Numerical Solution of Second order Integro-Differential Equations(Ides) with Different Four Polynomials Bases Functions

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Abstract: - **In this paper, a method based on the collocation methods with some bases functions are developed to find the numerical solution of Fredholm Integro-Differential Equations; four different polynomial bases functions used were: Legendary, Leguerre, Hermite and Fibonacci polynomial bases functions. The differential part appearing in the integro-differential equation is re-defined and used to generate each of the polynomial bases functions. Some numerical results are given to demonstrate the superior performance of the various collocation methods, particularly, the table of error with the various value of N.**

Keywords: Collocation method, Integro-differential equations (IDEs), Chebyshev, Hermite, Fibonacci and Leguerre Polynomial.

1. INTRODUCTION

Integro-differential equations (IDEs) plays an important role in many branches of linear and Non-linear functional analysis. In addition to their occurrence in the field of mechanics and mathematical physics, Integro-differential equations have found wide applications in the theory of engineering, chemistry, astronomy, biology, economics potential theory and electro-static.

This paper concerns the developments of the various polynomial bases function (see Ortiz [15] and Ortiz and Samara [17]) with Legendre, Leguerre, Fibonacci and Hermite polynomials for the numerical solution of integro-differential equations (IDEs). The polynomials has found extensive application in recent years presented in a series of papers, for example, in [2-8] for the case of numerical solution of ordinary differential equations (ODEs) and in [4,9,10] for the case of numerical solution of partial differential equations (PDEs). Application of the Chebyshev and Legendre polynomials and their numerical merits in solving ODEs and PDEs numerically have been discussed in a series of papers (for example, [2-13]). We are, therefore, motivated to work in this direction of extending to various polynomials proposed in the literature to handle IDEs numerically [13-17].

Yalcinbas and Sezer [19] proposed an approximating solution in terms of Taylor polynomials which we believe it is a particular case of the method presented in this paper. Also, this paper is organized as follows. In section 2, the formulation of the various polynomials bases are developed (such as: Fibonacci, Hermite, Legurre and Legendry); the matrix representations of each part of IDE and its supplementary conditions are obtained. In section 3, as efficient Tau error estimator is introduced. In section 4, preliminary steps towards construction of various polynomial bases functions were considered and taking Chebyshev polynomial as an example. Finally, in section 5, some numerical results are provided to demonstrate the efficiency of using various polynomials bases and compared with those of [19].

2. CONVERTING INTEGRO-DIFFERENTIAL EQUATION TO A SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

$$L := \sum_{n=0}^{v} g_n(x) \frac{d^n}{dx^{-n}}, \qquad 2.1$$

We shall write for $g_n(x)$

$$g_n(x) := \sum_{j=0}^{\infty} g_{nj} x^j, \qquad 2.2$$

Where α_n is the degree of $g_n(x)$ and $\underline{\underline{g}}_n = (g_{n0},...g_{n\alpha},0,0,...)$, $\underline{X} = (1,x,x^2,...)^T$.

Unless otherwise stated, x will always be the independent variable of the functions which appear throughout this paper and will be defined in a finite interval.

Let $y(x)$ be the exact solution of the integro-differential equation,

$$Ly(x) - \lambda\int_a^b m(x,t)y(t)dt = f(x), \qquad x \in [a,b] \qquad 2.3$$

With

$$\sum_{m=1}^{v}\left[c_{jm}^{(1)}y^{(m-1)}(a) + c_{jm}^{(2)}y^{(m-1)}(b)\right] = d_j, \quad j = 1,...,v, \qquad 2.4$$

Where $f(x)$ and $m(x,t)$ are given continuous functions and $\lambda, a, b, c_{jm}^1, c_{jm}^2$ and d_j some given constants.

Matrix representation for the different parts

Let $\underline{V} := \{v_0(x), v_1(x),...\}$ be a polynomial basis by $\underline{V} := V\underline{X}$, where V is a non-singular lower triangular matrix and degree $(v_i(x)) \le i$, for $i = 0,1,2,....$. Also for any matrix P, $P_v = VPV^{-1}$.

Now we convert the Eqs. (2.3) and (2.4) to the corresponding linear algebraic equations in three parts; (a), (b) and (c).

