Modified Picard Method for Solving Boundary Value Problems with Nonlinear and Robin Boundary Conditions

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Abstract— **The objective of this article is to enlarge the applications of Boundary Value Problems Picard Method (BVPP) for ODES with nonlinear and Robin boundary conditions. For this purpose, we propose a modified BVPP method in order to study three case studies, two ODES with nonlinear boundary conditions and one ODE with Robin boundary conditions. In a sequence, we will see that the suggested methodology can generate both, exact and handy accurate approximate solutions. Finally, the residual error of 0.0001115271482 obtained for our third case study shows that the modified BVPP method is potentially useful even for the case of nonlinear problems defined with conditions different of the usual Dirichlet boundary conditions.**

Keywords— ***Linear and nonlinear Differential Equation; Picard Method; Approximate and Exact Solutions; Boundary Value Problems; Nonlinear Boundary Conditions; Robin Boundary Conditions.***

I. INTRODUCTION

It is well known that research on nonlinear ODEs is important because many phenomena, are simply nonlinear. As it is well known, physical phenomena with frequency are modeled through the nonlinear above mentioned equations. The case of BVP for nonlinear ODEs is a branch of higher mathematics which includes several important cases such as Michaelis Menten problem [1], which explains the kinetics of enzyme-catalyzed reactions, Gelfand's differential equation [2,3] that describes combustible gas dynamics, Troesch's equation [4-9], which is employed in the research of confinement of a plasma column by a radiation pressure. Among the proposed methods to study linear and nonlinear problems we are particularly interested in the Picard Iteration Method (PIM). PIM [10-12] is an iterative method used mainly in order to establish some theorems for the existence and uniqueness for ODEs. On the other hand, its practical usefulness is relatively small because the slow convergence of the method and the integration procedure which rapidly becomes cumbersome [10-18].

With the purpose to remedy the above, in [13] BVPP was proposed to take advantage of the fortress of PIM, remedying in the best possible way its drawbacks, in order to use it as a novel tool to get approximate solutions of Dirichlet BVP problems for linear and nonlinear ODEs. Unlike BVPP, this article proposes the Nonlinear Boundary Value Problems Picard Method (NLBVPP) in order to enlarge applications of BVPP for the case of Robin and nonlinear boundary conditions.

The rest of the article is arranged as follows. Section 2 provides a brief review of the basic idea of Picard method. Section 3 provides the elements of NLBVPP method applied to nonlinear and Robin boundary conditions. Additionally, Section 4 presents three cases study, of which two corresponds to exact solutions. Section 5 offers a discussion on the results. Finally, Section 6 presents briefly our conclusion.

II. PICARD ITERATION METHOD

One of the relevant points of this method is the reformulation of the problem

$$y''(t) = f(t, y(t), y'(t)); \qquad y(t_0)=A,\ y'(t_0) = B \quad (1)$$

in terms of the integral equation [13-17].

$$y(t) = A + Bt + \int_{t_0}^{t}\int_{t_0}^{t} f(t', y(t'), y'(t'))dt'dt, \qquad (2)$$

The proposal of PIM consists in express the solution for (2) as the limit of functions $\{y_n(t)\}$, in the limit $n \to \infty$, in accordance with the recurrence formula

$$y_n(t) = A + Bt + \int_{t_0}^{t}\int_{t_0}^{t} f(t', y_{n-1}(t'), y'_{n-1}(t'))dt'dt, \quad n = 1,2,3.. \quad (3)$$

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A relevant point of (1) comes about when $f(t, y(t), y'(t))$ is continuous in all its arguments, with continuous first partial derivatives respect to y and y' in a neighborhood of the initial conditions of (1). In this case we know that independently of the selected initial function $y_0(t)$, the sequence $y_n(t)$ caused by the iterative procedure (3), converges to a solution of (1) [13-17].

