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Non- Linear Peristaltic Transport Of A Conducting Prandtl Fluid In A Porous Asymmetric Channel

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Abstract: The effects of both MHD and porous medium on peristaltic transport of a Prandtl fluid in an asymmetric channel have been studied under the assumptions of long wave length and low-Reynolds number. The flow is examined in a wave frame of reference moving with the velocity of the wave. The channel asymmetry is proposed by choosing the peristaltic wave train on the walls to have different amplitudes and phase. The expressions for stream function, velocity and pressure rise are obtained. The effects of different parameters like magnetic parameter, permeability parameter and Prandtl parameters on velocity and pressure rise are discussed numerically and explained graphically.

Keywords: Peristaltic transport, Conducting Prandtl fluid, asymmetric channel, porous medium

1 Introduction

Peristalsis is a mechanism of pumping viscous fluids in ducts against an increased pressure rise in general, by means of a series of moving contractile rings on the wall. Now the technique of peristalsis is well-known to physiologists as one of the major mechanisms for fluid transport in many biological systems. In particular, this mechanism is involved in urine transport from kidney to bladder, swallowing of food through the esophagus, movement of chyme in the gastro-intestinal tract, the transport of spermatozoa in the ducts efferentes of the male reproductive tract and in the movement of ovum in the fallopian tubes and in the vasomotion of small blood vessels. In addition, peristaltic pumping has many industrial applications involving biomechanical systems. The literature on this topic is quite extensive. Some significant contributions describing Newtonian and non- Newtonian fluid peristaltic transport are given in [1-11].

Recently it is a well accepted fact that the peristaltic flows of magnetohydrodynamic (MHD) fluids are important in medical sciences and bioengineering. The MHD characteristics are useful in the development of magnetic devices, hyperthermia, blood reduction during surgeries and cancer tumor treatment. Also the effect of magnetic field on viscous fluid has been reported for treatment of the following pathologies: Gastroenric pathologies, rheumatisms, constipation and hypertension that can be treated by placing one electrode either on the back or on the stomach and the other on the sole of the foot; this location will induce a better blood circulation. Hence several scientists having in mind such importance extensively discussed the peristalsis with magnetic field effects. Mekheimer and Al-Arabi[12] analyzed the non-linear peristaltic transport of MHD flow through a porous medium. Ali et al.[13] studied the slip effects on the peristaltic transport of MHD fluid with variable viscosity. Hayat and Ali[14] discussed the Peristaltic motion of a Jeffrey fluid under the effect of a magnetic field in a tube. Kothandapani and Srinivas[15] studied the Peristaltic transport of a Jeffrey fluid under the effect of magnetic field in an asymmetric channel Srinivas et al. [16] investigated the influence of slip condition, wall properties and heat transfer on MHD peristaltic transport. Hayat et al. [17] discussed the MHD flow of a Carreau fluid in a channel with different wave forms.

Viscous flow through porous medium is of fundamental importance in ceramic engineering, ground water hydrology, petroleum technology, power metallurgy, industrial filtration and such other fields. In springs of the geothermal region, water is known to be an electrically conducting fluid. Flow through porous medium has been studied by a number of researchers Hayat et al.[18] investigated the influence of partial slip on the peristaltic flow in a porous medium. The Effect of heat transfer on the peristaltic flow of an electrically conducting fluid in a porous space has bean studied by Hayat et al.[19] . Vajravelu et al.[20] discussed the influence of heat transfer on the peristaltic transport of a Jeffrey fluid in a vertical porous stratum. Singh and Rathee[21] studied the analysis of non-Newtonian blood flow through stenosed vessel in porous medium under the effect of magnetic field.

Keeping the above discussion in view, the non- linear peristaltic transport of a conducting Prandtl fluid in a porous asymmetric channel is investigated. The governing equations of Prandtl fluid model have been simplified under the assumptions of long wavelength and low Reynolds number approximations and are solved by using perturbation technique. The expressions for stream function, pressure gradient and pressure rise have been obtained. The effects of various parameters on the velocity and pressure rise are discussed through graphs.

