More on Stability and Dual Stability of Submodules

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***Abstract*- The aim of this paper is to introduce an indicator for d-stability of any submodule in a module. If N is a submodule of a module M, the intersection of all kernels of the homomorphisms of M into the quotient of M by N is considered. Dually, the sum of all images of the homomorphisms from a submodule N into the module M, as an indicator for the stability of N in the module M, is considered, too.**

Keywords: d-stable; fully d-stable; quasi-projective; stable; fully stable; quasi-injective.

I. INTRODUCTION

Throughout this paper, modules are left unitary modules over an associative ring with identity.

A submodule N of a module M is said to be dual stable(shortly d-stable) if $N \subseteq \ker \alpha$ for each $\alpha \in Hom(M, M/N)$ [2]. A module is fully d-stable if all its submodules are d-stable [2]. These concepts were introduced and studied widely by M. Abbas and the author in many previous papers (see, [2], [3], [4], and [5]). The class of fully d-stable modules is a subclass of duo modules, and it is contained in the class of multiplication modules (the general version, over rings not necessary commutative, see [7]). In fact it lies strictly between those two classes ([2] and [5]). The intersection of the class of quasi-projective with the class of duo modules is a subclass of fully d-stable modules[2]. In certain conditions, the concepts, full d-stability and quasi-projectivity coincide[5].

The cases in which fully d-stable modules become multiplication, also, were discussed in [5]. So the importance of fully d-stable modules can be seen in view of its position among duo, multiplication and quasi-projective modules. In addition, full d-stability has many generalizations (see [3] and [4]), which react with known concepts, in particular "full pseudo d-stability coincide with a dual to the concept of terse modules (due to Weakley [8]), see [4].

In this paper an indicator for the d-stability of a submodule will be introduced, which will tend to new results and properties due to d-stability. This indicator, also, will help to give shorter proofs for old results.

Let N be a submodule of a module M. The d-stability indicator of N in M, denoted by $k_M(N)$, is defined by

$$k_M(N) = \bigcap_{\alpha \in Hom(M,M/N)} \ker \alpha .$$

In section 2, the properties of $k_M(N)$, and its relation with the d-stability of N, will be discussed. Many new results about d-stability, with the help of $k_M(N)$, also proved. As a sample, the intersection of any family of d-stable submodules is again d-stable. While in the case of sum, a condition of being quasi-projective (the module) is needed.

The dual of the new concept, will be studied in section 3. M. Abbas in1991 introduced the concept of stable submodule N in a module M. A submodule N of a module M is said to be stable if, $\alpha(N) \subseteq N$ for each $\alpha \in Hom(N, M)$ [1]. The indicator of stability of N, denoted by $I_M(N)$, will be defined in this paper by $I_M(N) = \sum_{\alpha \in Hom(N,M)} Im\,\alpha$.

As in section 2, the indicator of stability has many properties, and serves to prove new results related to stability. The sum of any family of stable submodules is again stable. The intersection of stable submodules in a quasi-injective module is stable, too.

II. INDICATOR OF D-STABILITY

A. Definition.

Let N be a submodule of a module M, we define $k_M(N) = \bigcap_{\alpha \in Hom(M,M/N)} \ker \alpha$.

B. Proposition.

Let N be a submodule of a module M, then $k_M(N)$ is a submodule of M and **(i)** $k_M(N) \subset N$.
(ii) N is d-stable if and only if $k_M(N) = N$.

(iii) M is fully d-stable if and only if $k_M(N) = N$ for each submodule N of M.

Proof: Immediate by definitions. ◊

C. Lemma

Let M be a module, N and L be submodules of M such that $L \le k_M(N)$, then $k_M(N)/L \le k_{(M/L)}(N/L)$.

Proof: Let $\alpha : M/L \to (M/L)/(N/L)$ be any homomorphism, $\pi : M \to M/L$ be the natural epimorphism, and $\varphi : (M/L)/(N/L) \to M/N$ be an isomorphism. Then an easy check shows that $\ker\alpha = \ker(\varphi \circ \alpha \circ \pi)/L$, that is, for each $\alpha \in Hom((M/L),(M/L/(N/L))$, there is a $\beta \in Hom(M, M/N)$ such that $\ker\alpha = \ker\beta/L$. This implies, $k_M(N)/L \le k_{(M/L)}(N/L)$. ◊

D. Proposition.

If N is a d-stable submodule of a module M and L $\le N$, then N/L is d-stable in M/L.