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III. BASIC IDEA OF BOUNDARY VALUE PROBLEMS PICARD METHOD APPLIED TO NONLINEAR AND ROBIN BOUNDARY CONDITIONS (NLBVPP)

[13] proposed a methodology able to get analytical approximate solutions for Dirichlet BVP, by which it introduces boundary values of the original problem, PIM method with initial conditions.

As a matter of fact [13] focused BVPP method with the purpose of approximating ODEs for the relevant case of boundary value problems, for which the values of the solution function are specified at two points t_0 and t_1, that is to say,

$$y''(t) = f(t, y(t), y'(t)); \quad y(t_0) = A, y(t_1) = C. \tag{4}$$

The aim of this paper is generalizing the applications for the case of boundary conditions, which includes derivatives evaluated on the boundary points [19-22].

Thus, unlike the rather simple boundary problem (4), we will consider general nonlinear BVP expressed as follows.

$$\begin{gathered} y''(t) = f(t, y(t), y'(t)); \\ g_1(y(t_0), y'(t_0), y(t_1), y'(t_1)) = 0, \\ g_2(y(t_0), y'(t_0), y(t_1), y'(t_1)) = 0, \end{gathered} \tag{5}$$

where g_1 and g_2 are in principle, arbitrary functions.

On the other hand, ODES with Robin boundary conditions refers to the case of (5) where functions g_1 and g_2 express linear combinations of the required solution and its derivative at two points.

Next we will provide the basic steps of NLBVPP method taking into account its application for the boundary value problem (5).

To start, NLBVPP proposes a polynomial function $P(t)$ as trial function which certain parameters $D, E, F, ..$ to be determined

$$y_0(t) = P(t, D, E, F, ..), \tag{6}$$

besides it employs, instead of (2), the following integral equation

$$y(t) = \alpha + \beta t + \int_{t_0}^{t}\int_{t_0}^{t} f(t', y(t'), y'(t'))dt'dt, \tag{7}$$

where, the value of β (and possibly the value of α) is unknown for the moment .

The solution of (7) is expressed as the limit of successive approximations $\{y_n(t)\}$, in the limit $n \to \infty$, in accordance with

$$y_n(t, \beta, D, E, F, ..) = \alpha + \beta t + \int_{t_0}^{t}\int_{t_0}^{t} f(t', y_{n-1}(t', D, E, F, ..), y'_{n-1}(t', D, E, F, ..))dt'dt,$$

$$n = 1,2,3 \ldots \tag{8}$$

Since it has been assumed the continuity of $f(t', y(t'), y'(t'))$ then, irrespective of (6), $\{y_n(t)\}$ converges to the solution of the problem (see section 2)

$$\begin{gathered} y''(t) = f(t, y(t), y'(t)); \\ y(t_0) = \alpha, y'(t_0) = \beta. \end{gathered} \tag{9}$$

In order to guarantee that the n-th iteration of NLBVPP (8) is an approximate solution of (5), the parameters $\beta, D, E, F, ..,$ are selected to assure that the approximate solution satisfies simultaneously the system of equations $g_1(y(t_0), y'(t_0), y(t_1), y'(t_1)) = 0$, $g_2(y(t_0), y'(t_0), y(t_1), y'(t_1)) = 0$ also, and therefore (5). We will see that, although (5) and (9) are related in this manner with the end of motivating NLBVPP convergence; in practice is not necessary to regard the auxiliary problem (9). It should be mention that BVPP proposed three methods to determinate the above parameters, with the purpose of accelerating the convergence and to get accurate approximate solutions. As it is explained in [13], Method 1 and Method 2 require the knowledge of the numerical solution, while third method proposes to employ the Least Squares Method (LSM).

Since sometimes NLBVPP algorithm will require the third of the above mentioned methods, we will provide a brief discussion of it [13].

Supposing that the nth approximation is enough, from (8) we write symbolically

$$y_n = H(t, \beta, D, ..), \tag{10}$$

where $H(t, \beta, D, ..)$, is derived from the iterative process mentioned assuming that there are still parameters to determine, after substituting the nth approximation (8) into nonlinear boundary conditions of (5).

