Nonlinear Free Vibration Analysis of Functionally Graded Materials Spherical Shell using Higher Order Shear Deformation Theory

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***Abstract*—To investigate nonlinear free vibration response of functionally graded materials (FGMs) spherical shell. It is subjected to uniform and non-uniform temperature changes with temperature independent (TID) material properties. The basic formulation is based on higher order shear deformation theory (HSDT) with Von-Karman nonlinear strains using modified C^0 continuity. A direct iterative based nonlinear finite element method combined with mean centered first-order perturbation technique (FOPT) proposed for the Functionally Graded Materials plate is extended for spherical shell subjected to thermo-mechanical loading. The present outlined approach will be validated with those available in literature.**

***Index Terms*— FGMs, Nonlinear free vibration, HSDT.**

INTRODUCTION

Laminated composite materials which have a strong discontinuity of mechanical properties across the interfaces of two layers cause several problems. An advantage of FGMs over laminated composites is that material properties vary continuously and smoothly through the thickness from one surface to other surface. This is achieved by continuously varying the volume fraction of constituent materials. These classes of materials can survive environment with high temperature gradient while maintaining the desired structural integrity. Hence it can be used in many applications like plasma facing for nuclear reactor, wear resistant lining in mineral processing industry, rocket heat shields, thermoelectric generators, dental implantation, and bone replacement, and electrically insulating metal/ceramic joints. A large number of literatures have been reported on linear and nonlinear free vibration of plates. K. R. Jagtap and Achchhe Lal [1] investigated effect of random material properties on free vibration response of functionally graded materials plate with cutouts in thermal environment by using higher order shear deformation theory with C^0 continuity K. R. Jagtap et al. [2] investigated stochastic nonlinear free vibration analysis of functionally graded material plate resting on elastic foundation in thermal environment by using higher order shear deformation theory with von Karman nonlinear strain kinematics with modified C^0 continuity. Achchhe Lal et al. [3] investigated Nonlinear bending response of laminated composite spherical shell panel with system randomness subjected to hygro-thermo-mechanical loading. A direct iterative based nonlinear finite element method combined with mean centered first-order perturbation technique (FOPT) for the plate is extended for the spherical shell panel subjected to hygro-thermo-mechanical loading. Hiroyuki Matsunaga [4] investigated free vibration and stability of functionally graded shallow shells according to a 2D higher-order deformation theory. Shankara CA, Iyenger NGR [5] investigated free vibration of laminated spherical panels with random material properties.

FORMULATION

Consider a FGM shell consist of ceramic and metal at top and bottom layer of length a, width b, and total thickness h.

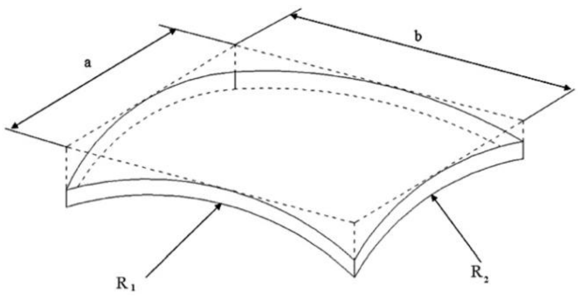

Fig.1 Geometry of spherical shell.

The properties of the FGMs shell are assumed to be varying through the thickness. The effective mechanical and thermal properties of the FGMs shell at an arbitrary point within the shell domain are expressed as [2],

$$
\begin{aligned}
E(z) &= E_b + [E_t - E_b] V_c(z) \\
\alpha(z) &= \alpha_b + [\alpha_t - \alpha_b] V_c(z) \\
\rho(z) &= \rho_b + [\rho_t - \rho_b] V_c(z) \\
k(z) &= k_b + [k_t - k_b] V_c(z)
\end{aligned} \qquad (1)
$$

Where, t and b represents the ceramic and metal constituents, E, α, ρ and k are the effective young modulus, thermal expansion coefficient, density and thermal conductivity VC is the volume fraction index, function of coordinate in the thickness direction (z)

$$V_c(z) = \left(0.5 + \frac{z}{h}\right)^n, \quad \frac{-h}{2} \le z \le \frac{h}{2}, \tag{2}$$

Where, n is volume fraction index and is always positive. For n=0, the shell is fully ceramic and when n=1, the composition of metal and ceramic is linear. The Poisson's ratio ν depends weakly on temperature change and is assumed to be a constant.

