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New oscillation criteria for second order advanced neutral differential equations

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Abstract

In this paper we present new criteria for oscillation of advanced neutral differential equations second order of the form

$$[r(t)[((a(t)x(t)+b(t)x(\tau(t))^{\alpha})']'+c(t)|x(\tau(t)|^{\alpha}=0$$

$$t \geq t_0 > 0 \qquad (1)$$

where the coefficient r(t) is nonnegative continuous function , a(t), b(t) and c(t) are continuous function which filled certain conditions .
The conclusion is based also on building functions where are involved coefficients of equation , positive functions $\rho(t)$ *and the positive function of Philo* $H(t,s)$ *.*
Here , by using the generalized Riccati technique we get a new oscillation criteria for (1).

Key words: *oscillation , differential equation, second order, interval, criteria etc.*

Introduction

Let consider and create new oscillation of advanced neutral differential equations second order of the form

$$[r(t)((a(t)x(t)+b(t)x(\tau(t))^{\alpha})']'+c(t)|x(\tau(t)|^{\alpha}=0$$

$$t > t_0$$

where α is a quotient of odd positive integers and α is a even number.
We assume that

A1) $a(t) > 0$, $c(t) \geq 0$, $0 \leq b(t) \leq 1$,

A2) $r(t) > 0$, $\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(s)} = \infty$

A3) $\tau(t) \in C^1([t_0,\infty),\Re)$, $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$.

In following we set $z(t) = a(t)x(t) + b(t)x(\tau(t))$. By a solution of equation (1) we consider a function $x(t),\ t \in [t_x,\infty) \subset [t_0,\infty\}$ which is twice continuously differentiable and satisfies equation (1) on the given interval. We consider only non-trivial solutions . A solution *x(t)* of (1) is said to be oscillatory if there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of points in the interval $[t_0,\infty\}$, such that $\lim_{n\to\infty} \lambda_n = \infty$ and $x(\lambda_n) = 0$, $n \in N$, otherwise it is said to be non-oscillatory. An equation is said to be oscillatory if all its solutions are oscillatory, otherwise it is considered that is non-oscillatory solution .

Lemma 1. If $x(t)$ is a positive solution of (1) then exists $t_1 \in [t_0,\infty)$ such the corresponding function

$$z(t) = a(t)x(t) + b(t)x(\tau(t)) \qquad (2)$$

satisfies

$$z(t) > 0,\quad z'(t) > 0,\quad z''(t) < 0 \text{ for } t > t_1$$

eventually.

Proof: Assume that the function $x(t)$ is a positive solution of (1) . Then from (1) follow that exists $t_1 \in [t_0,\infty)$ such that

$$((r(t)(z'(t))^{\alpha})' = -c(t)|x(\tau(t)|^{\alpha} < 0 \text{ for } t \geq t_1$$

from where we get that the function $r(t)(z'(t))^{\alpha}$ is decreasing for $t \geq t_1$ and we claim that $r(t)(z'(t))^{\alpha} > 0$ or

$r(t)(z'(t))^{\alpha} < 0$. If we let $r(t)(z'(t))^{\alpha} < 0$ on $t \geq t_1$ then exists $t_2 \geq t_1$, such that $r(t)(z'(t))^{\alpha} \leq r(t_2)(z'(t_2))^{\alpha} < 0$, for all $t \geq t_2$ from where

$$z'(t) \leq \frac{(r(t_2))^{\frac{1}{\alpha}} z'(t_2)}{(r(t))^{\frac{1}{\alpha}}}$$

Integrating this from t_2 to t we have

$$z(t) \leq z(t_2) + (r(t_2)^{\frac{1}{\alpha}} z'(t_2)) \int_{t_2}^{t} \frac{1}{(r(s))^{\frac{1}{\alpha}}} ds$$

we can see that $z(t) \to -\infty$, where $t \to \infty$. This contradicts because $z(t) > 0$ we have $r(t)(z'(t))^{\alpha} > 0$, from where $z'(t) > 0$.

From (1) we get

$$((r(t)(z'(t))^{\alpha})' < 0$$

$$r'(t)(z'(t))^{\alpha} + \alpha r(t)(z't))^{\alpha-1} z''(t) < 0$$

from where

$$z''(t) < 0.$$

This complete the proof.