To achieve that (10) corresponds to a precise approximate analytic solution of (5), we have to optimize $\beta, D, ..,$ substituting (10) into equation to solve $y''(t) = f(t, y(t), y'(t))$ (see (5)), from which results the following residual.

$$R(t, \beta, D, ..) = y''(t, \beta, D, ..) - f(t, y(t, \beta, D, ..), y'(t, \beta, D, ..)) \tag{11}$$

Next, it is applied the LSM, in order to minimize the square residual error [23].

$$I(\beta, D, ..) = \int_{t_0}^{t_1} R^2(t, \beta, D, ..)\, dt. \tag{12}$$

It is possible to identify the parameters $\beta, D, E, F, ..$ from the following equations

$$\frac{\partial I}{\partial \beta} = 0, \quad \frac{\partial I}{\partial D} = 0, \ldots \tag{13}$$

IV. CASE STUDIES.

This paper presents three case studies from which the first two show that NLBVPP has the potential to solve ODES even with analytical exact solutions; although we will propose the rather difficult nonlinear boundary conditions. Finally we will employ LSM explained in [13] in order to obtain an analytical approximate solution for a linear problem with Robin boundary conditions.

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Example 1.

This example shows the use of NLBVPP to get the exact solution for the following nonlinear ODE with nonlinear boundary conditions.

$$y''(x) + 2xy'^2(x) - 8xy = 2,\ \ 0 \le x \le 1,$$
$$y(0) = 0, y(1)y'(0) - \tfrac{1}{2}y'(1) = -1. \tag{14}$$

Equation (7) adopts the form

$$y(x) = \beta x + \int_0^x \int_0^x (-2xy'^2(x) + 8xy(x) + 2)dx'dx. \tag{15}$$

The recurrence formula is given by

$$y_n(x) = \beta x + \int_0^x \int_0^x (-2xy'^2{}_{n-1}(x) + 8xy_{n-1}(x) + 2)dx'dx,$$
$$n = 1,2,3.. \tag{16}$$

thus, the first iteration of the proposed method results in

$$y_1(x) = \beta x + \int_0^x \int_0^x (-2xy'^2{}_0(x) + 8xy_0(x) + 2)dx'dx. \tag{17}$$

Next, we select the simpler possible trial function, the initial condition of (14)

$$y_0(x) = 0, \tag{18}$$

thus by substituting (18) into (17), we obtain

$$y_1(x) = \beta x + x^2. \tag{19}$$

Evaluating (16) for$n = 2$, it is obtained

$$y_2(x) = \beta x + \int_0^x \int_0^x (-2xy_1'^2 + 8xy_1 + 2)dx'dx, \tag{20}$$

and after substituting (19) into (20), we get

$$y_2(x) = \beta x + x^2 - \tfrac{\beta^2 x^3}{3}. \tag{21}$$

Assuming that second iteration is sufficient, we substitute (21) into nonlinear boundary condition of (14) in order to assure that (21) satisfies it.

$$\left(\beta + 1 - \tfrac{\beta^2}{3}\right)\beta - \tfrac{1}{2}(\beta + 2 - \beta^2) = -1, \tag{22}$$

it is clear that $\beta = 0$ is a root of (22) and from (21) we get

$$y_2(x) = x^2. \tag{23}$$

In a sequence, by simple substitution we conclude that (23) is the exact solution for the nonlinear problem (14).

Example 2.