2 Mathematical formulation

We consider an incompressible MHD prandtl fluid in an asymmetric channel with porous medium of width $d_1 + d_2$. Let c be the speed by which sinusoidal wave trains propagate along the channel walls. Consider the rectangular coordinate system $(\overline{X}, \overline{Y})$ where $\overline{X}$ – axis and $\overline{Y}$ – axis are taken parallel and transverse to the direction of wave propagation respectively. The wall surfaces are modeled as (H_1 is the upper wall and H_2 is the lower wall)

$$\overline{Y} = H_1 = d_1 + a_1 Cos\left[\frac{2\pi}{\lambda}\left(\overline{X} - c\bar{t}\right)\right], \overline{Y} = H_2 = -d_2 - b_1 Cos\left[\frac{2\pi}{\lambda}\left(\overline{X} - c\bar{t}\right) + \phi\right] \qquad (1)$$

where a_1 and b_1 are the amplitudes of the waves, λ is the wave length, $\bar{t}$ is the time, ϕ is the phase difference varying in the range $0 \le \phi \le \pi, \phi = 0$ corresponds to symmetric channel with waves are out of phase and $\phi = \pi$ with waves are in phase, and further a_1, b_1, d_1, d_2 *and* ϕ satisfy the condition $a_1^2 + b_1^2 + 2a_1 b_1 \cos\phi \le (d_1 + d_2)^2$ so that walls will not intersect with each other.

The governing equations in the laboratory frame are

$$\frac{\partial \overline{U}}{\partial \overline{X}} + \frac{\partial \overline{V}}{\partial \overline{Y}} = 0 \qquad (2)$$

$$\rho\left[\frac{\partial \overline{U}}{\partial \bar{t}} + \overline{U}\frac{\partial \overline{U}}{\partial \overline{X}} + \overline{V}\frac{\partial \overline{U}}{\partial \overline{Y}}\right] = -\frac{\partial \overline{P}}{\partial \overline{X}} + \frac{\partial \overline{S}_{\overline{XX}}}{\partial \overline{X}} + \frac{\partial \overline{S}_{\overline{XY}}}{\partial \overline{Y}} - \frac{\mu}{k}\overline{U} - \sigma_e B_0^2 \overline{U} \qquad (3)$$

$$\rho\left[\frac{\partial\overline{V}}{\partial\bar{t}}+\overline{U}\frac{\partial\overline{V}}{\partial\overline{X}}+\overline{V}\frac{\partial\overline{V}}{\partial\overline{Y}}\right]=-\frac{\partial\overline{P}}{\partial\overline{Y}}+\frac{\partial\overline{S}_{\overline{YX}}}{\partial\overline{X}}+\frac{\partial\overline{S}_{\overline{YY}}}{\partial\overline{Y}}-\frac{\mu}{k}\overline{V} \tag{4}$$

where $\overline{U}$ and $\overline{V}$ are the velocities in the $\overline{X}$ and $\overline{Y}$ directions and $\overline{P}$ is the pressure in fixed frame, ρ is the density, μ is the coefficient of viscosity of the fluid, B_0 is a constant magnetic field and σ_e is the electrical conductivity. We introduce the following transformations between fixed and wave frames

$$\bar{x}=\overline{X}-c\bar{t},\ \bar{y}=\overline{Y},\ \bar{u}=\overline{U}-c,\ \bar{v}=\overline{V},\ \bar{p}\left(\bar{x}\right)=\overline{P}(\overline{X},\bar{t}) \tag{5}$$

where $\bar{u},\bar{v},\bar{p}$ are the velocity components and pressure in the wave frame respectively.