Proof: N is d-stable implies $k_M(N) = N$ (*B*,(ii)), hence $L \le k_M(N)$, then by (*C*) $N/L \le k_{(M/L)}(N/L)$. Therefore by (*B*, (i) and (ii)) N/L is d-stable in M/L. ◊

The converse of the above proposition is not true, that is, if N/L is d-stable in M/L, it is not necessary that N be d-stable in M. Note the following situation:

Let A be a fully d-stable module, M= $A \oplus A$, L= $A \oplus 0$ and N= $A \oplus B$, where B is a submodule of A, then N/L is d-stable in M/L while N is note d-stable in M (in fact N is even not fully invariant in M).

But the converse of Proposition *D* can be proved if we add certain condition.

E. Theorem.

Let M be a module, $L \le N \le M$. If N/L is d-stable in M/L and $L \le k_M(N)$, then N is d-stable in M.

Proof: Let $\alpha \in Hom(M, M/N)$, by the condition $L \le k_M(N)$, we can define a homomorphism β: M/L→ M/N, by $\beta(x+L) = \alpha(x)$. Then $\varphi\beta \in Hom((M/L),(M/L/(N/L))$, where φ is an isomorphism from M/N onto (M/L)/(N/L). By d-stability of N/L in M/L, N/L $\subseteq \ker(\varphi\beta) = \ker\beta$ which implies (by definition of β) $\alpha(N) = 0$, that is $N \subseteq \ker\alpha$. Therefore N is d-stable in M. ◊

It was proved in [2] , that " If M is fully d-stable and N is a submodule of M, then M/N is also fully d-stable". This result, now, can be concluded as a corollary of Proposition (*D*).

More precisely, Proposition (*D*) gives the following result.

F. Corollary.

If N is a submodule of a module M such that any containing submodule is d-stable in M. Then M/N is fully d-stable. ◊

G. Lemma.

If M is a module and $f \in End(M)$, then $f(k_M(N)) \subset N$.

Proof: Let $f \in End(M)$, π be the natural epimorphism of M onto $M/k_M(N)$, and $\alpha = \pi \circ f$, let $\varphi : M/k_M(N) \to M/N$ be defined by $\varphi(x + k_M(N)) = x + N$. φ is well defined homomorphism since $k_M(N) \subset N$, also $\varphi \circ \alpha = \varphi \circ \pi \circ f : M \to M/N$, so by definition of $k_M(N)$, $k_M(N) \subset \ker(\varphi \circ \pi \circ f) = f^{-1}(\pi^{-1}(\ker(\varphi))$, but $\ker\varphi = N/k_M(N)$ and $\pi^{-1}(N/k_M(N)) = N$, hence $k_M(N) \subset f^{-1}(N)$ which implies $f(k_M(N)) \subset N$. ◊

H. Lemma

For each $f \in End(M)$, the induced map $\bar{f} : M/k(N) \to M/N$, defined by $\bar{f}(x + k(N)) = f(x) + N$ is a well defined homomorphism.

Proof: Clear by Lemma *G*. ◊

I. Proposition.

Let N be a submodule of a module M. If M is quasi-projective then $k_M(N)$ is d-stable in M.

Proof: Let $\alpha \in Hom(M, M/k_M(N))$, and assume that $k_M(N) \not\subset \ker(\alpha)$, then there exists $x \in k_M(N)$ with $\alpha(x) = y + k_M(N)$ and $y \notin k_M(N)$, hence $\beta(y) \neq \bar{0}$ for some $\beta \in Hom(M, M/N)$. Since M is quasi-projective, there exists $f, g \in End(M)$ with $\nu \circ f = \alpha, \pi \circ g = \beta$ where ν and π are the natural epimorphisms of M onto $M/k_M(N)$ and M/N respectively. Define $\gamma : M/k_M(N) \to M/N$ by $\gamma(t + k_M(N) = g(t) + N$, γ is well defined by (2.8) and $\gamma \circ \alpha \in Hom(M, M/N)$, hence $(\gamma \circ \alpha)(x) = \bar{0}$ (since $x \in k(N)$), but $\gamma(\alpha(x)) = \gamma(y + k_M(N)) = g(y) + N = \beta(y) \neq \bar{0}$, a contradiction. ◊

J. Corollary.

If M is a quasi-projective module, then $k_M(k_M(N)) = k_M(N)$ for any submodule N of M. ◊

In [2], it was proved that that a quasi-projective module is fully d-stable if and only if it is duo. A more general result can be stated in the following, the proof is as in [2].

K. Remark.

If M is a quasi-projective module, and N is a submodule of M, then N is d-stable if and only if it is fully invariant in M.

Using the above remark and Proposition *I*, the following can be added.

