's are the 4D homogeneous defining polygon vertices for the non-rational 4D B-spline curve. $N_{i,p}(x)$ is the non-rational B-spline basis function, given in equations (4)& (5).

Projecting back into three-dimensional space by dividing through by the homogeneous coordinate yields the rational B-spline curve.

$$p(x) = \frac{\sum_{i=1}^{n+1} B_i h_i N_{i,p}(x)}{\sum_{i=1}^{n+1} h_i N_{i,p}(x)} = \sum_{i=1}^{n+1} B_i R_{i,p}(x) \tag{7}$$

Where the B_i 's are the 3D defining polygon vertices for the rational B-spline curve and the rational B-spline basis functions given by

$$R_{i,p}(x) = \frac{h_i N_{i,p}(x)}{\sum_{i=1}^{n+1} h_i N_{i,p}(x)} \tag{8}$$

Here, h_i 's are the homogeneous coordinates (occasionally called weights) provide additional blending capability. It is clear that when all h_i =1, $R_{i,p}(x)$ $=N_{i,p}(x)$,thus non-rational B-spline basis functions and curves are included as a special case of rational B-spline basis functions and curves.
NURBS derivatives:
Since the aim of the collocation Method is to compute approximation solution for differential Equations, so the derivatives of the NURBS basis functions needs to be calculated.NURBS basis functions are defined by equation (8). Equations (9) and (10) are the first and second derivatives of NURBS curve of order 3.

The first derivative of NURBS curve is

$$p'(x) = \sum_{i=1}^{n+1} B_i R'_{i.p}(x)$$

Where

$$R'_{i,p}(x) = \frac{h_i N'_{i,p}(x)}{\sum_{i=1}^{n+1} h_i N_{i,p}(x)} - \frac{h_i N_{i,p}(x) \sum_{i=1}^{n+1} h_i N'_{i,p}(x)}{\left(\sum_{i=1}^{n+1} h_i N_{i,p}(x) \right)^2} \tag{9}$$

The second derivative of NURBS curve is

$$p''(x) = \sum_{i=1}^{n+1} B_i R''_{i.p}(t)$$

Where

$$R''_{i,p}(x) = \frac{h_i N''_{i,p}(x)}{\sum_{i=1}^{n+1} h_i N_{i,p}(x)} - \frac{R_{i,p}(x) \sum_{i=1}^{n+1} h_i N''_{i,p}(x)}{\sum_{i=1}^{n+1} h_i N_{i,p}(x)} - 2 * \frac{R'_{i,p}(x) \sum_{i=1}^{n+1} h_i N'_{i,p}(x)}{\sum_{i=1}^{n+1} h_i N_{i,p}(x)} \tag{10}$$

From the above equations, the basis functions are defined as recursively in terms of previous degree basis function i.e. the p^{th} order basis functions is the combination of ratios of knots and (p-1) degree basis function. Again $(p-1)^{th}$ degree basis function is defined as the combination ratios of knots and (p-2) degree basis function. In a similar way every NURBS basis function of degree up to (p-(p-2)) is expressed as the combination of the ratios of knots and its previous basis functions.

NURBS derivative basis of second degree over uniform knot vector and equal weights for all control points (i.e., h_i =1) is shown graphically below in figures 2 and 3.

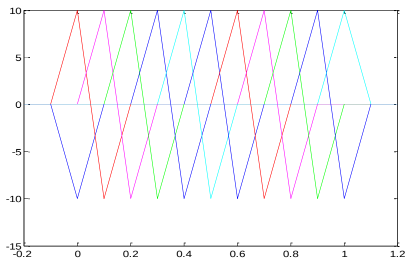

Figure 1: First derivative of second degree NURBS basis function with Knot vector X= {-0.2,-0.1, 0, 0.1, 0.2, 0.3, 0.4,0.5,0.6,0.7,0.8,0.9,1.0,1.1,1.2}