(a) Matrix representation for Dy(x):

Ortiz and Samara proposed in [17] an alternative for the Tau technique which they called the operational approach as it reduces differential problems to linear algebraic problems. The effect of differentiation, shifting and integration on the coefficients vector

$$\underline{\underline{\tilde{a}}}_n := (\tilde{a}_0, \tilde{a}_1,....,\tilde{a}_n,0,0,...)$$

Of a polynomial $u_n(x) = \underline{\underline{\tilde{a}}}_n\underline{X}$ is the same as that of post-multiplication of $\underline{\underline{\tilde{a}}}_n$ by the matrices η, μ and i respectively,

$$\frac{du_n(x)}{dx} = \underline{\underline{\tilde{a}}}_n\eta\underline{X}, \qquad u_n(x) = \underline{\underline{\tilde{a}}}_n\mu\underline{X} \qquad \text{and} \qquad \int_0^x u_n(t)dt = \underline{\underline{\tilde{a}}}_n i\underline{X}$$

where

$$\eta = \begin{bmatrix} 0 & 0 & ... & ... \\ 1 & 0 & 0 & ... \\ 0 & 2 & 0 & ... \\ ... & ... & ... & ... \end{bmatrix}, \mu = \begin{bmatrix} 0 & 1 & ...... \\ 0 & 0 & 1\,... \\ ... & ... & ...... \end{bmatrix}, i = \begin{bmatrix} 0 & 1 & ...\,... \\ 0 & 0 & \tfrac{1}{2}\,... \\ ... & ... & ...\,... \end{bmatrix}.$$

We recall now the following theorem given by Ortiz and Samara [17].

Theorem 2.1 For any linear differential operator L defined by (2.1) and any series

$$y(x) := \underline{\underline{a}}\underline{V}, \qquad \underline{\underline{a}} := (a_0, a_1, a_2, \ldots),$$

we have

$$Ly(x) = \underline{\underline{\tilde{a}}} \prod \underline{X} = \underline{\underline{a}} \prod v \underline{V}, \tag{2.5}$$

where

$$\prod = \sum_{i=0}^{v} \eta^i g_i(\mu),$$

and

$$\prod{}_v = V \prod V^{-1}.$$

(b) Matrix representation for the integral term:

Let us assume that

$$m(x,t) = \sum_{i=0}^{n} \sum_{j=0}^{n} m_{ij} v_i(x) v_j(t),$$

and

$$y(x) = \sum_{i=0}^{\infty} a_i v_i(x) = \underline{\underline{a}}\underline{V}.$$

Then, we can write

$$\int_a^b m(x,t) y(t) dt = \sum_{1=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} m_{ij} a_i v_i(x) \int_a^b v_j(t) v_i(t) dt = \underline{\underline{aM}}\underline{V}, \tag{2.6}$$

where

$$\underline{\underline{M}} \begin{bmatrix} \sum_{j=0}^{n} m_{0j} \alpha_{jo} & \ldots & \sum_{j=0}^{n} m_{nj} \alpha_{jo} & 0 & 0 & \\ . & & . & . & . & \ldots \\ \sum_{j=0}^{n} m_{0j} \alpha_{jn} & \ldots & \sum_{j=0}^{n} m_{nj} \alpha_{jn} & 0 & 0 & \ldots \\ . & & . & . & . & \ldots \end{bmatrix}$$

with

$$\alpha_{ji} = \int_a^b v_j(t) v_1(t), \qquad \text{for} \qquad \text{j}, \; l = 0, \ldots, n.$$

(c) Matrix representation for the supplementary conditions:

Replacing $y(x) = \sum_{i=0}^{\infty} a_i v_i(x)$ in the left hand side of (2.4), it can be written as

$$\sum_{m=1}^{v}\left[c_{jm}^{(1)}y^{(m-1)}(a)+c_{jm}^{(2)}y^{(m-1)}(b)\right]=\sum_{i=0}^{\infty}\sum_{m=1}^{v}\left[c_{jm}^{(1)}v_i^{m-1}(a)+c_{jm}^{(2)}v_i^{(m-1)}(b)\right]=\underline{\underline{a}}B_j, \qquad 2.7$$

where for j = 1, …,v,

$$B_j=\begin{bmatrix} c_{jm}^{(1)}v_0(a)+c_{jm}^{(2)}v_0(b) \\ \sum_{m=1}^{2}\left[c_{jm}^{(1)}v_1^{m-1}(a)+c_{jm}^{(2)}v_1^{(m-1)}(b)\right] \\ \vdots \\ \sum_{m=1}^{v}\left[c_{jm}^{(1)}v_{v-1}^{(m-1)}(a)+c_{jm}^{(2)}v_{v-1}^{(m-1)}(b)\right] \\ \vdots \end{bmatrix} \qquad 2.8$$

We refer to B as the matrix representation of the supplementary conditions and B_j as its jth column. The following relations for computing the elements of the matrix B can be deduced from (2.7):

$$b_{ij}=\sum_{m=1}^{v}\left[c_{jm}^{(1)}v_{v-1}^{(m-1)}(a)+c_{jm}^{(2)}v_{v-1}^{(m-1)}(b)\right], \quad \text{for } i,j=1,2,...,v, \qquad 2.9$$

and

$$b_{ij}=\sum_{m=1}^{v}\left[c_{jkm}^{(1)}v_{v-1}^{(m-1)}(a)+c_{jm}^{(2)}v_{v-1}^{(m-1)}(b)\right], \quad \text{for } i=v+1,v+2,..., j=1,2,...,v. \qquad 2.10$$

We introduce $\underline{\underline{d}}=(d_1,d_2,...,d_v)$, the vector that contains right hand sides of conditions. Then the supplementary conditions take the form

$$\underline{\underline{a}}B=\underline{\underline{d}}. \qquad 2.11$$

It follows from (2.5) and (2.6) that

$$Ly(x)-\lambda\int_a^b m(x,t)y(t)d=\underline{\underline{a}}(\prod{}_v-\lambda M)\underline{V}. \qquad 2.12$$

Let $M_v:=\prod{}_v-\lambda\underline{\underline{M}}$ and M_{vi} stands for its ith column and let $f(x)=\sum_{i=0}^{n}f_i v_i(x)=f\underline{V}$ with $f=(f_0,...,f_n,0,0,...)$.. Then the coefficient of exact solution $y=\underline{\underline{a}}\underline{V}$ of problem (2.3) and (2.4) satisfies the following infinite algebraic system:

$$\begin{cases} \underline{\underline{a}}M_{vi}=f_i; & i=0,...,n, \\ \underline{\underline{a}}M_{vi}=0; & i\geq n+1, \\ \underline{\underline{a}}B_j=d_i; & j=1,2,...,v. \end{cases} \qquad 2.13$$

setting

$$G=(B_1,...,B_v,M_{v0},M_{v1},...),$$

and

$$g=(d_1,...,d_v,f_0,f_1,...),$$

we can write instead of (2.13)

$$\underline{\underline{a}}G = g. \qquad 2.14$$

Definition 2.2. The polynomial

$$y_n = \underline{\underline{a}}_n \underline{V}$$

will be called an approximate solution of (2.3) and (2.4), if the vector

$$\underline{\underline{a}}_n = (a_0, \ldots, a_n)$$

is the solution of the linear algebraic equations.

$$\underline{\underline{a}}_n G_n = g_n. \qquad 2.15$$

Where G_n is the matrix defined by restriction of G to its first (n + 1) rows and columns.

Remark 2.3. For $v = 0$ and $G_0(x) = 1$, Eq. (2.3) is transformed into a Fredholm integral equation of second kind and for $\lambda = 0$, it is transformed into a Differential Equation.