The extra tensor $\overline{S}$ for Prandtl fluid is given by

$$\overline{S}=\frac{A\sin^{-1}\left\{\frac{1}{c}\left[\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2+\left(\frac{\partial\bar{v}}{\partial\bar{x}}\right)^2\right]^{1/2}\right\}}{\left[\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2+\left(\frac{\partial\bar{v}}{\partial\bar{x}}\right)^2\right]^{1/2}}\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right) \tag{6}$$

in which A and C are material constants of prandtl fluid model.

The non–dimensional quantities and the expressions for stream functions are given by

$$\left.\begin{array}{l} x=\frac{2\pi\bar{x}}{\lambda},\ y=\frac{\bar{y}}{d_1},\ u=\frac{\bar{u}}{c},\ v=\frac{\bar{v}}{\delta c},\ \delta=\frac{2\pi d_1}{\lambda},\ p=\frac{2\pi d_1{}^2\bar{p}}{\mu c\lambda},\ t=\frac{2\pi\bar{t}}{\lambda},\ h_1=\frac{H_1}{d_1}, \\ h_2=\frac{H_2}{d_1},\mathrm{Re}=\frac{\rho c d_1}{\mu},d=\frac{d_2}{d_1},a=\frac{a_1}{d_1},b=\frac{b_1}{d_1},M^2=\frac{\sigma_e B_0{}^2 d_1{}^2}{\mu},S=\frac{\overline{S}d_1}{\mu c},\sigma=\frac{1}{\sqrt{k}} \end{array}\right\} \tag{7}$$

and $$u=\frac{\partial\psi}{\partial y},\ v=-\frac{\partial\psi}{\partial x} \tag{8}$$

where R is the Reynolds number, δ is the dimensionless wave number, M is the magnetic parameter and σ permeability parameter .

After using the equations (7), (8) and long wave length and low Reynolds number approximation the basic equations reduces to

$$\frac{dp}{dx}=\frac{\partial}{\partial y}\left[\alpha\frac{\partial^2\psi}{\partial y^2}+\beta\left(\frac{\partial^2\psi}{\partial y^2}\right)^3\right]-(\sigma^2+M^2)\left(\frac{\partial\psi}{\partial y}+1\right) \tag{9}$$

$$0=\frac{\partial^2}{\partial y^2}\left[\alpha\frac{\partial^2\psi}{\partial y^2}+\beta\left(\frac{\partial^2\psi}{\partial y^2}\right)^3\right]-(\sigma^2+M^2)\frac{\partial^2\psi}{\partial y^2} \tag{10}$$

where $\alpha = \dfrac{A}{\mu C}$, $\beta = \dfrac{\alpha c^2}{C^2 d_1}$

corresponding non-dimensional boundary conditions are

$$\psi = \frac{F}{2}, \frac{\partial \psi}{\partial y} = -1 \quad at \quad y = h_1 = 1 + a\cos x$$

$$\psi = -\frac{F}{2}, \frac{\partial \psi}{\partial y} = -1 \quad at \quad y = h_2 = -d - ba\cos(x+\phi) \tag{11}$$

The flux at any axial station in the fixed frame is

$$Q = \int_{h_2}^{h_1} (u+1)dy = h_1 - h_2 + q \tag{12}$$

The average volume flow rate over one period of the peristaltic wave is defined as

$$\Theta = \frac{1}{T}\int_0^T Q dt = \frac{1}{T}\int_0^T (h_1 - h_2 + q)dt = q + 1 + d \tag{13}$$

3 Series solution

We note that the resulting equation (10) is highly non linear, hence we expand the flow quantities in a power series of the small parameter β as follows:

$$\left.\begin{aligned} \psi &= \psi_0 + \beta\psi_1 + O(\beta^2) \\ F &= F_0 + \beta q_1 + O(\beta^2) \\ \frac{dp}{dx} &= \frac{dp_0}{dx} + \beta\frac{dp_1}{dx} + O(\beta^2) \end{aligned}\right\} \tag{14}$$

Using the above relation in equations (10) and (11), we obtain a system of equations of different orders.