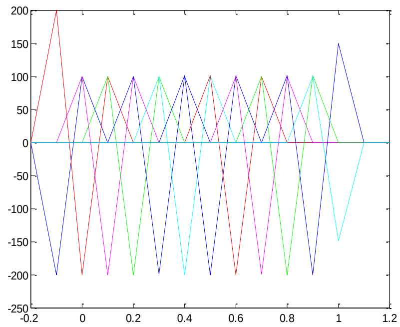

Figure 2: Second derivative of second degree NURBS basis function with Knot vector X= {-0.2,-0.1, 0, 0.1, 0.2, 0.3, 0.4, 0.5, 6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2}

3. COLLOCATION METHOD:

Collocation method is used widely in approximation methods particularly solving differential equations. The collocation method together with NURBS (Non-Uniform Rational Basis Spline) approximations represents an economical alternative since it only requires the evaluation of the unknown parameters at the grid points or nodes or collocation points. In normal collocation method we use polynomials whereas in NURBS collocation method we use NURBS basis functions. The selection of nodes or collocation points is arbitrary. The basis functions vanishes at the boundary values. The success of this Collocation method is dependent on the choice of basis. The main aim is to analyse the efficiency of the NURBS based collocation method for such problems with sufficient accuracy.

Formulation of NURBS Collocation Method:

As mentioned earlier NURBS functions are used as basis in collocation method whereas the base functions which are used in the normal collocation method are polynomials, vanishes at the nodes. Let [*a*, *b*] be the domain of the governing differential equation and is partitioned as $X=\{a=x_1, x_2, \ldots\ldots x_{n-1}, x_n=b\}$with non-uniform values between [a b] with *n* sub domains. for a particular homogeneous coordinate(weights) i.e., h_i's. The x_i's are known as nodes, these nodes are treated as knots in collocation NURBS method, where NURBS basis functions are defined and those nodes are used to make the residue equal zero to determine unknowns C_i's in equation (2). Extra knot values are taken into consideration both sides of the domain of problem when evaluating the second degree NURBS basis functions at the nodes. These extra knots are taken to satisfy the partition of unity property and to get accurate NURBS basis functions.

First derivative of approximation function (2) is

$$\frac{dU^h(x)}{dx} = \sum_{i=-2}^{n-1} C_i R'_{i,p}(x) \tag{11}$$

Second derivative of approximation function (2) is

$$\frac{d^2U^h}{dx^2} = \sum_{i=-2}^{n-1} C_i R''_{i,p}(x) \tag{12}$$

Substituting, the approximate solution (2) in (1) we have,

$$\frac{d^2U^h}{dx^2} + k_1 P(x)\frac{dU^h}{dx} + k_2 Q(x) U^h = F(x) \tag{13}$$

Substituting the Approximation function and its derivatives (2),(11) and (12) in the equation (13), we have

$$\sum_{i=-2}^{n-1} C_i R''_{i,p}(x) + k_1 P(x)\sum_{i=-2}^{n-1} C_i R'_{i,p}(x) + k_2 Q(x)\sum_{i=-2}^{n-1} C_i R_{i,p}(x) = F(x)$$

(14) Expanding the equation (14)

$$\begin{aligned}&[C_{-2}R''_{-2,p}(x) + C_{-1}R''_{-1,p}(x) + C_1R''_{1,p}(x) + \ldots\ldots + C_{n-1}R''_{n-1,p}(x)] + \\ &k_1P(x)[C_{-2}R'_{-2,p}(x) + C_{-1}R'_{-1,p}(x) + C_1R'_{1,p}(x) + \ldots\ldots + C_{n-1}R'_{n-1,p}(x)] + \\ &k_2Q(x)[C_{-2}R_{-2,p}(x) + C_{-1}R_{-1,p}(x) + C_1R_{1,p}(x) + \ldots\ldots + C_{n-1}R_{n-1,p}(x)] = F(x)\end{aligned} \tag{15}$$