In this example, NLBVPP is employed in order to obtain the exact solution for the following nonlinear ODE with nonlinear boundary conditions.

$$y''(x) + y'^2(x) - \tfrac{1}{2}y' = 0, 0 \le x \le 1, 4y(1) - 2y'(1) = 5,$$
$$4y'^2(0) + 8y'(1) = 13. \tag{24}$$

The above differential equation can be rewritten as the following integral equation

$$y(x) = \alpha + \beta x + \int_0^x \int_0^x \left(-y'^2(x) + \tfrac{1}{2}y'(x)\right)dx'dx \tag{25}$$

The recurrence formula for (25) is

$$y_n(x) = \alpha + \beta x + \int_0^x \int_0^x \left(-y'^2{}_{n-1}(x) + \tfrac{1}{2}y'_{n-1}(x)\right)dx'dx,$$
$$n = 1,2,3.. \tag{26}$$

Thus, the first iteration of NLBVPP results in

$$y_1(x) = \alpha + \beta x + \int_0^x \int_0^x \left(-y'^2{}_0 + \tfrac{1}{2}y'_0\right)dx'dx. \tag{27}$$

Next, we select as trial function, the constant function

$$y_0(x) = A. \tag{28}$$

Thus, after substituting (28) into (27), we obtain

$$y_1(x) = \alpha + \beta x. \tag{29}$$

Evaluating (26) for $n = 2$, it is obtained

$$y_2(x) = \alpha + \beta x + \int_0^x \int_0^x \left(-y'^2{}_1(x) + \tfrac{1}{2}y'_1(x)\right)dx'dx, \tag{30}$$

so that, after substituting (29) into (30), we get

$$y_2(x) = \alpha + \beta x + \tfrac{1}{2}\left(-\beta^2 + \tfrac{1}{2}\beta\right)x^2. \tag{31}$$

Assuming that third order approximation is sufficient, next we evaluating (26) for $n = 3$, so that

$$\begin{aligned} y_3(x) = \alpha + \beta x &+ \frac{1}{2}\left(-\beta^2 + \frac{1}{2}\beta\right)x^2 \\ &+ \frac{1}{6}\left(-\beta^2 + \frac{1}{2}\beta\right)\left(-2\beta + \frac{1}{2}\right)x^3 \\ &- \frac{1}{12}\left(-\beta^2 + \frac{1}{2}\beta\right)^2 x^4. \end{aligned} \tag{32}$$

We substitute (32) into boundary conditions of (24) in order to assure (32) satisfies them, thus we get the following algebraic system

$$4\alpha + 2\beta = 5, \quad 8\alpha + 10\beta = 13, \tag{33}$$

with solution

$$\alpha = 1, \quad \beta = 1/2. \tag{34}$$

The substitution of (34) into (32) gives place to

$$y_3(x) = 1 + \tfrac{1}{2}x. \tag{35}$$

Likewise by simple substitution we conclude that (35) is the exact solution for the nonlinear problem (24).

Example 3.

With the purpose to show the application of the proposed method to ODES with Robin boundary conditions, NLBVPP will be used with the purpose of obtaining an approximate solution for the following nonlinear ODE with variable coefficients.

$$y''(x) - \varepsilon x^2 y' = 0, \quad 0 \le x \le 1,$$
$$y(0) = 0, \quad 2y'(1) - y'(0) = 1, \tag{36}$$

Where ε is a parameter.

Just as the cases presented above, we can rewrite (36) in the following integral equation

$$y(x) = \beta x + \varepsilon \int_0^x \int_0^x x^2 y'(x)dx'dx, \tag{37}$$

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and the corresponding iterative formula is given by

$$y_n(x) = \beta x + \varepsilon \int_0^x \int_0^x x^2 y'_{n-1}(x) dx' dx. \qquad (38)$$

Next, we select as trial function

$$y_0(x) = cx, \qquad (39)$$

where c is a constant parameter.