Lemma 2. Let $\varphi(\omega) = Bw - Aw^{\frac{\alpha+1}{\alpha}}$, *A>0, and B are constants*, α is a quotient of odd positive integers. Then function φ attains its maximum value on $\Re$ at

$$w_{\max} = \frac{\alpha^{\gamma}}{(\alpha+1)^{\alpha}} \frac{B^{\alpha}}{A^{\alpha}} \quad \text{and}$$

$$\max(w) = \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^{\alpha}}.$$

Proof.: From

$$\varphi'(w) = B - \frac{\alpha+1}{\alpha} A w^{\frac{1}{\alpha}} \quad \text{and}$$

$\varphi'(w) = 0$, we get

$$w = \frac{\alpha^{\gamma}}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^{\alpha}}.$$

Since $\varphi''(w) = -\frac{\alpha+1}{\alpha^2} A w^{\frac{1-\alpha}{\alpha}} < 0$, we have that the function $\varphi(w)$ attains to max value on $\Re$ at $w_{\max}$, i. e. $\varphi(w_{\max})$ is a max value of function $\varphi(w)$ and

$$\varphi(w_{\max}) = \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^{\alpha}}$$

and we can write the inequality

$$Bw - Aw^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^{\gamma}}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^{\alpha}}.$$

Consider (2) we have

$$x(t) = \frac{1}{a(t)}[z(t) - b(t)x(\tau(t))]$$

from where

$$x(\tau(t)) = \frac{1}{a(\tau(t))}[z(\tau(t)) - b(\tau(t))x(\tau(\tau(t)))]$$

for $x(t) > 0$, $\tau(t) \leq t$ and $x'(t) > 0$, also from (2) we get

$$x(\tau(t)) \leq x(t) \text{ and}$$

$$x(\tau(t)) < z(\tau(t))$$

finally

$$x(\tau(t)) \geq \frac{1}{a(\tau(t))}[z(\tau(t)) - b(\tau(t))z(\tau(t))]$$

$$x(\tau(t)) \geq \frac{1}{a(\tau(t))}[z(\tau(t))(1 - b(\tau(t))].$$

Now define

$$w(t) = v(t)\frac{r(t)(z'(t))^{\alpha}}{z^{\alpha}(\tau(t))}, \text{ for}$$

$$t \geq t_0 > 0 \qquad (3)$$

differenting (3) and using (1) we see that

$$w'(t) = v'(t)$$

$$\frac{r(t)(z'(t))^{\alpha}}{z^{\alpha}(\tau(t))} + v(t)\frac{(r(t)z'^{\alpha}(t))}{z^{\alpha}(\tau(t))} -$$

$$\alpha v(t)\frac{r(t)z'^{\alpha}(t)z^{\alpha-1}(t)z'(\tau(t))\tau'(t)}{z^{2\alpha}(\tau(t))}$$

$$w'(t)=\frac{v'(t)}{v(t)}w(t)$$
$$-v(t)\frac{c(t)(1-p(\tau(t))^{\alpha}z'^{\alpha}(\tau(t))}{a^{\alpha}(\tau(t))z^{\alpha}(\tau(t))}-\alpha w(t)\frac{z'(\tau(t))\tau'(t)}{z(\tau(t))}$$

$$w'(t)=\frac{v'(t)}{v(t)}w(t)$$
$$-v(t)\frac{c(t)(1-p(\tau(t))^{\alpha}}{a^{\alpha}(\tau(t))}-\alpha w(t)w^{\frac{1}{\alpha}}(t)\frac{\tau'(t)}{r^{\frac{1}{\alpha}}(\tau(t))v^{\frac{1}{\alpha}}(t)}$$

for $L(t)=\dfrac{c(t)(1-p(\tau(t))^{\alpha}}{a^{\alpha}(\tau(t))}>0$

we obtain

$$w'(t)=\frac{v'(t)}{v(t)}w(t)$$
$$-v(t)L(t)-\alpha w^{\frac{\alpha+1}{\alpha}}(t)\frac{\tau'(t)}{r^{\frac{1}{\alpha}}(\tau(t))v^{\frac{1}{\alpha}}(t)} \qquad (4)$$