L. Corollary.

If M is quasi-projective module, N a submodule of M, then

(i) $k_M(N)$ is fully invariant in M.

(ii) $k_M(N)$ is the largest submodule L of M contained in N with the property $f(L) \subset N$ for each $f \in End(M)$.

Proof: (i) clear by (*I*) and the above remark.

(ii) Let $k_M(N) \subset L \subset N$, with $f(L) \subset N$ for each $f \in End(M)$. If $\alpha \in Hom(M, M/N)$ and $f \in End(M)$ be such that $\alpha = \pi \circ f$, where π is the natural epimorphism of M onto M/N, (M is quasi-projective module), then $\alpha(L) = \pi(f(L)) = \bar{0}$, that is, $L \subset \ker \alpha$ for each $\alpha \in Hom(M, M/N)$, hence $L \subset k_M(N)$. ◊

M. Theorem.

The intersection of any family of d-stable submodules of a module is again d-stable.

Proof: Let $\{N_i\}_{i \in I}$ be a family of d-stable submodules of a module M, $N = \bigcap_{i \in I} N_i$ and let $\alpha \in Hom(M, M/N)$. For each $i \in I$, define $\varphi_i \in Hom(M/N, M/N_i)$ by $\varphi_i(x + N) = x + N_i$, then φ_i are well-defined since $N \subset N_i$. Let $\alpha_i = \varphi_i \circ \alpha$, $i \in I$. Since N_i are d-stable, it follows $N_i \subset \ker \alpha_i = \alpha^{-1}(\ker \varphi_i) = \alpha^{-1}(N_i/N)$. So $\alpha(N_i) \subset N_i/N$ for each $i \in I$. Now $\alpha(N) \subset \bigcap_{i \in I} \alpha(N_i) \subset \bigcap_{i \in I}(N_i/N) = \bigcap_{i \in I} N_i \big/ N = 0$. This means $N \subset \ker \alpha$, and hence N is d-stable in M. ◊

In [2], a lemma was proved to serve finding an example of a fully d-stable module which is not quasi-projective. The lemma says " If M is an R-module having exactly three nontrivial submodules, N_1, N_2 and $N_1 \cap N_2$, with M/N_1 not isomorphic to M/N_2, then M is a fully d-stable module which is not quasi-projective." (Lemma 3.7, [2]). Now a shorter proof for the first part of this lemma (the proof of full d-stability) can be given in the following .

<u>**Proof:**</u> By the hypothesis N_1 and N_2 are maximal submodules with M/N_1 not isomorphic to M/N_2, hence by (a note after Example 3.5, [3]) both N_1 and N_2 are d-stable. Then by (Theorem *M*), $N_1 \cap N_2$ is d-stable too. Hence M is fully d-stable.

The sum of d-stable submodules of a module need not be d-stable, as it is seen in the following example.

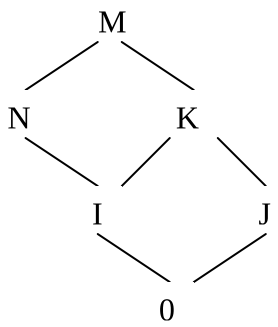
N. Example.

Referring to an example of Hallett [6], where R is an algebra over $Z/2Z$ having basis $\{e_1, e_2, e_3, n_1, n_2, n_3, n_4\}$ with the following multiplication table:

	e_1	e_2	e_3	n_1	n_2	n_3	n_4
e_1	e_1	0	0	n_1	n_2	0	0
e_2	0	e_2	0	0	0	0	0
e_3	0	0	e_3	0	0	n_3	n_4
n_1	0	n_1	0	0	0	0	0
n_2	0	0	n_2	0	0	0	0
n_3	n_3	0	0	0	0	0	0
n_4	0	n_4	0	0	0	0	0

Let $M = Re_1 \oplus Rn_2$, then M has the following submodules:

$N = Re_1$, $K = Rn_2 \oplus Rn_3$, $I = Rn_3$, and $J = Rn_2$. The lattice of these submodules is



Note that N and K are maximal in M, with $M/N \cong M/K$, hence N and K both are not d-stable (see Corollary 3.4, [3]). While I and J are d-stable submodules of M (easy check), and I+J=K which is not d-stable.

With quasi-projectivity, d-stability will be closed under sum of submodules as is shown in the following.

O. Theorem.

If M is a quasi-projective module, then the sum of any family of d-stable submodules of M is again a d-stable submodule of M.