Thus, by substituting (39) into (38) for $n = 1$, we obtain

$$y_1(x) = \beta x + \frac{\varepsilon c x^4}{12}. \qquad (40)$$

Evaluating (38) for $n = 2$, it is obtained

$$y_2(x) = \beta x + \varepsilon \int_0^x \int_0^x (x^2 y'_1(x)) dx' dx. \qquad (41)$$

The substitution of (40) into (41) and after performing the involved elementary integration process we obtain

$$y_2(x) = \beta x + \varepsilon\left(\frac{\beta x^5}{20} + \frac{\varepsilon c x^7}{126}\right). \qquad (42)$$

After substituting (42) into the Robin boundary condition $2y'(1) - y'(0) = 1$(see (36)) it is obtained

$$y_2(x) = \beta x + \frac{\varepsilon \beta x^5}{20} + \left(\frac{1}{14} - \left(\frac{\beta}{14} + \frac{\varepsilon\beta}{28}\right)\right) x^7. \qquad (43)$$

We note that (43) yet depends on β. In order to optimize its value we will employ the LSM; with this purpose we substitute differential equation (36) into (12) to obtain.

$$I = \int_0^1 (y''^2(x) - 2\varepsilon x^2 y'(x) y''(x) + \varepsilon^2 x^4 y'^2(x))\, dx. \qquad (44)$$

Following LSM algorithm, we identify the value of β from the condition

$$\frac{\partial I(\beta)}{\partial \beta} = 0, \qquad (45)$$

where $I(\beta)$ turns out by substituting (43) into integral (44). Thus we get the following expression for β in terms of ε

$$\beta = \frac{117(43052 + 2605\varepsilon)}{80865\varepsilon^2 + 1429480\varepsilon + 14365494}. \qquad (46)$$

Finally, substituting (46) into (43) we get

$$y_2(x) = \frac{117(43052 + 2605\varepsilon)x}{80865\varepsilon^2 + 1429480\varepsilon + 14365494} + \frac{117\varepsilon(43052 + 2605\varepsilon)x^5}{20(80865\varepsilon^2 + 1429480\varepsilon + 14365494)} + \left(\frac{1}{14} - \frac{117(43052 + 2605\varepsilon)}{14(80865\varepsilon^2 + 1429480\varepsilon + 14365494)} - \frac{117\varepsilon(43052 + 2605\varepsilon)}{28(80865\varepsilon^2 + 1429480\varepsilon + 14365494)}\right) x^7. \qquad (47)$$

We note that (47) provides a solution which is valid in principle, for any value of ε.

V. DISCUSSION.

This work proposed an extension of BVPP, the Nonlinear Boundary Value Problems Picard Method (NLBVPP) with the end of finding analytical approximate solutions for Boundary Value problems with nonlinear and Robin boundary conditions. One of the relevant results, which follows from the precision of the obtained results, is that the slow convergence of PIM, is due to an inadequate selection of trial function (some authors suggest always begin PIM, by using the initial condition of the ODE to solve as trial fuction). Although [13] proposed a methodology able to find accurate analytical approximate solutions for BVP. Boundary Value Problems Picard Method studied the most known case of Dirichlet boundary conditions. Therefore this work proposed NLBVPP to involve derivatives into boundary conditions. Thus, we applied NLBVPP method to study three nonlinear differential equations, from which the first two corresponded to nonlinear boundary conditions and the third one to Robin boundary conditions.

We note that our first two nonlinear problems provide an exact solution, and the obtained results by the proposed method showed that in those cases where a problem has an exact solution, NLBVPP is potentially useful to identify it and provide it. Thus, at the first case study, NLBVPP employed its freedom in order to select as a trial function the initial condition of the proposed problem $y_0(x) = 0$. After substituting the proposed analytical approximate solution (21) into nonlinear boundary condition from (14), the unknown initially condition was easily identified as $\beta = 0$. As a matter of fact, by simple substitution we concluded that this value gives place to exact solution for the nonlinear problem (14) (see (23)).

For case study 2, NLBVPP method was employed in order to study the nonlinear problem (24). Unlike the previous case, both boundary conditions are nonlinear. Following the algorithm NLBVPP we proposed as a trial function a constant $y_0(x) = A$. Assuming sufficient the third iteration, NLBVPP method provided a fourth order polynomial provided with two parameters, which correspond to the initially unknown values $y(0) = \alpha, y'(0) = \beta$ (see (32)).