3.1 Zero- order System

The governing equations of the zero-order are

$$0 = \frac{\partial^2}{\partial y^2}\left[\alpha \frac{\partial^2 \psi_0}{\partial y^2}\right] - (\sigma^2 + M^2)\frac{\partial^2 \psi_0}{\partial y^2} \tag{15}$$

The appropriate boundary conditions are

$$\psi_0 = \frac{F_0}{2}, \frac{\partial \psi_0}{\partial y} = -1 \quad at \quad y = h_1 = 1 + a\cos x$$
$$\psi_0 = -\frac{F_0}{2}, \frac{\partial \psi_0}{\partial y} = -1 \quad at \quad y = h_2 = -d - ba\cos(x+\phi) \tag{16}$$

The solution of equation (14) in terms of stream function, is given by

$$\psi_0 = c_1 + c_2 y + c_3 \cosh\sqrt{\frac{\sigma^2 + M^2}{\alpha}}\, y + c_4 \sinh\sqrt{\frac{\sigma^2 + M^2}{\alpha}}\, y \tag{17}$$

and the axial velocity is

$$u_0 = c_2 + \sqrt{\frac{\sigma^2 + M^2}{\alpha}}\left(c_3 \sinh\sqrt{\frac{\sigma^2 + M^2}{\alpha}}\, y + c_4 \cosh\sqrt{\frac{\sigma^2 + M^2}{\alpha}}\, y\right) \tag{18}$$

3.2 First- order System

The governing equations of the first-order are

$$0 = \frac{\partial^2}{\partial y^2}\left[\alpha \frac{\partial^2 \psi_1}{\partial y^2} + \left(\frac{\partial^2 \psi_0}{\partial y^2}\right)^3\right] - (\sigma^2 + M^2)\frac{\partial^2 \psi_1}{\partial y^2} \tag{19}$$

The corresponding boundary conditions are

$$\psi_1 = \frac{F_1}{2}, \frac{\partial \psi_1}{\partial y} = 0 \quad at \quad y = h_1$$
$$\psi_0 = -\frac{F_1}{2}, \frac{\partial \psi_1}{\partial y} = 0 \quad at \quad y = h_2 \tag{20}$$

The solution of equation (19) in terms of stream function, is given by

$$\begin{aligned}\psi_1 = {} & c_5 + c_6 y + c_7 \cosh\sqrt{\frac{\sigma^2 + M^2}{\alpha}}\, y + c_8 \sinh\sqrt{\frac{\sigma^2 + M^2}{\alpha}}\, y \\ & - \frac{\sigma^2 + M^2}{8\alpha^2}\left(L_6 \cosh 3\sqrt{\frac{\sigma^2 + M^2}{\alpha}}\, y + L_7 \sinh 3\sqrt{\frac{\sigma^2 + M^2}{\alpha}}\, y\right. \\ & + \left(\frac{\sigma^2 + M^2}{\alpha^2}\right)^{5/2} \frac{L_8}{\alpha}\, y \left(c_3 \sinh\sqrt{\frac{\sigma^2 + M^2}{\alpha}}\, y + c_4 \cosh\sqrt{\frac{\sigma^2 + M^2}{\alpha}}\, y\right)\end{aligned} \tag{21}$$

The axial velocity and the corresponding pressure gradient from the momentum equation are