Now let the coefficients of C_{-2}, C_{-1}, $C_1\ldots\ldots$, C_{n-1} are assumed as $R_{-2}(x)$, $R_{-1}(x)$, $R_1(x)\ldots R_{n-1}(x)$, now we have the equation 15, as

$$[R_{-2}(x)]C_{-2} + [R_{-1}(x)]C_{-1} + [R_1(x)]C_1 + \ldots\ldots + [R_{n-1}(x)]C_{n-1} = F(x) \tag{16}$$

In matrix form, we have

$$[R_{-2}(x) \quad R_{-1}(x) \quad R_1(x). \quad . \quad . \quad . R_{n-1}(x)]\begin{bmatrix} C_{-2} \\ C_{-1} \\ C_1 \\ . \\ . \\ . \\ C_{n-1}\end{bmatrix} = F(x) \tag{17}$$

Equation (16) is evaluated at x_i's, i=1,2,3,….n-1 gives the system of (n-1)×(n+1) equations in which (n+1) arbitrary constants are involved.

The Matrix (17) can be written as

$$\begin{bmatrix} R_{-2}(1) & R_{-1}(1) & R_1(1)\ldots.. & R_{n-1}(1) \\ R_{-2}(2) & R_{-1}(2) & R_1(2)\ldots.. & R_{n-1}(2) \\ R_{-2}(3) & R_{-1}(3) & R_1(3)\ldots.. & R_{n-1}(3) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ R_{-2}(n-1) & R_{-1}(n-1) & R_1(n-1)\ldots.. & R_{n-1}(n-1)\end{bmatrix}\begin{bmatrix} C_{-2} \\ C_{-1} \\ C_1 \\ . \\ . \\ . \\ C_{n-1}\end{bmatrix} = \begin{bmatrix} F(1) \\ F(2) \\ F(3) \\ . \\ . \\ . \\ F(n-1)\end{bmatrix} \tag{18}$$

Applying boundary conditions to approximate the solution, we have

$$\sum_{i=-2}^{n-1} C_i R_{i,p}(a) = d_1 \qquad \sum_{i=-2}^{n-1} C_i R_{i,p}(b) = d_2 \tag{19}$$

A square matrix of size (n+1)x(n+1) is obtained from equations (18), (19)

$$\begin{bmatrix} R_{-2}(a) & R_{-1}(a) & R_1(a). & . & . & . & R_{n-1}(a) \\ R_{-2}(1) & R_{-1}(1) & R_1(1). & . & . & . & R_{n-1}(1) \\ R_{-2}(2) & R_{-1}(2) & R_1(2). & . & . & . & R_{n-1}(2) \\ R_{-2}(3) & R_{-1}(3) & R_1(3). & . & . & . & R_{n-1}(3) \\ . & . & .. & . & . & . & . \\ . & . & .. & . & . & . & . \\ . & . & .. & . & . & . & . \\ R_{-2}(n-1) & R_{-1}(n-1) & R_1(n-1). & . & . & . & R_{n-1}(n-1) \\ R_{-2}(b) & R_{-1}(b) & R_1(b). & . & . & . & R_{n-1}(b) \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_1 \\ . \\ . \\ . \\ . \\ . \\ C_{n-1} \end{bmatrix} = \begin{bmatrix} d_1 \\ F(1) \\ F(2) \\ F(3) \\ . \\ . \\ . \\ F(n-1) \\ d_2 \end{bmatrix} \tag{20}$$

It is in the form of $[R][C] = [F]$

The matrix $[R]$ is diagonally dominated square matrix of size (n+1) because of local support of basis functions. So that the system of equations are easily solved for arbitrary constants C_i's.

We have

$$[C] = [F][R]^{-1} \tag{21}$$

The approximate solution becomes as known solution is obtained. Now the final approximation solution obtained by substituting these constants in equation(2). This approximate solution is used to evaluate the field variable at each node(Collocation point) in the considered domain. The exact solution is also evaluated at these points and the result values are compared with each other to find out the accuracy of the NURBS Collocation Method.

4. Test Problem:

A numerical example is considered to study the efficiency and convergence of the Collocation Method.