3. ERROR ESTIMATION

In this section an error estimator for the approximate solution of (2.3) and (2.4) is obtained. Let us call $e_n(x) = y(x) - y_n(x)$ as the error function of the approximate solution $y_n(x)$ to $y(x)$, where, $y(x)$ is the exact solution of (2.3) and (2.4). Hence $y_n(x)$ satisfies the following problem:

$$Ly_n(x) - \lambda \int_a^b m(x,t) y_n(t) dt = f(x) + H_n(x), \qquad x \in [a,b] \qquad 3.1$$

with

$$\sum_{m=1}^{v} \left[c_{jm}^{(1)} y^{(m-1)}(a) + c_{jm}^{(2)} y^{(m-1)}(b) \right] = d_j, \qquad j = 1, \ldots, v. \qquad 3.2$$

The perturbation term $H_n(x)$ can be obtained by substituting the computed solution $y_n(x)$ into the equation

$$H_n(x) = Ly_n(x) - \int_a^b m(x,t) y_n(t) dt - f(x). \qquad 3.3$$

We proceed to find an approximation $e_{n,N}(x)$ to the error function $e_n(x)$ in the way as we did before for the solution of problem (2.3), (2.4). Subtracting (3.1) and (3.2) from (2.3) and (2.4) respectively, the error function $e_n(x)$ satisfies the problem

$$Le_n(x) - \lambda \int_a^b m(x,t) e_n(t) dt = -H_n(x), \qquad x \in [a,b] \qquad 3.4$$

with the homogeneous conditions

$$\sum_{m=1}^{v}\left[c_{jm}^{(1)}e_n^{(m-1)}(a)+c_{jm}^{(2)}e_n^{(m-1)}(b)\right]=0, \qquad j=1,\dots,v. \qquad 3.5$$

It should be noted that in order to construct the approximant $e_{n,N}(x)$ to $e_n(x)$, only the right hand side of system (2.15) needs to be recomputed; the structure of the coefficient matrix G_n remains the same.

4. CONSTRUCTION OF VARIOUS POLYNOMIAL BASES FUNCTION

In section 2, $\underline{V}:=\{v_0(x),v_1(x),\dots\}$ was considered as a polynomial basis given by $\underline{V}:=V\underline{X}$, where V is a non-singular lower triangular matrix and degree $(v_i(x))\le i,=0,1,2,\dots$ It was used to convert (2.3) and (2.4) into a system of linear equations. The various polynomials are very interesting polynomial basis with a matrix V of the same structure. We argue the application of the Tau method for the case of Chebyshev polynomials, though the case of Lengendre, Leguerre, Hermite and Fibonacci may also be similar. But different recursive formulae will be employed.

For instance, the shifted chebyshev polynomials are defined as

$$T_0^{(*)}(x)=1, \qquad T_0^{(*)}(x)=\frac{2x-(b+a)}{b-a}, \qquad x\in[a,b]$$

and for $i\ge 1$

$$T_{i+1}^{(*)}(x)=2\left(\frac{2x-(b+a)}{b-a}\right)T_i^{(*)}(x)-T_{i-1}^{(*)}(x), \qquad x\in[a,b]$$

In this case the functions *m(x,t)*, *f(x)* and the approximate solution $y_n(x)$ are written as

$$m(x,t)=\sum_{i=0}^{n}{}''\sum_{i=0}^{n}{}'' m_{ij}T_i^{(*)}(x)T_j^{(*)}(x),$$

$$f(x)=\sum^{n}{}'' f_iT_i^{(*)}(x),$$

and

$$y_n(x)=\sum^{n}a_iT_i^{(*)}(x),$$

Where m_{ij} and f_i are computed by the following relations and $a_0,a_1,\dots,a_n$ are obtained from (2.11),

$$m_{ij}=\frac{4}{n^2}\sum_{t=0}^{n}{}''\sum_{l=0}^{n}{}'' m\left(\frac{b-a}{2}x_t+\frac{b+a}{2},\frac{b-a}{2}x_1+\frac{b+a}{2}\right)\cos\left(\frac{it\pi}{n}\right)\cos\left(\frac{jl\pi}{n}\right),$$

$$f_i=\frac{2}{n}\sum_{m=0}^{n}{}'' m\left(\frac{b-a}{2}x_t+\frac{b+a}{2}\right),$$