$$\begin{aligned}
u_1 = c_6 &+ \sqrt{\frac{\sigma^2+M^2}{\alpha}}\left(c_7 \sinh\sqrt{\frac{\sigma^2+M^2}{\alpha}}\, y + c_8 \cosh\sqrt{\frac{\sigma^2+M^2}{\alpha}}\, y\right) \\
&- \frac{3}{8\alpha}\left(\frac{\sigma^2+M^2}{\alpha^2}\right)^{5/2}\left(L_6 \sinh 3\sqrt{\frac{\sigma^2+M^2}{\alpha}}\, y + L_7 \cosh 3\sqrt{\frac{\sigma^2+M^2}{\alpha}}\, y\right) \\
&+ \frac{L_8}{\alpha}\left(\frac{\sigma^2+M^2}{\alpha^2}\right)^{3} y\left(c_3 \cosh\sqrt{\frac{\sigma^2+M^2}{\alpha}}\, y + c_4 \sinh\sqrt{\frac{\sigma^2+M^2}{\alpha}}\, y\right) \\
&+ \frac{L_8}{\alpha}\left(\frac{\sigma^2+M^2}{\alpha^2}\right)^{5/2}\left(c_3 \sinh\sqrt{\frac{\sigma^2+M^2}{\alpha}}\, y + c_4 \cosh\sqrt{\frac{\sigma^2+M^2}{\alpha}}\, y\right)
\end{aligned} \tag{22}$$

Finally the expressions for axial velocity, the corresponding pressure gradient and pressure rise are given by

$$u = u_0 + \beta u_1 \tag{23}$$

The pressure gradient is obtained from the dimensionless equation of motion as

$$\frac{dp}{dx} = \frac{dp_0}{dx} + \beta\frac{dp_1}{dx} \tag{24}$$

In which

$$\frac{dp_0}{dx} = -(c_2+1)\left(\sigma^2+M^2\right) \text{ and } \frac{dp_1}{dx} = -c_6\left(\sigma^2+M^2\right) \tag{25}$$

$$\Delta p = \int_0^1 \frac{dp}{dx}\,dx \tag{26}$$

4. Results and Discussion

The expression for velocity in terms of y is given by the equation (23). Velocity profiles are plotted in Fig.1 to study the effects of the different parameters such as the magnetic parameter M, permeability parameter σ, phase difference ϕ, volume flow rate Θ and Prandtl parameters α and β on the velocity distribution. Fig.1a. and Fig.1b. are drawn to study the effect of M and σ on the velocity. We noticed that the velocity profiles are parabolic. We found that the velocity increases with increasing M, σ and meet at -0.5 and then opposite behavior is observed. From Fig.1c. we seen that the velocity decreases with an increase in ϕ and coincide at a point between -0.5 and 0 after that opposite behavior is observed. From Fig.1d. we observe that the velocity increases with increasing Θ. Fig.1e. and Fig.1f. are plotted to study the effect of α and β on the velocity. We observed that the velocity decreases with increasing α, β and meet at -0.5 and then opposite behavior is observed.

We have calculated the pressure rise Δp in terms of the mean flow Θ from equation (26). The variation of pressure rise with the mean flow for different M and σ is shown in figures 2a. and 2b. It is noticed that the pressure rise decreases with the increase in Θ. We found that for a given Θ, pressure rise increases with increasing M, σ and coincide at a point between 0 and 0.5 after that opposite behavior is observed. From Fig. 2c. we observed that for a given

Θ, pressure rise increases with increasing ϕ. Fig.2d. and Fig.2e are plotted to study the effects of Prandtl parameters α and β. We noticed that the pressure rise increases with increasing α and β.