for

$$B=\frac{v'(t)}{v(t)},\ A=\frac{\tau'(t)}{v^{\frac{1}{\alpha}}(t)r^{\frac{1}{\alpha}}(\tau(t))}$$

we have

$$w'(t)\le -v(t)L(t)+B(t)w(t)-A(t)w^{\frac{\alpha+1}{\alpha}(t)}$$

now to consider lemma 2, we have

$$w'(t)\le -v(t)L(t)+\frac{\alpha^{\alpha}B^{\alpha+1}}{(\alpha+1)^{\alpha+1}A^{\alpha}}$$

from where

$$w'(t)\le -v(t)L(t)+\frac{(v'(t))^{\alpha+1}r(\tau(t))}{(\alpha+1)^{\alpha+1}v^{\alpha}(t)(\tau'(t))^{\alpha}}$$
$$t\ge t_0>0 \qquad (5)$$

We say that a function $H(t,s)$ belongs to the class X if

i) $H\in C(D,[0,\infty))$;

ii) $H(t,t)=0$ and $H(t,s)>0$, for $-\infty<s<t<+\infty$;

iii) H has continuous partial derivatives on first and second variable

$$\frac{\partial H(t,s)}{\partial t}=h_1(t,s)\sqrt{H(t,s)} \quad \text{and}$$
$$\frac{\partial H(t,s)}{\partial s}=-h_2(t,s)\sqrt{H(t,s)}$$

Teorem 1. Assumed that A1) – A3) hold .Assume that exists a positive differentiable function *v(t)* and a function $H(t,s)\in X$ and if there exist $(a,b)\subseteq[t_0,\infty), c\in(a,b)$, such that

$$\frac{1}{H(c,a)}\int_a^c[H(t,s)v(s)L(s)-\frac{H(t,s)v'(s)+h_1(t,s)\sqrt{H(t,s)}}{(\alpha+1)^{\alpha+1}v^{\alpha}(s)H^{\alpha}(t,s)\tau'^{\alpha}(s)}]ds$$
$$\frac{1}{H(b,c)}\int_c^b[H(b,s)v(s)L(s)-\frac{H(b,s)v'(s)-h_2(b,s)\sqrt{H(b,s)}}{(\alpha+1)^{\alpha+1}v^{\alpha}(s)H^{\alpha}(b,s)\tau'^{\alpha}(s)}]ds>0 \qquad (6)$$

then every solution of eq. (1) is osillatory.

Proof: Suppose to the contrary, that *x(t)* be a non-oscillatory solution of (1) , say $x(t)\neq 0$ on $[t_0,\infty)$ from where $z(t)\neq 0$ on $[t_0,\infty)$.

If inequation (5) multiplying with $H(t,s)$ and integrate from c to t where $t\in(c,b), s\in(c,t)$ we have

$$\int_c^t H(t,s)v(s)L(s)ds\le -\int_c^t H(t,s)w'(s)ds+$$
$$\int_c^t H(t,s)\frac{w'(s)}{w(s)}w(s)ds-\int_c^t\frac{W^{\frac{\alpha+1}{\alpha}}(s)\tau'(s)H(t,s)}{r^{\frac{1}{\alpha}}(\tau(s))v^{\frac{1}{\alpha}}(s)}ds$$

$$\int_c^t H(t,s)v(s)L(s)ds\le -w(s)H(t,s)\Big|_c^t-$$
$$\int_c^t h_2(t,s)\sqrt{H(t,s)}w(s)ds+$$
$$+\int_c^t H(t,s)\frac{w'(s)}{w(s)}w(s)ds-\int_c^t\frac{W^{\frac{\alpha+1}{\alpha}}(s)\tau'(s)H(t,s)}{r^{\frac{1}{\alpha}}(\tau(s))v^{\frac{1}{\alpha}}(s)}ds$$

$$\int_c^t H(t,s)v(s)L(s)ds \le w(c)H(t,c) + \int_c^t [(H(t,s)\frac{v'(s)}{v(s)} -$$

$$-h_2(t,s)\sqrt{H(t,s)})w(s) - \frac{W^{\frac{\alpha+1}{\alpha}}(s)\tau'(s)\alpha H(t,s)}{r^{\frac{1}{\alpha}}(\tau(s))v^{\frac{1}{\alpha}}(s)}]ds$$