With the purpose to calculate them, we substituted (32) into nonlinear boundary conditions of (24) to get the simple algebraic system (33) with solution given by (34). Finally by substituting (34) into (32) we obtain the exact solution (35).

It is worth mentioning the ease of NLBVPP to calculate the above mentioned parameters and finally finding the solution for the proposed nonlinear problems. This contrast with the complexity of other methods employed to study such problems. The case of problems with nonlinear boundary conditions have been analyzed for other authors. Thus, [19] proposed the reduction of a studied nonlinear boundary value problem to a parameterized boundary-value problem with linear boundary conditions containing some artificially inserted parameters. [20] introduces the theory about nonlinear boundary value problems for ODEs but from rather abstract point of view, emphasizing the numerical solutions of BVP. Moreover, [21] reduced the original nonlinear ODE to a family of problems with linear boundary conditions plus certain nonlinear algebraic determining equations. [22] introduced the theme of nonlinear boundary conditions of second order differential equations but emphasizing a rather theoretical point of view, about the existence of solutions and it is not adequate for applications. Unlike the above methods we noted that the proposed method in this article is straightforward, handy and useful for practical applications.

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Finally, third case study proposed the linear ODE with Robin boundary conditions given by (36) [24,25]. In order to employ NLBVPP algorithm, we proposed as trial the linear function (39). We note that (39) is provided with an adjustment parameter$c$so that, following the proposed algorithm, we obtained the second iteration approximate solution (42). With the same ease as the previous examples, the substitution of (42) into the Robin boundary condition (see (36)), let us to obtain a result which only depends on parameter β(see (43)). In order to optimize the value of β we employed the LSM through condition (45), in such a way that we obtained an expression for β in terms of ε and the approximate solution (47). We noted that equation (47) provides a general solution for the proposed linear problem (36); valid in principle for any value of ε.With the purpose to show the potentiality of the proposed method we regard four particular cases of (47) in order to demonstrate its good performance. Figure 1, show the cases of (47) for the values of $\varepsilon = 0.5$, $\varepsilon = 1$,$\varepsilon = 1.5$ and $\varepsilon = 2.5$.Although these figures visually show the good performance of NLBVPP, it is more reliable to use the square residual error (S.R.E) of (47) defined by$\int_a^b R^2\,(u(t))dt$, with a and b denoting the end points (see (12)),$R(u(t))$is the residual, defined in section 3, while $u(t)$ denotes an approximate solution to the equation of interest, in our case (47) [23]. As it is well known, the square residual error is a positive number, concerned with the total committed error, by using $u(t)$.S.R.E, would be zero only if $u(t)$turns out to be the exact solution of the ODE under study [26].

The resulting values for S.R.E were respectively 0.0001115271482, 0.0002895929165, 0.0004887128544 and 0.001163125442 which confirms the precision of NLBVPP.

Finally, we emphasize the good performance of the proposed method, since we consider only the second iteration of NLBVPP. However, as the values of ε increases or more accurate approximations are required, then we may need greater number of iterations of the proposed method.

VI. CONCLUSIONS.

This article proposed the NLBVPP method, as a natural extension of BVPP with the purpose of including the case of ODEs defined on finite intervals with both, nonlinear and

