Modern Mathematical Techniques for Analysis of Temperature Distribution in Straight Profile Fins.

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***Abstract* - A fin is an extended surface device attached to the surface of a structure protruding into the adjacent fluid, where its purpose is to increase the heat transfer between the solid surface and the fluid. Because of the wide ranging applications, the analysis of fin heat transfer is of great significance. The equations governing fins with temperature dependent conductivity are nonlinear diffusion type differential equations. Due to the mathematical complexity of these equations exact analytical solutions are not easily tractable. An approximate solution to the original problem can be obtained by using Numerical & Analytical methods or combination of both approaches. In the past, investigations were focused on analytical solutions because they lacked the advantages of today's powerful computers to develop numerical solutions. The present paper illustrates the advanced mathematical schemes for analyzing the temperature profile of the Conductive-Convective rectangular linear fin of straight profile which is the solution of second order differential equation. Linear Differential fin equations are solved through Bessel functions , which gave standard Exact solution then solved by Approximated method, Numerical methods either iterative or non- iterative i.e. Power Series Solution, Finite Difference Technique and modern methods like Differential Transformation Method (DTM) further comparison is made from results obtained by different method in tabular and graphical form. At the end, the paper recommends that method which is fast to calculate and also gives reliable as well as accurate result for calculation of linear fin.**

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***Keywords:* Heat Transfer Fin, Straight Profile, Bessel's Functions, Finite Difference, Power Series, DTM.**

1. INTRODUCTION

The ability to dissipate heat effectively with minimum cost has always been an engineering concern. A paper by **Harper and Brown (1922)** appeared as an NACA report. Which provide a mathematical analysis of the interesting interplay between convection and conduction in single extended surfaces? Harper and Brown called it a *cooling fin*, which

has known merely as a *fin*. Fins are widely used to augment the rate of heat transfer from the primary surface to the ambient medium in a large variety of thermal equipment. Typical components are found in such diverse applications are air–land–space vehicles and their power sources, in chemical, refrigeration, and cryogenic processes, in electrical and electronic equipment, in convention furnaces and gas turbines, in process heat dissipaters and waste heat boilers, and in nuclear-fuel modules. The main objective is to delineate three computational procedures for solving the descriptive equation for a rectangular fin of straight profile. The three computational procedures are the power series method, the finite-difference technique and the differential transformation method.

Fin Selection

The fins are designed and manufactured in many forms. The simplest type of fins for the analysis and manufacturing is the longitudinal fins of rectangular straight profile or constant thickness.

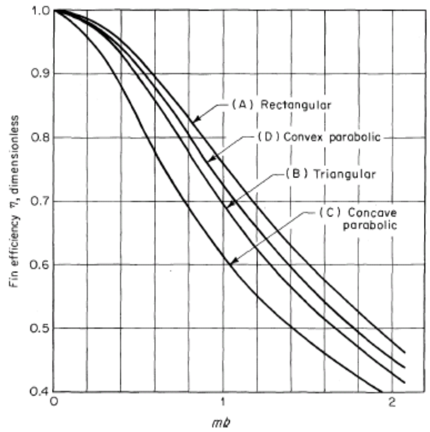

Figure 1 Graph for the efficiency of straight (Longitudinal) fins.

The thermal design of constant fin thickness is also relatively simple. A straight rectangular fin is an extended surface attached to a plane wall. It may be uniform in cross sectional area, or its area may vary along its length to form a triangular, parabolic or trapezoidal shape. Therefore the study on extended surfaces begins with this. The problem for selection of fins of maximum efficiency (shown in figure), less volume and weight are also overcome by its usage.

Modeling

The heat that is conducted through a body must frequently be removed by some convection process. So the analysis of combined conduction-convection system is very important from a practical standpoint.

Consider the one dimensional rectangular fin exposed to surrounding fluid at temperature T_∞ and at Tip of fin $T_0 = 0$,

as shown in figure, Assuming heat flow is unidirectional along b.
Energy In (left face) = Energy out (Conduction through material) + Energy out (by convection to surrounding).