From Lemma2 for $A = \frac{\tau'(s)\alpha H(t,s)}{r^{\frac{1}{\alpha}}(\tau(s))v^{\frac{1}{\alpha}}(s)}$,

$$B = H(t,s)\frac{v'(s)}{v(s)} - h_2(t,s)\sqrt{H(t,s)}$$

we have

$$\int_c^t H(t,s)v(s)L(s)ds \le w(c)H(t,c) +$$

$$\int_c^t \frac{H(t,s)v'(s) - h_2(t,s)\sqrt{H(t,s)}}{(\alpha+1)^{\alpha+1}v^{\alpha}(s)H^{\alpha}(t,s)\tau'^{\alpha}(s)}ds \qquad (7)$$

Let $t \to b_-$ in (7 and dividing it by $H(b,c)$ we get

$$\frac{1}{H(b,c)}\int_c^b H(b,s)v(s)L(s)ds \le w(c) +$$

$$\int_c^b \frac{H(b,s)v'(s) - h_2(b,s)\sqrt{H(b,s)}}{(\alpha+1)^{\alpha+1}v^{\alpha}(s)H^{\alpha}(b,s)\tau'^{\alpha}(s)}ds \qquad (8)$$

If (5)multiplying with $H(s,t)$ and integrate over (t,c) where $t \in (a,c), s \in (t,c)$ we get

$$\int_t^c H(s,t)v(s)L(s)ds \le -\int_t^c H(s,t)w'(s)ds +$$

$$\int_t^c H(s,t)\frac{w'(s)}{w(s)}w(s)ds - \int_t^c \frac{W^{\frac{\alpha+1}{\alpha}}(s)\tau'(s)H(s,t)}{r^{\frac{1}{\alpha}}(\tau(s))v^{\frac{1}{\alpha}}(s)}ds$$

$$\int_t^c H(s,t)v(s)L(s)ds \le -w(s)H(s,t)\Big|_t^c + \int_t^c h_1(t,s)\sqrt{H(s,t)}w(s)ds +$$

$$+\int_t^c H(s,t)\frac{w'(s)}{w(s)}w(s)ds - \int_t^c \frac{W^{\frac{\alpha+1}{\alpha}}(s)\tau'(s)H(s,t)}{r^{\frac{1}{\alpha}}(\tau(s))v^{\frac{1}{\alpha}}(s)}ds$$

$$\int_t^c H(s,t)v(s)L(s)ds \le -w(c)H(s,t) + \int_t^c [(H(s,t)\frac{v'(s)}{v(s)} +$$

$$+h_1(s,t)\sqrt{H(s,t)})w(s) - \frac{W^{\frac{\alpha+1}{\alpha}}(s)\tau'(s)\alpha H(s,t)}{r^{\frac{1}{\alpha}}(\tau(s))v^{\frac{1}{\alpha}}(s)}]ds$$

From Lemma2 for $A = \frac{\tau'(s)\alpha H(s,t)}{r^{\frac{1}{\alpha}}(\tau(s))v^{\frac{1}{\alpha}}(s)}$,

$$B = H(s,t)\frac{v'(s)}{v(s)} + h_1(s,t)\sqrt{H(s,t)}$$

we have

$$\int_t^c H(s,t)v(s)L(s)ds \le w(c)H(s,t) +$$

$$\int_t^c \frac{H(s,t)v'(s) + h_1(s,t)\sqrt{H(s,t)}}{(\alpha+1)^{\alpha+1}v^{\alpha}(s)H^{\alpha}(s,t)\tau'^{\alpha}(s)}ds \qquad (9)$$

Let $t \to a^+$ in (9)) and dividing it by $H(c,a)$ we obtain

$$\frac{1}{H(c,a)}\int_a^c H(t,s)v(s)L(s)ds \le w(c) +$$

$$\frac{1}{H(c,a)}\int_a^c \frac{H(t,s)v'(s) + h_1(t,s)\sqrt{H(t,s)}}{(\alpha+1)^{\alpha+1}v^{\alpha}(s)H^{\alpha}(t,s)\tau'^{\alpha}(s)}ds \quad (10)$$