A. Displacement field model

Higher order shear deformation theory with C0 continuity has been used to find displacement field model

$$\bar{u} = u(1 + \frac{z}{R_2}) + f_1(z)\phi_x + f_2(z)\theta_x,$$

$$\bar{v} = v(1 + \frac{z}{R_1}) + f_1(z)\phi_y + f_2(z)\theta_y,$$

$$\bar{w} = w. \tag{3}$$

Where $(\bar{u}, \bar{v}, \bar{w})$ denote the displacement of a point along the (X, Y, Z) coordinates axe, (u, v, w) are corresponding displacements of a point on the mid plane, ϕ_x and ϕ_y are the rotations at Z=0 of normal to the mid surface with respect to X and Y axes, θ_x and θ_y are the slopes along X and Y directions $\theta_x = \frac{dw}{dx}$ and $\theta_y = \frac{dw}{dy}$. The function $f_1(z)$ and $f_2(z)$ can be written as,

$$f_1(z) = C_1(z) - C_2(z)^3; \; f_2(z) = -C_4(z)^3,$$

With $C_1 = 1$, $C_2 = C_4 = \frac{4h^2}{3}$

The displacement vector for the modified model can be written as,

$$\{\Lambda\} = [u \; v \; w \; \theta_y \; \theta_x \; \phi_y \; \phi_x]^T \tag{4}$$

B. Strain Displacement Relations

The strain vector consisting of strains in terms of mid-plane deformation, rotations of normal and higher order terms associated with displacement for isotropic layer is,

$$\{\varepsilon\} = \{\varepsilon_l\} + \{\varepsilon_{nl}\} - \{\bar{\varepsilon}_t\} \tag{5}$$

Where, $\{\varepsilon_l\}$, $\{\varepsilon_{nl}\}$ and $\{\bar{\varepsilon}_t\}$ are the linear and nonlinear strain vectors (Von-Karman sense), thermal strain vector respectively. The nonlinear strain vector can be written as [2],

$$\varepsilon_{nl} = \frac{1}{2}[A_{nl}]\{\theta\} \tag{6}$$

Where,

$$A_{nl} = \frac{1}{2} \cdots \begin{bmatrix} w_{,x} & 0 \\ 0 & w_{,y} \\ w_{,x} & w_{,y} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \theta = \begin{Bmatrix} w_{,x} \\ w_{,y} \end{Bmatrix}.$$

The thermal strain vector $\{\bar{\varepsilon}_t\}$ is represented as,

$$\{\bar{\varepsilon}_t\} = \begin{Bmatrix} \bar{\varepsilon}_x \\ \bar{\varepsilon}_y \\ \bar{\varepsilon}_{xy} \\ \bar{\varepsilon}_{yz} \\ \bar{\varepsilon}_{zx} \end{Bmatrix} = \Delta T \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_{12} \\ 0 \\ 0 \end{Bmatrix} \tag{7}$$

Where α_1, α_2 and α_{12} are the coefficient of thermal expansion in the x, y and z directions respectively which can be obtained from the thermal coefficient in the longitudinal α_1 and transverse α_2 directions of the ceramic and metal using the transformation matrix and ΔT is the uniform and nonuniform temperature change. The temperature field for nonuniform temperature change is expressed as,

$$\Delta T = T(z) - T_0 \tag{8}$$

Where $T(z)$ is expressed as,

$$T(z) = T_b + (T_t - T_b)\eta(z).$$

Where $T(z)$ is the temperature distribution along z direction,

T_t And T_b, are temperature of top and bottom surface, and

Parameter $\eta(z)$ is defined as,

$$\eta(z) = \frac{1}{c}\Bigg[\left(0.5 + \frac{z}{h}\right) - \frac{k_{tb}}{(n+1)k_b}\left(0.5 + \frac{z}{h}\right)^{n+1} + \frac{k_{tb}^2}{(2n+1)k_b^2}\left(0.5 + \frac{z}{h}\right)^{2n+1} - \frac{k_{tb}^3}{(3n+1)k_b^3}\left(0.5 + \frac{z}{h}\right)^{3n+1} + \frac{k_{tb}^4}{(2n+1)k_b^4}\left(0.5 + \frac{z}{h}\right)^{4n+1} - \frac{k_{tb}^5}{(2n+1)k_b^5}\left(0.5 + \frac{z}{h}\right)^{5n+1}\Bigg] \tag{9}$$

Where, $c = \left[1 - \frac{k_{tb}}{(n+1)k_b} + \frac{k_{tb}^2}{(2n+1)k_b^2} - \frac{k_{tb}^3}{(3n+1)k_b^3} + \frac{k_{tb}^4}{(2n+1)k_b^4} - \frac{k_{tb}^5}{(2n+1)k_b^5}\right]$

With $k_{tb} = k_t - k_b$ and k is defined as thermal conductivity.