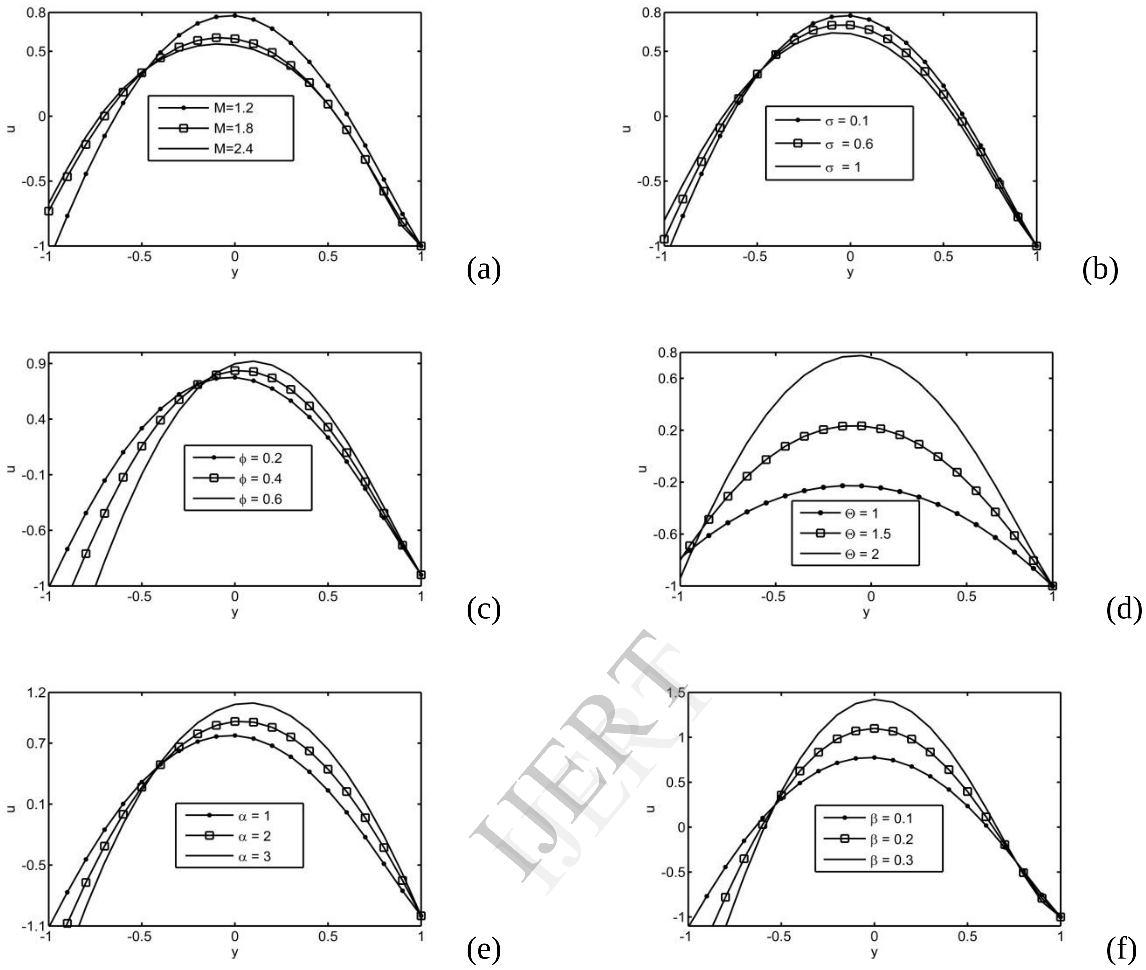
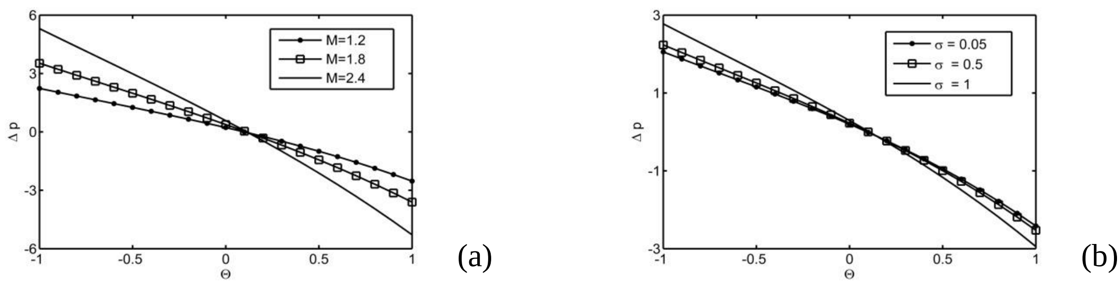


