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Modelling Recursive Surplus Dynamics Via Multidimensional Forward-Backward Stochastic Differential Utility for Solvency, Pricing and Portfolio Allocation

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Abstract: In the present paper, we consider a mash-up of surplus stochastic differential utility and multidimensional forward-backward stochastic differential utility equations. The model allows us to add the stuff of the real world such as jumpy claims, liquidity shocks, liability driven investing, ruin probability constraint stuff and Conditional Value-at-Risk. We use Epstein-Zin recursive preferences to separate risk aversion from intertemporal substitution. How those preferences enjoy the dynamics of surpluses, premium pricing and portfolio allocation. We demonstrate that our system has a unique solution and well-defined forward surplus process and backward utility process. Overall, the stochastic differential utility with multi-dimensional forward-backward stochastic differential utility system is a mathematically sound, yet practical tool for insurers and financial institutions that have to deal with wildly uncertain and incomplete markets.

Keywords: Surplus stochastic differential utility, multidimensional forward–backward stochastic differential utility, conditional value-at-risk, liability-driven investment, solvency.

1. INTRODUCTION

The concepts of recursive preferences and surplus stochastic differential utility (SSDU) have become super-important in continuous time finance, which are widely used in the fields of portfolio choice, asset pricing and long-term solvency problems. One of the major contributions was from Epstein- Zin preferences. These preferences distinguish between being risk averse and the elasticity of intertemporal substitution. This separation alters the optimal consumption and investment decisions. The authors (Kraft et al., 2018), (Björk et al., 2019) and (Biagini et al., 2019) have shown the effect of recursive utility on portfolio choice in incomplete markets. Their work gave behavioral underpinning for models with persistent risk. Backward stochastic differential equations (BSDEs) have been an important part of the modeling of recursive utility. They relate the utility processes to dynamic programming and Hamilton -Jacoby - Bellman equations. The author (DeLong, 2019) gave a good presentation of BSDE methods in finance, while (Hu et al., 2018) illustrated the way forward/backward SDE systems where they capture the interaction between wealth dynamics and recursive utility. This framework is significant in a situation when surplus and utility are dependent on each other. Recent studies have brought mathematical bases of these models such as (Possamaï et al., 2020) existence and uniqueness results for multidimensional quadratic backstopped SDEs, (Cheridito et al., 2020) generalized BSDE theory to the presence of jumps. These results prove to be important for insurance as claim processes often involve sudden shocks (Han and Wong, 2022). Such developments enable recursive utility models to deal with discontinuities and multi-sources of risk. In the field of actuarial science, the dynamics of surplus and probabilities of ruin are important topics. The classical ruin theory gives static estimates of the insolvency risk and the author (Lin and Li, 2019) gave generalized ruin theory to models with stochastic investment. Ruins bound for controlled surplus processes were obtained by (Hillairet and Renaud, 2020) and ruins probability under stochastic environment was also studied by (Liang and Yu, 2022). These studies shifted from static constraint measures of ruin to dynamic constraint measures of ruin as part of optimization problems. Liability-driven investment (LDI) and asset - liability management (ALM) are also popular fields for study. The author (Lin and Young, 2020) demonstrated the inclusion of liabilities as a direct model, which alters the optimal portfolio allocation and (Devolder and Mitric, 2020) have

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stressed the existence of the need for solvency-consistent pricing in incomplete markets. The results point to the fact that decisions regarding price and capital need to be made with consideration of liability risk. Tail risk measure like conditional value -at -risk (CVaR) is also incorporated into insurance models and (Chen and Yang, 2018) gave a portfolio optimization with CVaR under recursive utility. The author (Ni and Sun, 2019) studied the optimal reinsurance and investment under the CVaR constraints. These studies indicate that the control for tail risk affects investment behaviors and capital allocation. Recursive preference has also recently been taken in account in incomplete and equilibrium markets (Fadina and Schmidt, 2021). These works emphasize on the importance of the hedge demand and ambiguity in deciding the premium rates and asset allocation. The computational methods such as high dimensional BSDE's have become solvable by deep learning. The nonlinear BSDE efficient algorithms were developed by (Kapllani and Teng, 2024). These techniques render forward/backward systems large enough to be practical for real insurance problems. Recursive models of utility typically disregard explicit constraints of ruin and ruin models are typically based on traditional assumptions of utility. Therefore, the models including liquidity shocks, multidimensional jumps, CVaR and recursive aggregation in one system still are limited. There is a need for a framework with focus on solvency. This is the motivation for coming up with an integrated approach of surplus stochastic differential utility and multidimensional forward-backward stochastic differential utility (MFBSDU). This research paper underscores that the literature reveals that

(i) Solid mathematics for multidimensional BSDE and FBSDE systems of jumps, quadratic growth.
(ii) A mature actuarial literature for ruin, CVaR and LDI.
(iii) Fast moving computational techniques for solving high dimensional BSDEs.