The uniform temperature change Eq. can be written as,

$$T(z) = T_0 + (T_t - T_b) \tag{10}$$

Where, T_0 is initial temperature.

C. Stress strain Relation

The stress strain relation accounting thermal effect can be written as

$$\{\sigma\} = \{\bar{Q}\}\{\varepsilon\} \, or \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}$$

$$= \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} & 0 & 0 \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{Q}_{44} & 0 \\ 0 & 0 & 0 & 0 & \overline{Q}_{55} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix} - \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_{12} \\ 0 \\ 0 \end{Bmatrix} \right\} \quad (11)$$

$$\lambda_1 = \overline{Q}_{11}\alpha_1 + \overline{Q}_{12}\alpha_2 + \overline{Q}_{16}\alpha_{12}$$

Where, $\lambda_2 = \overline{Q}_{12}\alpha_1 + \overline{Q}_{22}\alpha_2 + \overline{Q}_{26}\alpha_{12}$

$$\lambda_{12} = \overline{Q}_{16}\alpha_1 + \overline{Q}_{26}\alpha_2 + \overline{Q}_{66}\alpha_{12}$$

Where $\left\{\overline{Q}_{ij}\right\}$, $\{\sigma\}$ and $\{\varepsilon\}$ are the transformed stiffness matrix, stress and strain vectors for isotropic shell respectively. For FGM material the elastic constants are defined as,

$$\left\{\overline{Q}_{11}\right\} = \left\{\overline{Q}_{22}\right\} = \frac{E(z,T)}{1-\upsilon^2}, \left\{\overline{Q}_{12}\right\} = \frac{\upsilon E(z,T)}{1-\upsilon^2},$$

$$\left\{\overline{Q}_{44}\right\} = \left\{\overline{Q}_{55}\right\} = \left\{\overline{Q}_{66}\right\} = \frac{E(z,T)}{2(1+\upsilon)}.$$

D. Strain energy of the shell

The strain energy of the FGM shell is given by,

$$U = \frac{1}{2}\int_v \{\varepsilon\}^T \{\sigma\}\, dv \quad (12)$$

Above equation can be expanded as,

$$U = \frac{1}{2}\int_A \left\{\overline{\varepsilon}_l\right\}^T [D]\left\{\overline{\varepsilon}_l\right\} dA + \frac{1}{2}\int_A \left\{\overline{\varepsilon}_l\right\}^T [D_3]\{A\}\{\theta\}\, dA + \frac{1}{2}\int_A \left\{\overline{\varepsilon}_l\right\}[D_4]\{A\}^T\{\theta\}^T dA + \frac{1}{2}\int_A \{A\}\{\theta\}[D_5]\{A\}^T\{\theta\}^T dA \quad (13)$$

Where $[D]$, $[D_3]$, $[D_4]$ and $[D_5]$ are shell stiffness matrices and $\left\{\overline{\varepsilon}_l\right\}$ is the linear mid-plane vector. The strain energy function calculated for each element above can be summed to get the total strain energy.

$$U = \sum_{e=1}^{NE} U^e = \sum_{e=1}^{NE} \{\Lambda\}^{(e)^T} [K_l + K_{nl}(\Delta)]^e \{\Lambda\}^{(e)} = \{q\}^T [K_l + K_{nl}(q)]\{q\}$$

(14)

Where $[K_l]$, $[K_{nl}]$ and {q} are defined as global linear, nonlinear stiffness matrix and displacement vector respectively.

E. Work done

Because of uniform and nonuniform temperature change, pre-buckling stresses in FGM shell are generated. The in-plane pre-buckling stress resultant per unit length is reason for buckling. The work done (W) by in-plane stress resultants in producing out of plane displacements 'w' can be expressed as,

$$W = \frac{1}{2}\int_A [N_x(w_{,x})^2 + N_y(w_{,y})^2 + 2N_{xy}(w_{,x})^2(w_{,y})^2]dA = \frac{1}{2}\int_A \begin{Bmatrix} w_{,x} \\ w_{,y} \end{Bmatrix}^T \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \begin{Bmatrix} w_{,x} \\ w_{,y} \end{Bmatrix} dA \quad (15)$$

Where N_x, N_y and N_{xy} are thermal in plane, thermal compressive and stress resultant per unit length. Using finite element method and summing over the entire element above equation can be written as,

$$W = \sum_{e=1}^{NE} W^{(e)} = \sum_{e=1}^{NE} \{\Lambda^{(e)}\}^T \lambda_T [K_g^{(e)}]\{\Lambda^{(e)}\} = \lambda_T \{q\}^T [K_g]\{q\} \quad (16)$$

Where λ_T and $[K_g]$ are defined as critical thermal buckling temperature and global geometric stiffness matrix.

