

Natural Absolutely Continuous and Mutually Singular Measures

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Abstract: a linear space can be identified as an integer measure either finite or infinite. Its respective subspaces can be either finite or infinite depending on the dimension of the linear space. An arbitrary subspace will have a mutually singular subspace in the linear space such that the sum of the integer measures of the direct sum of these subspaces will be the integer measure of the linear space. $C[0,1]$ is a linear space of infinite integer measure. $\{0\}$ is a linear space of zero measure. So, every subspace of a linear space can be appropriately suited to the integer measure. A linear operator from a linear space into itself can also be suited with an integer measure that is the integer measure of its range space. The range and null spaces of the linear operator are subspaces of a linear space, it can be taken to task the dimension theorem. It allows the absolute continuous and mutually singular measures partition over the given integer measure that naturally holds for every linear space and every linear operator.

CHAPTER 1:

The basic linear algebra topics of linear space, subspace, linear transformation, range space, null space and their dimensions are considered. Usually the linear operator is defined from a linear space into another such that both are defined over the same field. In the present discussion, the linear space can be finite or infinite dimensional and the linear operators are considered on a subspace into itself. The subspace can either be finite or infinite dimensional.

Definition 1.1: $(F, +, \cdot)$ is a field and $V(F)$ is a linear space or simply V , the dimension of V is the non negative integer m is the integer measure of V .

Definition 1.2: if $T:V(F) \rightarrow V(F)$ is a linear operator and W is a subspace of V , $\mu:V(F) \rightarrow Z^+ \cup \{0\}$ is defined by $\mu(W) = m$ the integer measure of W .

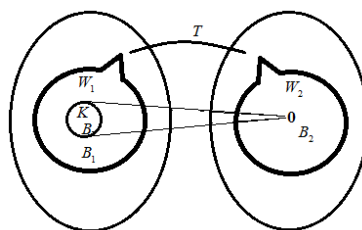
Definition 1.3: if $T:V(F) \rightarrow V(F)$ is a linear operator, $T(W_1) = W_2$ for every pair of subspaces W_1 , and W_2 , then $\mu(T) = \mu(W_2)$

Definition 1.4: if $T(W_1) = W_2$, then $K = \{v \in W_1 : T(v) = 0 \in W_2\}$ is the subspace of W_1 .

CHAPTER 2:

In real analysis, the measure of a set is defined to be a non negative real number suitable to every set of real numbers. In the present case, the word 'non negative' is taken. However, the real number is replaced by an integer as follows.

Definition 2.1: if the basis of one subspace W_1 of $V(F)$ cannot span the other subspace W_2 of $V(F)$, and vice versa, $\mu(W_1)$ and $\nu(W_2)$ are the integers measures of W_1 and W_2 respectively, then μ is said to be mutually singular with ν and is denoted by $\mu \perp \nu$



Definition 2.2: if $T(W_1) = W_2$ is a linear operator with the null space K , $T(W_1) = \mu_1(W_1)$ and $T(W_2) = \mu_2(W_2)$, B_1 and B are the bases of W_1 and K respectively, such that the vectors of W_2 are spanned by the basis vectors of $B_1 \sim B$, then we say μ_2 is absolutely continuous with respect to μ_1 and is denoted by $\mu_2 \ll \mu_1$

Definition 2.3: A property is said to hold **almost everywhere** (a.e.), if the set of points where it fails to hold is a set of integer measure 0.

Working 2.1: $\square [0,1]$ is a linear space. $W_1 = \{a_0 + a_1t + a_2t^2 + a_3t^3 \mid a_i \in \square, 0 \leq i \leq 3\}$ is a subspace of $\square [0,1]$.

$T: W_1 \rightarrow \square [0,1]$ is defined by $T(f(t)) = 0 + a_1t + a_2t^2 + a_3t^3$ is linear.

$T(W_1) = W_2 = \{0 + a_1t + a_2t^2 + a_3t^3 \mid a_i \in \square, 1 \leq i \leq 3\}$ is a subspace.

Clearly, $\dim W_1 = \mu(\square [0,1]) = 4$, $\dim T = \nu_1(\square [0,1]) = 3$ and $K = \{f(t) \in W_1 \mid T(f(t)) = \mathbf{0}\}$

See that if $f(t)$ is a constant polynomial, then it is in K . So, $\nu_0(\square [0,1]) = 1$

Now, the basis of W_1 is $B_1 = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$, basis of K is $B = \{(1,0,0,0)\}$ and the basis of W_2 is

$B_2 = \{(0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

No vector of $L(B)$ can be written as a linear combination of B_2 and vice versa.

This allows $L(B) = K$ is mutually singular with $L(B_2) = W_2$ in terms of integer measure.

So, it can be expressed as $\nu(L(B)) = \nu_0(K) \perp \mu(L(B_2))$

On the other hand, $L(B_2)$ can be spanned by B_1 .

This satisfies $\nu(L(B_2)) = \nu_1(W_2) \ll \nu(W_1)$.

Or $\nu_1 \ll \nu$

This type of working can be done on an arbitrary linear space and a linear operator. Keeping such an idea in view, it can be proceeded as follows. The motive of signed measure is utilized to establish the absolute continuity and mutual singularity of the measures upon the linear spaces.

Chapter 3:

Theorem: there exists a natural decomposition on a linear space $V(F)$ of the integer measure ν such that $\nu_0 \perp \mu$, and $\nu_1 \ll \mu$ such that $\nu_0 + \nu_1 = \nu$ for some integer measure μ over $V(F)$ satisfying $\mu = \nu$ almost everywhere.

Proof: if $V(F)$ is a linear space, then there can be infinitely many linear transformations possible on $V(F)$ satisfying the dimension theorem.

That is, if $T: V(F) \rightarrow V(F)$ is a linear operator, then $\dim R(T) + \dim N(T) = \dim V(F)$ (3.1)

Remember that $R(T)$ and $N(T)$ are the subspaces of $V(F)$ and the dimension of a linear space is termed as the integer measure of the linear space nothing but the number of vectors in the basis of $V(F)$.

The vectors in the basis of the $N(T)$ are linearly independent of that of $R(T)$ and vice versa that satisfies the dimension theorem.

Therefore, $\dim N(T) \perp \dim R(T)$ for every $T: V(F) \rightarrow V(F)$

Since $\dim N(T) = \nu_0(T)$ and $\dim R(T) = \nu_1(T)$

These things satisfy the said statement.

See that $\dim V(F) = \nu(V(F))$ and so, by (3.1), it follows $\nu(V(F)) = \nu_0(V(F)) + \nu_1(V(F))$

In other words $\nu = \nu_0 + \nu_1$ such that $\nu_0 \perp \nu$ and $\nu_1 \ll \nu$

Further, $\nu(W) = \mu(W)$ for almost all subspaces W of $V(F)$.

Hence the proof.

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