Proof: Let $\{N_j\}_{j\in J}$ be a family of d-stable submodules of a module M, then by Remark *K* each N_j is fully invariant. If $f \in End(M)$, then $f(N_j) \subset N_j, j \in J$, hence $f(\sum_{j\in J} N_j) = \sum_{j\in J} f(N_j) \subset \sum_{j\in J} N_j$, that is, $\sum_{j\in J} N_j$ is fully invariant, and (again by (*K*)) it is d-stable in M. ◊

P. Theorem.

If M is a quasi-projective module, N, K are submodules of M and $\{N_i\}_{i\in I}$ is a family of submodules of M then

(i) If $N \subset K$ then $k_M(N) \subset k_M(K)$.

(ii) $k_M(N)$ is the largest d-stable submodule of M contained in N.

(iii) $k_M(\bigcap_{i\in I} N_i) = \bigcap_{i\in I} k_M(N_i)$ and $k_M(\sum_{i\in I} N_i) \supset \sum_{i\in I} k_M(N_i)$.

(iv) If $\{L_j\}_{j\in J}$ is the family of all d-stable submodules of M contained in N then $k_M(N) = \sum_{j\in J} L_j$.

Proof: (i) Let $N \subset K$ and $x \notin k_M(K)$, then there exists $\alpha \in Hom(M, M/K)$ with $\alpha(x) \neq \bar{0}$. Since M is quasi-projective, there exists $f \in End(M)$ such that $\alpha(x) = f(x) + K$, hence $f(x) \notin K$. Let $\beta \in Hom(M, M/N)$ defined by $\beta = \pi \circ f$, where π is the natural epimorphism of M onto M/N, then $\beta(x) = f(x) + N \neq \bar{0}$ (since $N \subset K$), hence $x \notin \ker \beta$, and so, $x \notin k_M(N)$. Therefore $k_M(N) \subset k_M(K)$.

(ii) Assume that $k_M(N) \subset L \subset N$, where L is d-stable, then by (i) $k_M(L) \subset k_M(N)$, but $k_M(L) = L$, since L is d-stable, hence $L \subset k_M(N)$, and so. $L = k_M(N)$.

(iii) By (i) $k_M(\bigcap_{i\in I} N_i) \subset k_M(N_i)$ for each $i \in I$, Hence $k_M(\bigcap_{i\in I} N_i) \subset \bigcap_{i\in I} k_M(N_i)$. On the other hand $\bigcap_{i\in I} k_M(N_i)$ is a d-stable submodule contained in $\bigcap_{i\in I} N_i$ (Proposition *I* and Theorem *M*), hence $\bigcap_{i\in I} k_M(N_i) \subset k_M(\bigcap_{i\in I} N_i)$ (by (ii)), therefore $k_M(\bigcap_{i\in I} N_i) = \bigcap_{i\in I} k_M(N_i)$.

Also by (i) $k_M(N_i) \subset k_M(\sum_{i\in J} N_i)$, which implies $\sum_{i\in I} k_M(N_i) \subset k_M(\sum_{i\in I} N_i)$.

(iv) Since $k_M(N)$ is a d-stable submodule contained in N, $k_M(N) \in \{L_j\}_{j\in J}$ and so $k_M(N) \subset \sum_{j\in J} L_j$, on the other hand, by(ii) and Proposition(2.15), $\sum_{j\in J} L_j \subset k_M(N)$. therefore, $k_M(N) = \sum_{j\in J} L_j$. ◊

III. THE INDICATOR OF STABILITY

A. Definition.

Let N be a submodule of a module M, we define $I_M(N) = \sum_{\alpha\in Hom(N,M)} Im\,\alpha$.

B. Proposition.

Let N be a submodule of a module M, then $I_M(N)$ is a submodule of M and **(i)** $N \subset I_M(N)$.

(ii) N is stable if and only if $I_M(N) = N$.

(iii) M is fully stable if and only if $I_M(N) = N$ for each submodule N of M.

Proof: Immediate by definitions. ◊

C. Lemma.

If M is a module and $N \le K$ are submodules of M, then $I_K(N) \subset I_M(N)$.

Proof: It is clear since it can be considered that $Hom(N,K) \subset Hom(N,M)$. ◊

D. Proposition.

If $N \le K$ are submodules of a module M, and if N is stable in M, then it is stable in K.

Proof: Clear by Lemma *C* and Proposition *B*, (ii). ◊

E. Remark.

If x is an element of a module M over a ring R, then $I_M(Rx)$ will be simply denoted by $I_M(x)$. It is clear by Proposition(3.2) and (Corollary 1.5 in [1]) that : M is fully stable if and only if $I_M(x) = Rx$ for each x in M.

F. Proposition.

If N is a submodule of a module M, then $I_M(N)$ is stable in M.