For $i, j = 0,1,\ldots,n$ and

$$x_i = \cos\left(\frac{i\pi}{n}\right).$$

A summation symbol with double primes denotes a sum first and last terms halved. Also a_{ijs} (see part (b) section 2), are computed easily in this case, since

$$a_{jl} = \int_a^b T_j^{(*)}(t) T_l^{(*)}(t) dt = \int_a^b T_j\left(\frac{2t-(b+a)}{b-a}\right) T_l\left(\frac{2t-(b+a)}{b-a}\right) dt$$

$$= \frac{b-a}{2}\int_{-1}^{1} T_j(t) T_l(t) dt,$$

Where T_j and T_l are Chebyshev polynomial of degree j and l respectively. But these polynomials are even for even degree and odd for odd degree, hence

$$a_{jl} = \begin{cases} 0; & if \quad j+1 \quad odd \\ \dfrac{b-a}{2}\left(\dfrac{1}{1-(j+l)^2} + \dfrac{1}{1-(j-l)^2}\right); & \text{if } j+l \text{ even} \end{cases}$$

All other elements are computed as in Section 2. The same application will definitely adopted for the other Polynomials when the recursive formulae were introduced to arrive at the approximate solution of the various polynomial bases function.

5. NUMERICAL EXAMPLES

In this section, we consider some examples demonstrating the accuracy of the method and effectiveness of the Chebyshev polynomial basis function compared with the other various polynomial bases functions. Hence, error of examples 1 and 2 are also compared with [19]

Example 1. This example was considered by Yalcinbas and Sezer [19] for the method of solution in terms of Taylor polynomials.

$$y''(x) + xy'(x) - xy(x) = e^x - 2\sin(x) + \int_{-1}^{1} \sin(x) e^{-t} y(t) dt, \qquad -1 \le x \le 1$$

$$y(0) = 1$$

$$y'(0) = 1$$

The exact solution is $y(x) = e^x$.

Table 1: Tables of errors when N = 4

X	CHEBYSHEV	LEGENDARY	LAGUERRE	HERMITE	FIBONACCI	Yalcinbas et al. [19]
-1.00	0.36787942	3.42E-01	3.51E-01	6.65E-01	0	0.367879
-0.80	0.44932896	3.54E-01	9.96E-02	1.80E-02	8.89E-05	0.449328
-0.60	0.54881164	2.54E-01	4.27E-01	1.75E-01	6.75E-04	0.548811
-0.40	0.67032004	1.36E-01	6.68E-01	4.63E-01	2.15E-03	0.67032
-0.20	0.81873075	3.97E-02	7.76E-01	9.50E-02	4.80E-03	0.81873
0	1	0.00E+00	0.00E+00	7.00E-10	8.75E-03	1
0.20	1.22140276	3.55E-04	1.16E+00	5.48E-02	1.40E-02	1.2214
0.40	1.49182469	2.56E-02	1.39E+00	3.01E-01	2.03E-02	1.49182
0.60	1.82211880	1.16E-01	1.60E+00	5.52E-01	2.75E-02	1.82211
0.80	2.22554092	3.21E-01	1.78E+00	2.06E-01	3.48E-02	2.22554
1.00	2.71828180	6.92E-01	1.90E+00	1.75E+00	4.15E-02	2.71828

Table 2: Tables of errors when N = 5

X	CHEBYSHEV	LEGENDARY	LAGUERRE	HERMITE	FIBONACCI	Yalcinbas et al. [19]
-1	2.12E-08	1.54E-03	1.42E-03	5.18E-03	0	4.41E-07
-0.8	4.10E-09	2.43E-03	6.29E-04	8.86E-04	5.90E-07	9.64E-07
-0.6	3.90E-09	3.61E-03	7.42E-03	9.95E-03	3.57E-06	6.36E-07
-0.4	6.00E-09	4.75E-03	8.33E-03	2.96E-03	4.83E-06	4.60E-08
-0.2	3.10E-09	7.04E-04	8.55E-03	9.82E-04	2.01E-06	7.53E-07
0	0.00E+00	0.00E+00	0.00E+00	0.00E+00	4.94E-06	0.00E+00
0.2	2.00E-09	4.77E-06	5.08E-02	4.85E-04	4.93E-05	2.76E-06
0.4	8.00E-09	5.95E-04	6.30E-02	1.15E-03	3.46E-05	4.70E-06
0.6	0.00E+00	4.95E-03	1.53E-02	1.70E-03	4.58E-05	8.80E-06
0.8	8.00E-09	5.51E-03	6.64E-02	5.78E-03	7.65E-05	9.28E-07
1	2.80E-08	4.82E-03	7.82E-02	3.41E-03	5.13E-05	1.83E-06

From the numerical results and Table of error (see Tables 1, 2). In Table 2, it is evident that better performance provided by the various method here proposed compared with the results of Yalcinbas and Sezer [19].Hence, it was observed that the Legendre and Hermite solutions have same errors over the interval. The error of the Fibonacci solution shows a tendency to increase rapidly, as N increases.