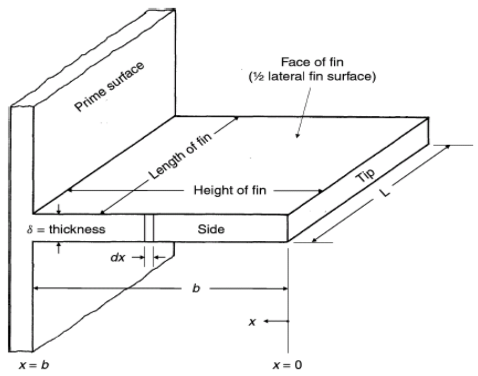

Figure 2: Terminology for the Rectangular fin of straight profile

$$\frac{d^2\theta}{dx^2} - \frac{hP}{KA_C}\theta = 0$$

$$\frac{d^2\theta}{dx^2} - M^2\theta = 0 \qquad (1)$$

Where $\frac{hP}{KA_C} = M^2$

$\frac{T - T_\infty}{T_o - T_\infty}$ (Dimensionless temperature)

Boundary conditions

X=0 $\quad \frac{d\theta}{dX} = 0$;

X=1 $\quad \theta=1$;

By solving differential equation (1) and applying boundary conditions

$$\theta = \theta_0 \frac{\cosh m(b-x)}{\cosh mL} \qquad (2)$$

Mathematical Analysis

The mathematical schemes for obtaining solution of the temperature distribution along the length of Heat transfer Fins which is Conductive-Convective type are as follows.

i) Exact Analytical Solution.
ii) Approximate analytical solution.
iii) Numerical solution schemes.

These solutions are based on the assumption that the Thermal Conductivity (K) and heat transfer coefficient (h) are constant throughout the length of the fin.

Exact Analytical solution

The profile of the fin is governed in the form of second order differential equations which via a transformation can be converted into the Bessel Equation and compared with generalized form of Bessel's equation. Then solved it using the Bessel functions (Theoretical Method) which give standard exact solution. This method found to be important because it represent the solutions to the many differential equations.

(1)	(2)	(3)	(4)	(5)
X	M^2=1	M^2=1.5	M^2=2.0	M^2=6.572
0	0.648054	0.540981	0.459185	0.153145
0.2	0.661059	0.55729	0.47767	0.17372
0.4	0.700594	0.607202	0.534612	0.240971
0.6	0.768246	0.693726	0.634595	0.372969
0.8	0.86673	0.822079	0.78567	0.605181
1	1	1	1	1

Table 1- Temperature distribution by theoretical method

Finite Difference Techniques

To utilize this numerical method, equation of finite approximation written for each node within the material and resultant system of equation solved for the temperature at the various nodes.

Fin region $0 \le X \le 1$ is derived into I equal intervals of size $\Delta X = \frac{1-0}{I}$,resulting in I+1 nodes for i=0,1,2,3……….I as in figure.

Derivative for II order equation can be written as

$$\frac{d^2\theta}{dR^2}\Big|_i = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta X)^2}$$

Truncation error is of order $(\Delta X)^2$

Substitute the values in differential equation

$$\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta X)^2} - M^2\theta_i = 0$$

this equation is equation generator which gives a system of i algebraic equations for node temperatures θ_i , where i=1,2,3………I. Two additional equations are needed to equalize number
of equation in system of equation to the number of node temperature (θ_i).these two relations come from two boundary conditions.

(a) At Fin base (i=0) θ_0 =1, at X=1

(b) At Fin tip (i=I) $\frac{d\theta}{dX} = 0$ at X=0

Incorporate these Boundary Conditions in finite difference form θ_0 =1 at last node i=I, derivative represented by two point central formulation $\frac{d^2\theta}{dR^2}\Big|_I = \frac{\theta_{i+1} + \theta_{i-1}}{2\Delta X}$ with truncation error of $(\Delta X)^2$ Set off equality $\theta_{I+1} = \theta_{I-1}$ and put in equation, Convective fin of rectangular profile I divided into four equal interval of size

ΔX = (1-0 /5)=0.2

i=0 1 2 3 4 5
------------------ *----------*-----------*-----------*
X=0 X_1 X_2 X_3 X_4 X_5

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From equation generator three nodes temperature θ_1 ,θ_2 ,θ_3 regulated by following algebraic equation.

i=1 : $\theta_{i+1} - \left[2 + M^2(\Delta X)\right]\theta_1 + \theta_{i-1} = 0$

We obtained five equation of temperature profile which is representing here in matrix form.