Adding (8) and (10) we have the following inequality

$$\frac{1}{H(c,a)}\int_a^c [H(c,s)v(s)L(s) -$$

$$\frac{H(c,s)v'(s) + h_1(c,s)\sqrt{H(c,s)}}{(\alpha+1)^{\alpha+1}v^{\alpha}(s)H^{\alpha}(c,s)\tau'^{\alpha}(s)}]ds +$$

$$\frac{1}{H(b,c)}\int_c^b [H(b,s)v(s)L(s) -$$

$$\frac{H(b,s)v'(s) - h_2(b,s)\sqrt{H(b,s)}}{(\alpha+1)^{\alpha+1}v^{\alpha}(s)H^{\alpha}(b,s)\tau'^{\alpha}(s)}]ds \le 0$$

Which contradict to the condition (6) , therefore , every solution of equation (1) be oscillatory . The proof is complete.

Corollary 1: Let assume that A_1, A_2, A_3 hold. If

$$\limsup_{t\to\infty}\frac{1}{H(t,a)}\int_k^t [H(t,s)v(s)L(s)-\frac{H(t,s)v'(s)+h_1(t,s)\sqrt{H(t,s)}}{(\alpha+1)^{\alpha+1}v^{\alpha}(s)H^{\alpha}(t,s)\tau'^{\alpha}(s)}]ds>0 \quad (11)$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,c)}\int_k^t [H(t,s)v(s)L(s)-\frac{H(t,s)v'(s)-h_2(t,s)\sqrt{H(t,s)}}{(\alpha+1)^{\alpha+1}v^{\alpha}(s)H^{\alpha}(t,s)\tau'^{\alpha}(s)}]ds>0 \quad (12)$$

for any $H\in X$, $v\in C^1([t_0,\infty),(0,\infty))$ and for all $k\geq t_0$, then every solution of (1) is oscillatory.

Proof: For $k\geq t_0$, from (11) if we take $k=a$, and $c>a$, we get

$$\limsup_{t\to\infty}\frac{1}{H(c,a)}\int_a^c [H(t,s)v(s)L(s)-\frac{H(t,s)v'(s)+h_1(t,s)\sqrt{H(t,s)}}{(\alpha+1)^{\alpha+1}v^{\alpha}(s)H^{\alpha}(t,s)\tau'^{\alpha}(s)}]ds \quad (13)$$

From (12) for $k=c$ and for any $b>c$

$$\limsup_{t\to\infty}\frac{1}{H(t,c)}\int_c^b [H(b,s)v(s)L(s)-\frac{H(t,s)v'(s)-h_2(t,s)\sqrt{H(t,s)}}{(\alpha+1)^{\alpha+1}v^{\alpha}(s)H^{\alpha}(t,s)\tau'^{\alpha}(s)}]ds>0$$

If adding (12) to (13) ,we obtain the inequality of the theorem 1. Now, the proof is complete.

If for $H(t,s)=(t-s)$, $t\geq s\geq t_0$, we have the following corollary.

Corollary 1. Let assume that A_1, A_2, A_3 hold. If

$$\limsup_{t\to\infty}\frac{1}{t}\int_k^t [(t-s)v(s)L(s)-\frac{(t-s)v'(s)+1}{(\alpha+1)^{\alpha+1}v^{\alpha}(s)(t-s)^{\alpha}\tau'^{\alpha}(s)}]ds \quad (14)$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\int_k^t [(t-s)v(s)L(s)-\frac{(t-s)v'(s)-1}{(\alpha+1)^{\alpha+1}v^{\alpha}(s)(t-s)^{\alpha}\tau'^{\alpha}(s)}]ds \quad (15)$$

for any $H\in X$, $v\in C^1([t_0,\infty),(0,\infty))$ and for all $k\geq t_0$, then every solution of (1) is oscillatory.

Proof: From (14) and (15) for $\frac{\partial H(t,s)}{\partial t}=1$, $\frac{\partial H(t,s)}{\partial s}=-1$ we have (14) respectively (15). The proof is complete.

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