Fig. 1. Velocity distribution $(a=0.5, b=0.5, \mathrm{d}=1, \mathrm{x}=1)$

$(a)\Theta=2,\phi=0.2,\alpha=1;\beta=0.1,\sigma=0.1;(b)M=1.2,\Theta=2,\phi=0.2,\alpha=1,\beta=0.1;(c)M=1.2,\Theta=2,\sigma=0.1,\alpha=1,\beta=0.1;$
$(d)M=1.2,\phi=0.2,\sigma=0.1,\alpha=1,\beta=0.1;(e)M=1.2,\Theta=2,\phi=0.2,\sigma=0.1,\beta=0.1;(f)M=1.2,\Theta=2,\phi=0.2,\sigma=0.1,\alpha=1;$



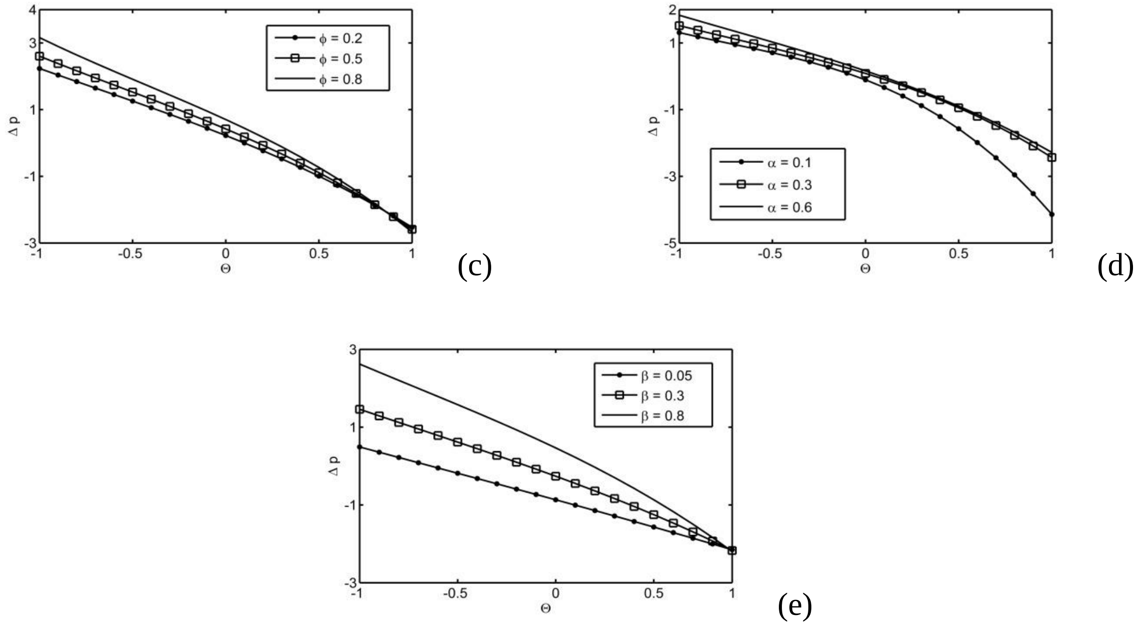


Fig.2 .graph of Pressure gradient $(a=0.5, b=0.5, \mathrm{d}=1)$

$(a)\phi=0.2, \alpha=1; \beta=0.5, \sigma=0.5; (b)M=1.2, \phi=0.2, \alpha=1, \beta=0.5; (c)M=1.2, \sigma=0.5, \alpha=1, \beta=0.5;$
$(d)M=1.2, \phi=0.2, \sigma=0.5, \beta=0.5; (e)M=1.2, \phi=0.2, \sigma=0.5, \alpha=1;$