F. Kinetic energy of FGM shell

The kinetic energy (T) of the vibrating FGM shell can be expressed as,

$$T = \int_V \rho^{(k)} \{\dot{u}\}^T \{\dot{u}\} dV \quad (17)$$

Where ρ and $\{\dot{u}\} = \{\dot{\bar{u}} \;\; \dot{\bar{v}} \;\; \dot{\bar{w}}\}$ are the density and velocity vector of the shell respectively, above equation can be expressed as,

$$T = \sum_{e=1}^{NE} T^{(e)} = \sum_{e=1}^{NE} \{\dot{\Delta}\}^T [m]\{\dot{\Delta}\} = \{\dot{q}\}[M]\{\dot{q}\} \quad (18)$$

Where, [M] is the global mass matrix.

EQUATION OF MOTION AND ITS SOLUTION

The governing equation for thermally induced nonlinear free vibration of the shell analysis can be derived using Lagrange's equation of motion.

$$\delta \int_{t_1}^{t_2} (U - W - T) dT = 0 \quad (19)$$

Substituting the values and obtaining in the form of nonlinear generalized eigenvalue problem as,

$$[K]\{q\} + [M]\{\ddot{q}\} = 0 \quad (20)$$

Where, $[K] = \{[K_l] + [K_{nl}(q)] + [K_{fl}] + [K_{fnl}(q)] - \lambda_T [K_g]\}$

The above equation is nonlinear free vibration equation which can be solved as a linear eigenvalue problem assuming that the shell is vibrating in its principal made in each iteration, the above equation can be expressed as generalized eigenvalue problem as,

$$[[K] - \lambda[M]]\{q\} = 0 \quad (21)$$

Where $\lambda = \omega^2$ with ω is natural frequency of the shell. The nonlinear eigenvalue problem is solved by employing a direct iterative based C^0 nonlinear finite element method in conjunction with perturbation technique.

G. Solution- perturbation technique

FOPT gives results with desired accuracy for problems with low variability. Hence the variance of the eigenvalue using FOPT can be expressed as,

$$Var(\lambda_i) = \sum_{j=1}^{q} \sum_{k=1}^{q} \lambda_{i,j}^d \lambda_{i,k}^d Cov\left(b_j^{\,r}, b_k^{\,r}\right) \quad (22)$$

RESULT AND DISCUSSION

A nine noded Lagrange isoparametric element with 63 DOFs per element for the present HSDT model has been used for discretizing the laminate and (4 × 4) mesh has been used for the study. Typical results are presented for functionally graded spherical shell made of ceramic and metal of the different compositions with their volume fractions index following the power law distribution through the shell thickness as shown in

Fig 1. The results are compared with those in literatures. The dimensionless mean natural frequency of the FGM shell is defined as,

$$\varpi = \omega h\sqrt{\rho_c / E_c} \quad (23)$$

Where, ω is dimensionalized natural frequency of FGM shell. Table 1 gives the material properties used for computation.

Table 1

Material	E (Mpa)	$\rho(kg/m^3)$	$\alpha(1/c)$
Aluminum	70	2707	0.0000023
Alumina	380	3800	0.0000075

Boundary Condition:
All edges simply supported (SSSS):

$$v = w = \theta_y = \psi_y = 0,\ at\ x = 0,\ a$$

$$u = w = \theta_x = \psi_x = 0\ at\ y = 0,\ b$$

Table 2

Validation table for frequency parameter ($\varpi = \omega h\sqrt{\rho_c / E_c}$), for Al / Al_2O_3 FGM shell with a/b=1.

a/R_1	b/R_2	n	Present	Matsunaga [4]
0.5	0.5	0	0.0752	0.07514
		0.5	0.0653	0.06569
		1	0.0594	0.06006
1	1	0	0.1124	0.1095
		0.5	0.0976	0.09782
		1	0.0887	0.09047

RESULT AND DISCUSSION

Natural frequencies of simply supported FGMs spherical shell have been obtained by using higher order shear deformation theory for arbitrary values of volume fraction index. For case of isotropic (fully ceramic/metal) spherical shell, fundamental frequencies have been compared with existing results and it has been shown that the present results are in excellent agreement with the existing result. It is shown that the higher order shear deformation theory can provide accurate results for natural frequencies of FGM spherical shell.

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