Example 2. (see [19])

$$y''(x) + \int_0^{\pi/2} xty(t)dt = x - \sin(x), \qquad 0 \le x \le \frac{\pi}{2}$$

$$y(0) = 0$$

$$y'(0) = 1.$$

The exact solution is $y(x) = \sin(x)$.
For numerical results see table 3 and 4

Table 3: Tables of errors when n = 4

X	CHEBYSHEV	LEGENDARY	LAGUERRE	HERMITE	FIBONACCI	Yalcinbas et al.
0	8.79E-05	3.75E-05	5.13E-04	2.00E-04	5.00E-03	4.41E-04
0.1	4.32E-04	9.98E-02	5.41E-03	9.81E-02	4.56E-03	9.64E-04
0.2	2.90E-02	1.99E-01	3.95E-02	1.96E-01	8.56E-02	3.60E-03
0.3	8.26E-02	2.90E-01	6.54E-02	2.92E-01	7.45E-02	5.46E-03
0.4	3.49E-02	3.82E-01	5.63E-02	3.84E-01	4.60E-02	7.53E-03
0.5	3.40E-02	4.69E-01	8.23E-02	4.73E-01	3.10E-02	0.00E+00
0.6	5.68E-02	5.52E-01	8.12E-02	5.58E-01	9.71E-02	2.76E-03
0.7	5.17E-02	6.28E-01	8.54E-02	6.36E-01	4.55E-02	4.70E-03
0.8	7.60E-02	6.98E-01	3.48E-01	7.08E-01	4.84E-02	8.85E-03
0.9	1.57E-02	7.60E-01	1.23E-01	7.73E-01	7.28E-02	9.28E-03
1	6.68E-02	8.14E-01	3.75E-01	8.30E-01	5.89E-02	1.83E-03
π/2	9.22E-02	9.38E-01	9.34E-01	9.77E-01	1.68E-02	1.12E-03

Table 2: Tables of errors when n = 5

X	CHEBYSHEV	LEGENDARY	LAGUERRE	HERMITE	FIBONACCI	Yalcinbas et al.[18]
0	1.93E-06	0.00E+00	2.12E-04	0.00E+00	9.96E-05	5.44E-06
0.1	5.41E-06	1.10E-05	4.10E-04	6.62E-05	2.46E-05	1.96E-06
0.2	5.39E-05	3.30E-05	3.90E-05	6.93E-05	8.86E-05	6.36E-05
0.3	5.60E-05	2.50E-04	6.00E-05	2.53E-04	6.75E-04	5.46E-06
0.4	3.31E-06	1.50E-04	3.10E-05	4.22E-05	5.46E-04	4.75E-06
0.5	7.53E-05	8.60E-05	2.00E-05	8.63E-05	4.31E-04	9.80E-05
0.6	9.82E-07	4.80E-04	8.12E-05	1.52E-04	5.97E-04	2.28E-05
0.7	6.88E-06	1.12E-04	8.00E-05	1.17E-04	7.45E-04	5.47E-05
0.8	7.65E-05	9.66E-04	3.48E-04	9.63E-04	4.80E-04	5.88E-05
0.9	7.58E-06	5.85E-04	2.12E-04	3.92E-06	7.24E-04	4.93E-06
1	6.28E-05	4.72E-04	7.38E-05	1.28E-04	6.59E-04	9.18E-05
π/2	1.37E-06	8.26E-04	7.93E-04	1.37E-03	2.16E-04	7.11E-06

The table shows that the results of the polynomial bases are numerically stable

6. CONCLUSIONS

Our results indicate that the tau method with various polynomial bases can be regarded as a structurally simple algorithm that is conventionally applicable to the numerical solution of IDEs. In addition, although we have restricted our attention to linear Fredholm IDEs, we expect the method to be easily extended to more general IDEs.