Consider a loaded typical concrete pier of a bridge with varying cross section, to study the variation of displacement along the pier. the geometry and loads of a pier are shown in figure 3[J.N. Reddy *et al*, 2005]. The load 20(kN/m^2) represents the weight of the bridge and an assumed distribution of traffic on the bridge. The concrete weighs approximately 25(kN/m^3) and its modulus is E=28*10^6(kN/m^2),the displacement along the pier at different points is analyzed.

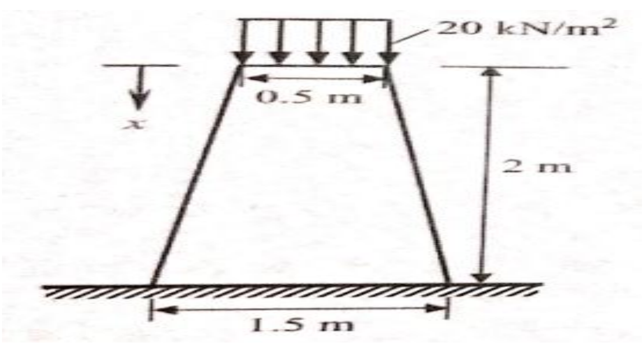

Figure 3: A typical concrete pier

Governing equation is

$$\frac{d^2u}{dx^2} + \frac{1}{(1+x)}\frac{du}{dx} = \frac{25}{E} \quad \text{for} \quad 0 \le x \le 2 \tag{22}$$

With boundary conditions u(2)=0, $\left[\frac{(1+x)E}{4}\frac{du}{dx}\right]_{x=0} = -5$ and having a exact solution

$$u(x) = \frac{56.25 - 6.25(1+x)^2 - 7.5\ln\left[\frac{1+x}{3}\right]}{E} \tag{23}$$

Comparing the given differential equation with equation (1), we have

F(x)=25/E, a=0, b=2 and d_1=0, d_2=−20/(E(1+x)), P(x)=1/(1+x).

Taking the approximation function from the equation (2), it can be written as

$$U^h(x) = \sum_{i=-2}^{n-1} C_i R_{i,p}(x)$$

Taking number of intermittent segments (or sub domains) as 11 (i.e. n=11), order of NURBS curve as 3 (i.e. p=3). X={a=X_0=0, X_1, X_2, …. X_{n-1}, X_n=b} with non-uniform values between [a b],for a homogeneous coordinates(weights) h_1=1.12, h_i=1,for i≠1 and knot vector having 15 elements or knot values. Now the above equation can be modified as

$$U^h(x) = \sum_{i=-2}^{10} C_i R_{i,3}(x) \tag{24}$$

Substituting the approximation function in governing equation, we have

$$\sum_{i=-2}^{10} C_i R''_{i,3}(x) + \frac{1}{1+x}\sum_{i=-2}^{10} C_i R'_{i,3}(x) = \frac{25}{E} \tag{25}$$

Knot vector is x_i={0, 0.0093, 0.1564, 0.2133, 0.8854, 1.5498, 1.6346, 1.7374, 1.9238, 1.9923, 2}.

By solving the set of equations we get the constants C_i where i=0,1,2,....,9 and by substituting these constant values in approximation solution equation then we get the final solution for the given problem.

Now the final approximation solution is evaluated at each node (Collocation point)i.e. x_i={0, 0.0093,0.1564,0.2133, …,1.9923} and the values field variable u(x) at each node are calculated. The exact solution also evaluated at these points and result values of field variable u(x) are compared with each other to find out the accuracy of the NURBS Collocation Method and shown in table 1, and by increasing the degree of NURBS basis from second to third degree (i.e., order from p=3 to 4) the accuracy of method increases, result obtained is compared with exact and second degree solution. The values are tabulated as below in table 2.