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Appendix

$$L_1 = \sinh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1 - \sinh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_2, \quad L_2 = \cosh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1 - \cosh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_2,$$

$$L_3 = L_2 - \sqrt{\frac{\sigma^2+M^2}{\alpha}}(h_1-h_2)\sinh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1, \quad L_4 = L_1 - \sqrt{\frac{\sigma^2+M^2}{\alpha}}(h_1-h_2)\cosh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1,$$

$$L_5 = c_3\cosh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1 + c_4\sinh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1, \quad L_6 = c_3\left(\frac{c_3{}^2+3c_4{}^2}{4}\right), \quad L_7 = c_4\left(\frac{c_4{}^2+3c_3{}^2}{4}\right),$$

$$L_8 = \frac{3}{2}\left(\frac{c_4{}^2-c_3{}^2}{4}\right), \quad L_9 = \frac{\sigma^2+M^2}{8\alpha^2}\left(L_6\cosh 3\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1 + L_7\sinh 3\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1\right),$$

$$L_{10} = \left(\frac{\sigma^2+M^2}{\alpha}\right)^{5/2}\frac{L_8 h_1}{\alpha}\left(c_3\sinh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1 + c_4\cosh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1\right),$$

$$L_{11} = \frac{\sigma^2+M^2}{8\alpha^2}\left(L_6\cosh 3\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_2 + L_7\sinh 3\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_2\right),$$

$$L_{12} = \left(\frac{\sigma^2+M^2}{\alpha}\right)^{5/2}\frac{L_8 h_2}{\alpha}\left(c_3\sinh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_2 + c_4\cosh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_2\right),$$

$$L_{13}=\left(\frac{\sigma^2+M^2}{\alpha}\right)^{5/2}\frac{3}{8\alpha}\left(L_6\sinh 3\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1+L_7\cosh 3\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1\right)$$

$$L_{14}=\left(\frac{\sigma^2+M^2}{\alpha}\right)^{5/2}\frac{L_8\,h_1}{\alpha}\left\{\left(c_3\cosh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1+c_4\sinh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1\right)\sqrt{\frac{\sigma^2+M^2}{\alpha}}+\left(c_3\sinh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1+c_4\cosh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1\right)\right\}$$

$$L_{15}=\left(\frac{\sigma^2+M^2}{\alpha}\right)^{5/2}\frac{3}{8\alpha}\left(L_6\sinh 3\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_2+L_7\cosh 3\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_2\right),$$

$$L_{16}=\left(\frac{\sigma^2+M^2}{\alpha}\right)^{5/2}\frac{L_8\,h_2}{\alpha}\left\{\left(c_3\cosh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_2+c_4\sinh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_2\right)\sqrt{\frac{\sigma^2+M^2}{\alpha}}+\left(c_3\sinh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_2+c_4\cosh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_2\right)\right\}$$

$$L_{17}=L_{11}+L_{10}-(L_9+L_{12}),\quad L_{18}=L_{15}+L_{14}-(L_{13}+L_{16}),\quad L_{19}=L_{17}-(h_1-L_2)(L_{14}-L_{13}),$$

$$L_{20}=L_{19}L_1-\frac{L_{18}L_3}{\sqrt{\frac{\sigma^2+M^2}{\alpha}}},\quad L_{21}=L_4L_1-L_3L_2,\quad L_{22}=L_{17}+c_8L_1+c_7L_2,$$

$$L_{23}=c_7\cosh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1+c_8\sinh\sqrt{\frac{\sigma^2+M^2}{\alpha}}h_1,$$

$$c_1=\frac{F_0}{2}-c_2h_1-L_5,\quad c_2=\frac{F_0-c_3L_2-c_4L_1}{h_1-h_2},\quad c_3=\frac{-c_4L_2}{L_1},\quad c_4=\frac{L_1\left(F_0+(h_1-h_2)\right)}{L_1L_4-L_2L_3},$$

$$c_5=\frac{F_1}{2}-c_6h_1-L_{23}-L_{10}+L_9,\quad c_6=\frac{F_1-L_{22}}{h_1-h_2},\quad c_7=\frac{-L_{18}-c_8L_2\sqrt{\frac{\sigma^2+M^2}{\alpha}}}{L_1\sqrt{\frac{\sigma^2+M^2}{\alpha}}},\quad c_8=\frac{F_1L_1-L_{22}}{L_{23}}$$

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