[A] [θ] = [S]

By using MATLAB solution of this matrix for the temperature profile θ at different location of the fin from base (X=1) to Tip (X=0) at interval of 0.2. By taking the various values of constant M^2 and solved by MATLAB we get solution which are represent here in tabular form.

		M²=6.452	*M²=2.0*	*M²=1.5*	*M²=1.0*
S no	*X*	*FDM*	*FDM*	*FDM*	*FDM*
1	0.0	0.1572	0.4610	0.5423	0.6489
2	0.2	0.1779	0.4784	0.5586	0.6619
3	0.4	0.2454	0.5362	0.6084	0.7013
4	0.6	0.3774	0.6354	0.6947	0.7688
5	0.8	0.6087	0.7865	0.8227	0.8671
6	1.0	1	1	1	1

Table 2 - Temperature distribution by Finite difference

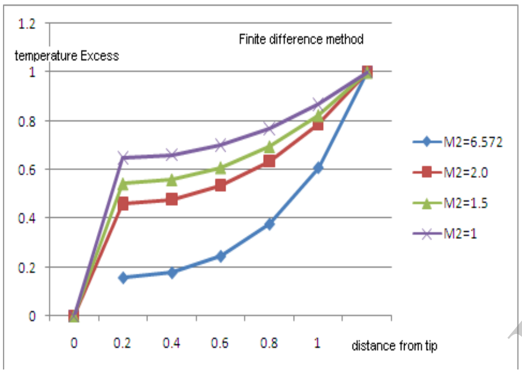

Figure-3: Temperature excess at various values of M^2

Clearly the larger the number of nodes, the more complex and time consuming the solution, even with high speed computer. . Several Software packages are available for solution of simultaneous equations,

including MathCAD, TK solver, Mat Lab and Microsoft Excel.

Power Series Method

The method is a robust analytical method for solving linear differential equations ,Let variable coefficient second order linear homogeneous equation is

$$a_o(x)y'' + a_1(x)y' + a_2(x)y = 0$$

$$y'' + \frac{a_1(x)}{a_o(x)}y' + \frac{a_2(x)}{a_o(x)}y = 0 \quad ; \text{Where} \quad a_0(x) \neq 0$$

$$y'' + p(x)y' + q(x)y = 0 \qquad (3)$$

in this equation the variable coefficient are most likely polynomials represented by power series. A solution of this equation in terms of power series form

$$y(x) = c_o + c_1(x - x_0) + c_2(x - x_0)^2 + ... = \sum_{m=0}^{\infty} c_m (x - x_0)^m \qquad \dots (4)$$

Where the constants c_1, c_2, c_3, c_4............ are the coefficient and x_0 is the center of power series.

$$\frac{dy}{dx} = \sum_{m=1}^{\infty} mc_m (x - x_0)^{m-1}$$

$$\frac{d^2y}{dx^2} = \sum_{m=2}^{\infty} m(m-1)c_m (x - x_0)^{m-2}$$

Substitute values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in differential equation. Then equate the equation to zero the co-efficient of lowest power of x, this gives a quadratic equation in m which is known as ***indicial equation.*** Equate to zero the coefficients of other powers of x to find c_1, c_2, c_3......... in terms of c_0 .Substitutes the values of constants in equation (4) to get the series solution of the equation (3) .It is the iterative method which is used truncation of the series.