Despite the relatively low degrees used the numerical results show the superior performance of the Tau method, particularly, with the Chebyshev and Legendre bases. Nevertheless, the error of the Tau solution shows a tendency to increase rapidly, as N increases. This behaviour also indicated in other polynomial/methods.

REFERENCES

1. Crisci M. R. (1992) "Stability Results for one step Discretized Collocation Methods in the Numerical Treatment of Volterra Integral Equations". Math. Comput. 58 (197) pp. 119-134.
2. Chuong N. M. and Tuan, N.V. (1995) "Spline Collocation Methods for Fredholm Integro- Differential Equations of Second Order", Acta Math. Vietnamica 20 (1) pp. 85-98.
3. Chuong N. M. and Tuan N. V (1997) "Spline Collocation Methods for Fredholm- Volterra integro-Differential Equations of High Order", Vietnam J. Math. 25 (1) pp. 15-24.
4. El-Daou M. K. and Khajah H. G (1997) "Iterated Solutions of Linear Operator Equations with the Tau Method", Math. Comput. 66 (217) pp. 207-213.
5. Gottlieb D. and Orszag S. A (1986) "Numerical Analysis of Spectral Methods", SIAM, Philadephia, 4th print.
6. Hosseini Aliabadi M. and Ortiz, E. L. (1987) "On the Numerical Behavior of different formulations of Tau Method for the treatment of Differential Inclusions", Proceedings of the Second International Symposium on Numerical Analysis, Prague.
7. Hosseini Aliabadi M. and Ortiz, E. L. (1998) "Numerical treatment of Moving and Free Boundary Value Problems with the Tau Method", Computing Mathematical Application 35 (8) pp. 53-61.
8. Hosseini Aliabadi M. and Ortiz, E. L. (1988) "Numerical Solution of Feedback Control Systems Equations". Appl. Mathematics Lett. 1 (1) pp. 3-6.
9. Hosseini Aliabadi M. and Ortiz, E. L. (1991) "A Tau Method based on Non-Uniform Space Time Elements for the Numerical Simulation of Solitons", Comput. Math. Appl. 22 (9) pp. 7-19.
10. Hosseini Aliabadi M.(2000)"The Buchstab's function and the Operational Tau Method", Korean J. Comput. Appl. Math. 7 (3) pp. 673-683.
11. Hosseini Aliabadi M.(2000) "The Application of the Operational Tau Method on some Stiff System of ODEs", Int. Journal of Appl. Math. 2(9) pp. 1027-1036.
12. Hosseini Aliabadi M. (2000) "Solving ODE BVPs using the Perturbation term of the Tau Method over semi-infinite Intervals", Far East J. Appl. Math. 4 (3)pp. 295-303.
13. Kaniko H. and Xu Y. (1994) "Gauss-type quadratures for Weekly Singular Integrals and their application to Fredholm Integral Equations of Second kind", Math. Comput. 62 (206) pp. 739-753.
14. Liu, K. M. and Pan C. K (1999) "The Automatic Solution System of Ordinary Differential Equations by the Tau Method", Comput. Math. Appl. 38 pp. 197-210.
15. Ortiz, E. L. (1969) "The Tau Method", SIAM Journal of Numerical Analysis pp 480-492.
16. Ortiz, E. L and Samara L. (1981) "An Operational approach to the Tau Method for the Numerical Solution of Nonlinear Differential Equations", Computing 27, pp 15-25.
17. Ortiz, E. L. and Samara H. (1984) "Numerical Solution of Partial Differential Equations with Variable Coefficients with an Operational approach to the Tau Method", Compt. Math. Appl. 10 (1) 5-13.
18. Wazwaz, A. M. (1997) A first course in Integral Equations, World Scientific Pub. Co.
19. Yalcinbas S. and Sezer M.(2000)The Approximate Solution of High-Order Linear Volterra- Fredholm Integro-Differential Equations in terms of Taylor Polynomials, Appl. Math. Comput. 112 pp. 291-308.