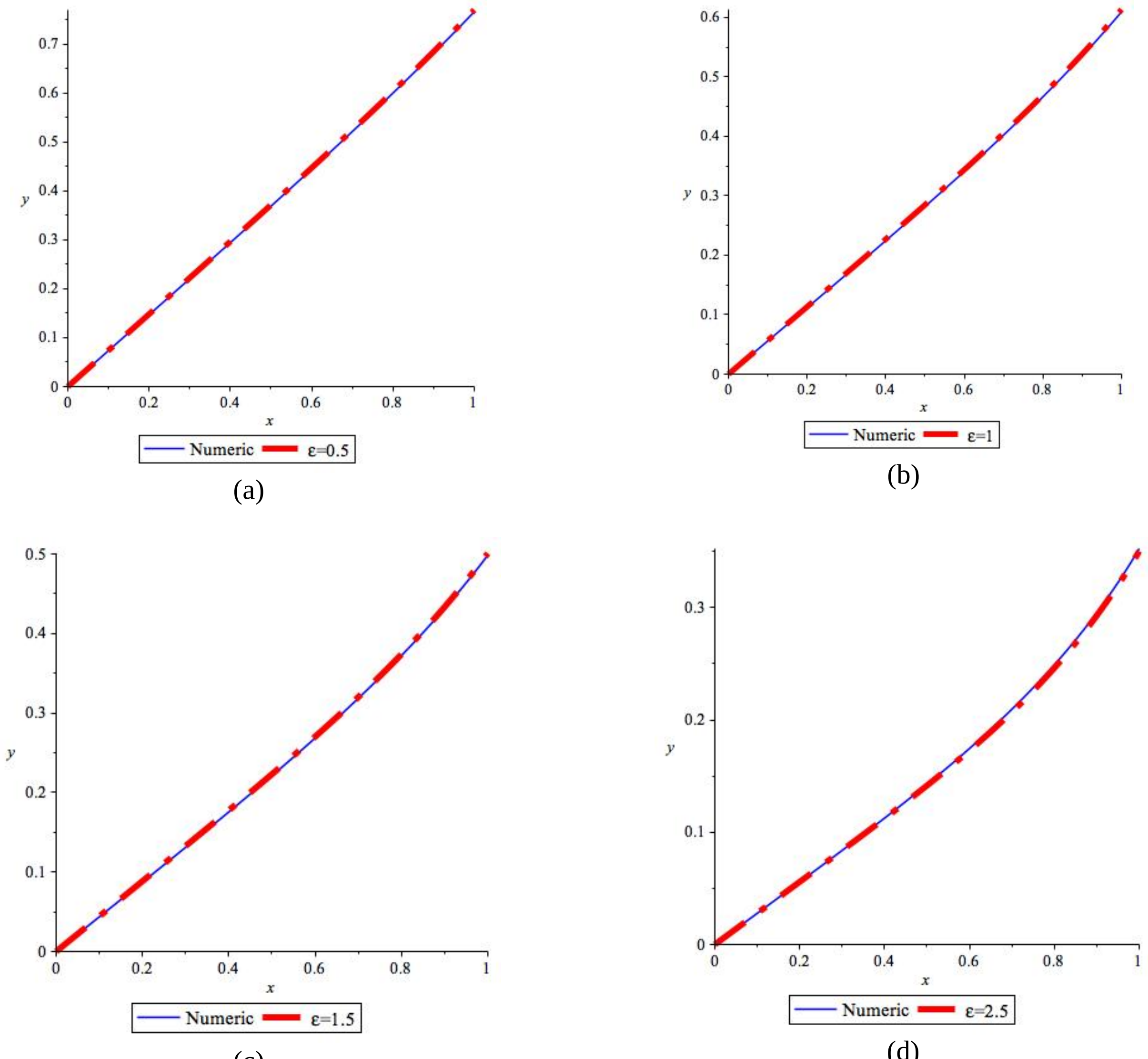

Figure 1. Exact numerical solution (solid line) and approximate solutions (47) for $\varepsilon = [0.5, 1, 1.5, 2.5]$ (a,b,c,d respectively) represented by dash-dot lines.

Robin boundary conditions. Unlike [13] where BVPP was employed to get analytical approximate solutions, this work proposed two cases study with exact solution. We emphasize that NLBVPP had the potential and sensibility to provide the exact solutions even with the additional difficulty to deal with nonlinear boundary conditions problems. Finally NLBVPP faced successfully a nonlinear problem depending on a parameter, with Robin boundary conditions. It is important to note, the method provides a general solution for this case, valid in principle for arbitrary values of the parameter, showing potentiality for this kind of problems.

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