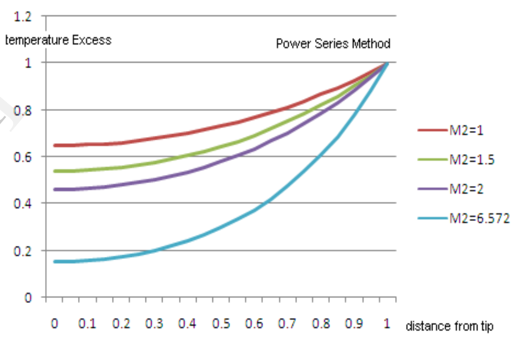

Figure-4: temperature distribution at various M^2.

Differential Transformation Method (DTM)

J.K Zhou proposed concept of DTM. This method Construct an analytical solution in the form of data functions. Differential transform method can easily be applied to DAEs and series solutions are obtained. After the transformation, here, we have formulated series coefficients very simply for the considered problems. Steady state heat conduction in fin profile using a exact series method of solution called differential transformation method. It converges with only six or less terms .it does not require use of Bessel or other special functions.

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Original function y(x)	Transformed function Y(k)
u(x)±v(x)	U(k)±V(K)
cw(x)	cW(k)
$\frac{dw}{dx}$	(k+1)W(k+1)
$\frac{d^n w}{dx^n}$	(k+1)(k+2)…(k+n)W(k+n)
u(x)v(x)	$\sum_{l=0}^{k} U(l)V(k-l)$
x^n	$\delta(k-n)=\begin{cases}1, k=n\\0, k\neq n\end{cases}$

Table-3- Fundamental operations of 1D Differential Transformation Method.

These transformations are then applied to the differential equation given above. Differential transform of

$$\theta(x)=\frac{1}{k!}\left(\frac{d^k\theta}{dx^k}\right)_{x=0}=T(k)$$

Then inverse transformation $\theta(x)=\sum_{k=0}^{\infty} x^k T(k)$

$\theta(x)=T(0)+xT(1)+x^2T(2)+x^3T(3)+x^4T(4)$……… Now transform each term of equation For various values of T (k) , the dimensionless temperature distribution is given by

$$\theta(x)=T(0)+xT(1)+x^2T(2)+x^3T(3)+x^4T(4)+\ldots\ldots$$

$$\theta(x)=0.64863+0.32431x^2+0.0270x^4\ldots\ldots$$

(1)	(2)	(3)	(4)	(5)
X	***M^2=1***	***M^2=1.5***	***M^2=2***	***M^2=6.572***
0	0.648064	0.540997	0.459184	0.154324
0.2	0.661069	0.557309	0.477674	0.175057
0.4	0.700604	0.607226	0.534633	0.24282
0.6	0.768258	0.693758	0.634646	0.375715
0.8	0.866741	0.822115	0.785746	0.608579
1	1	1	1	1

Table 4 - Temperature distribution for Rectangular fin by DTM.

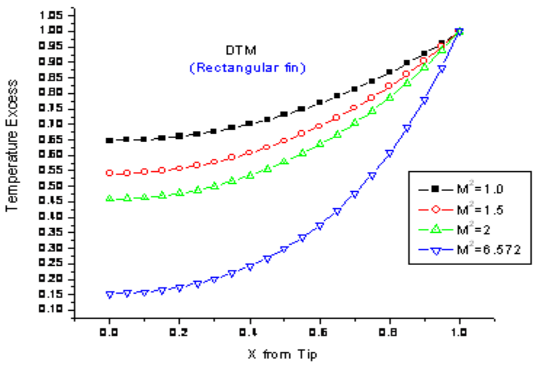

Figure 5: Temperature excess by DTM at various M^2

7. CONCLUSIONS

The differential Transformation technique is one of the numerical methods for ordinary deferential equations, which uses the form of polynomials as the approximation to the exact solutions that are sufficiently differentiable and it provides iterative procedure to obtain the high-order Taylor series. Taylor series method is computationally taken long time for large orders. The differential transform is an iterative procedure for obtaining Taylor series solutions of differential equations and system of PDEs. This method reduces the size of computational domain and applicable to many problems easily. Power series and FDM are powerful methods. But the modern method such as Differential Transformation Method (DTM) is very fast to calculate and gives reliable and accurate result.

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