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Table1:Comparison of field variable (u) with exact solutions for knot vector x_i ={0,0.0093,0.1564,0.2133,..... ..,109923,2} and weights h_i={1.12,1,1,1....1,1}

Node (Knot values)	Exact Solution ($*10^{-5}$)	NURBS Collocation Method Solution (unequal weights) ($*10^{-5}$)
0	0.2080	0.2016
0.0093	0.2073	0.2009
0.1564	0.1966	0.1903
0.2133	0.1923	0.1860
0.8854	0.1340	0.1292
1.5498	0.0601	0.0583
1.6346	0.0494	0.0480
1.7374	0.0361	0.0350
1.9238	0.0108	0.0105
1.9923	0.0011	0.0011
2.0	0.0	0.0

Table 2: Comparison of field varible u(x) with exact and NURBS Collocation Method for second ((p-1)=2)and third((p-1)=3) degree basis.

Node (Knot Values)	Exact Sol ($*10^{-5}$)	NURBSCM Sol. with unequal weights, for p=3(second degree) ($*10^{-5}$)	NURBSCM sol. with unequal weights, for p=4(third degree) ($*10^{-5}$)
0	0.2080	0.2016	0.2073
0.0093	0.2073	0.2009	0.2067
0.1564	0.1966	0.1903	0.1959
0.2133	0.1923	0.1860	0.1916
0.8854	0.1340	0.1292	0.1334
1.5498	0.0601	0.0583	0.0599
1.6346	0.0494	0.0480	0.0492
1.7374	0.0361	0.0350	0.0359
1.9238	0.0108	0.0105	0.0107
1.9923	0.0011	0.0011	0.0011
2.0	0	0	0

From the tables, it can be stated that the values of the field variable obtained by NURBS Collocation Method using unequal weights are nearer to the Exact solution values and by increasing the degree of the NURBS basis the accuracy of collocation method increases. Thus maximum modulus error is constantly decreasing as the degree of the basis functions increased.

The following figures compares the approximate solution obtained by NURBS Collocation Method with exact solution calculated at same collocation points.

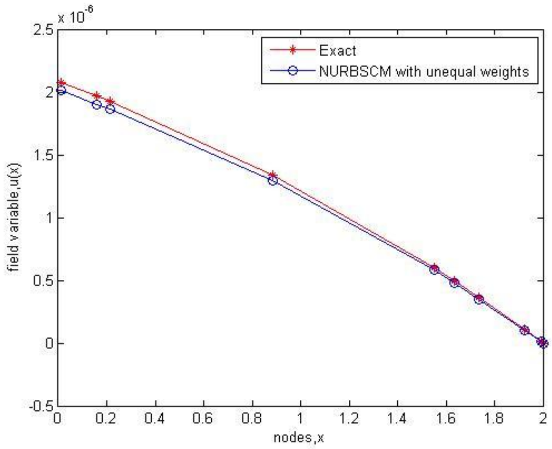

Figure4: Comparison of field variable (u) with exact solutions for non-uniform knot spacing

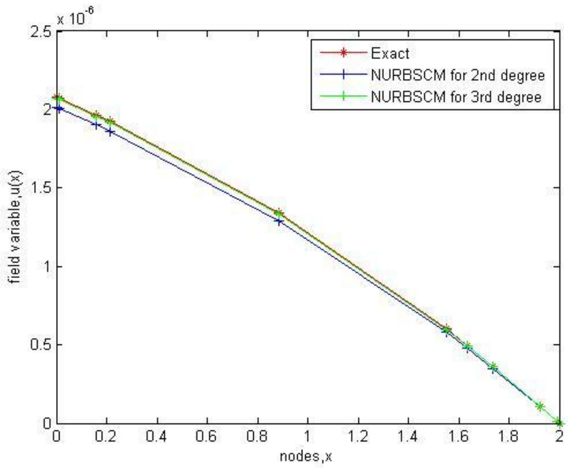

Figure5: Comparison of field varible u(x) with exact and NURBS Collocation Method for second and third degree basis for unequal weights

5. CONCLUSIONS:

In this work, an attempt is made to use the NURBS basis functions as the shape functions in the finite element method.NURBS basis functions are defined recursively and incorporated in the collocation method. The accuracy and efficiency of the present method is illustrated by a structural test problem. The NURBS Collocation Method solution is compared with exact solution and found to be in best fit approximation.

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