Proof: For simplicity denote $I_M(N)$ by I. Let $\beta: I \to M$ be any homomorphism, then $\beta\alpha \in Hom(N,M)$, for each homomorphism $\alpha: N \to M$, since $\alpha(N) \subset I$. Now,

$$\beta(I) = \beta(\sum_{\alpha\in Hom(N,M)} Im\,\alpha) = \sum_{\alpha\in Hom(N,M)} \beta(Im\,\alpha)$$
$$= \sum_{\alpha\in Hom(N,M)} Im(\beta\alpha) \subset \sum_{\delta\in Hom(N,M)} Im\,\delta = I$$

Therefore, I is stable in M. ◊

Remark. If $f \in End(M)$, and if J is a stable submodule containing a submodule N of M, then $f(N) \subset J$. (clear)

G. Corollary.

If N is a submodule of a quasi-injective module M, then $I_M(N)$ is the smallest stable submodule of M containing N.

Proof: Denote $I_M(N)$ by I. Let J be any stable submodule containing N.

$y \in I$ implies $y = \sum_{i=1}^{n} \alpha_i(x_i)$, $\alpha_i \in Hom(N,M)$, $x_i \in N$ and n a positive integer.

Let f_i be an extension of α_i to M for $i = 1,2,...,n$, then $\alpha_i(x_i) = f_i(x_i) \in J$ for $i = 1,2,...,n$ (since $x_i \in N \subset J$, and $f(N) \subset J$ by the above remark). Therefore $y \in I$ and then $I \subset J$. ◊

In [1], it was mentioned that the sum of any family of stable submodules of a module is again a stable submodule. In the following, the intersection will be discussed .

H. Theorem.

If M is a quasi-injective module, and $\{N_i\}_{i\in I}$ is a family of stable submodules of M, then $\bigcap_{i\in I} N_i$ is stable in M.

Proof: Let $\alpha : \bigcap_{i\in I} N_i \to M$, $f \in End(M)$ be an extension of α (M is quasi-injective), and let α_i be the restriction of f to N_i for each $i \in I$, then $\alpha(\bigcap_{i\in I} N_i) \subset \bigcap_{i\in I} \alpha(N_i) = \bigcap_{i\in I} \alpha_i(N_i) \subset \bigcap_{i\in I} N_i$. ◊

Corollary *G* and Theorem *H* will help to prove the following properties of $I_M(N)$.

I. Theorem.

If M is a quasi-injective module, N, K are submodules of M and $\{N_j\}_{j\in J}$ is a family of submodules of M then

(i) If $N \subset K$ then $I_M(N) \subset I_M(K)$.

(ii) $I_M(\bigcap_{j\in J} N_j) \subset \bigcap_{j\in J} I_M(N_j)$ and $I_M(\sum_{j\in J} N_j) = \sum_{j\in J} I_M(N_j)$.

(iii) If $\{L_j\}_{j\in J}$ is the family of all d-stable submodules of M containing N then $I_M(N) = \bigcap_{j\in J} L_j$.

Proof: (i) If $N \subset K$, then by Proposition *B* ,(i) , $N \subset K \subset I_M(K)$. But $I_M(N)$ is the smallest stable submodule containing N (Corollary *E*), hence $I_M(N) \subset I_M(K)$.

(ii) By Theorem *H*, $\bigcap_{j\in J} I_M(N_j)$ is stable, also $\bigcap_{j\in J} N_j \subset \bigcap_{j\in J} I_M(N_j)$, so $I_M(\bigcap_{j\in J} N_j) \subset \bigcap_{j\in J} I_M(N_j)$ by Corollary *G*. For the sum, by the note after *G*, $\sum_{j\in J} I_M(N_j)$ is a stable submodule, also it contains $\sum_{j\in J} N_j$, this implies $I_M(\sum_{j\in J} N_j) \subset \sum_{j\in J} I_M(N_j)$. On the other hand $I_M(N_j) \subset I_M(\sum_{j\in J} N_j)$ for each $j \in J$, since $N_j \subset \sum_{j\in J} N_j$ and by (i). Hence $\sum_{j\in J}(I_M(N_j)) \subset I_M(\sum_{j\in J} N_j)$, therefore $I_M(\sum_{j\in J} N_j) = \sum_{j\in J} I_M(N_j)$.

(iii) It is clear that $I_M(N) \subset \bigcap_{j\in J} L_j$ (by Corollary *G*) and (ii), since $N \subset \bigcap_{j\in J} L_j$).

On the other hand $I_M(N)$ is a member of the family $\{L_j\}_{j\in J}$, hence $\bigcap_{j\in J} L_j \subset I_M(N)$. ◊

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