There is a lack of a coherent solvency-oriented framework that combines surplus dynamics and jumps, Epstein-Zin recursive preferences, embedded ruin and CVaR controls and liability-based allocation of incomplete markets.

This paper demonstrates the impact of the SSDU-MFBSDU framework on premium loadings; solvency bounds and hedge demand in comparison to classical benchmarks and addresses the integrated approach.

2. METHODOLOGY

To ensure a common understanding, this paper defines several terms central to the study. Those are defined below:

- Probability space and Filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[o,T]}, \mathbb{p})$: We consider probability space is a triplet $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[o,T]}, \mathbb{p})$, where, Ω is the sample space, $\mathcal{F}$ is a σ-algebra of subsets of Ω and $\mathbb{p}$ is the probability measure. A σ-algebra $\mathcal{F}$ is a collection of subsets of Ω satisfying the following conditions
  i. $\Omega \in \mathcal{F}$
  ii. If $A \in \mathcal{F}$ then$A^c \in \mathcal{F}$
  iii. If $A_1, A_2, \cdots \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

  A probability filtration explicitly from random variables $X_1, X_2, \dots X_n$ for discrete andcontinuous time construction.
- Discrete and Continuous Time Canonical: The times $t \epsilon 0,1,2 \dots n$ with $t = 0$ initial the sample space is $\Omega \in \mathbb{R}^n, \omega = (\omega_1, \omega_2, \dots \omega_n)$ with the σ-field is $\mathcal{F} = B(\mathbb{R}^n)$is called Borel-σ-field. The probability measure is $\mathbb{p}$ on $\left(\mathbb{R}^n, B(\mathbb{R}^n)\right)$ this encodes the joint law of $(X_1, X_2, \dots, X_n)$. The canonical random variables define discrete if the natural filtration is $\mathcal{F} = \{\phi, \Omega\}$, $\mathcal{F} = \sigma(X_1, X_2, \dots X_t)(t); t = 1(1)n$
  $N = \mathcal{A} \epsilon \mathcal{F}\,; P(\mathcal{A}) = 0$ then $\bar{\mathcal{F}}_t = \sigma(\mathcal{F}_t \cup N)\ \forall\ \bar{\mathcal{F}}_{\square} = \bar{\mathcal{F}}_n$
  The canonical random variables define continuous if the natural filtration generated by the step process or right continuous, $\mathcal{F}_t^o = \sigma(X_s; 0 \le s \le t), t \in (0, T]$, $\hat{\mathcal{F}}_t\{u > t\} = \mathcal{F}_t$. This yields a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[o,T]}, \mathbb{p})$.
- Asset Dynamics: Let $X_1, X_2, X_3, \dots X_n$ random variables for asset prices n be a risky asset and one is risk-free asset i.e. $S_t^i : \Omega \longrightarrow R_+$ at time t. Each risky free asset is an $\mathcal{F}_t$-adapted stochastic process that is $S_t^i \epsilon L^2\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[o,T]}, \mathbb{p}\right) = e^{rt} = B_t\ \forall r \ge 0$ is called as risk-free rate. Let us consider Brownian Motion of driving uncertainty with satisfied filtration $\mathcal{F}_t$ is
$$S_t^i = S_0^i \exp\left[\left(\mu_i - \frac{1}{2}\sum_{j=1}^{d} \sigma_{ij}^2\right)t + \sum_{j=1}^{d} \sigma_{ij}^2\, W_t^j\right]$$
  S_t^i is called asset dynamics rigorously set on the probability space and filtration.
- Actuarial liabilities: Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[o,T]}, \mathbb{p})$ supporting an m-dimensional Brownian motion W and a Poisson random measure $N(dt, dz)$ on $[0, T]$ with predictable compensator $\Lambda(dt, dz) = \lambda_t v(dz)$. Assume $\int_{R_+}^{\square} \Lambda z^2 vdz < \infty$ and λ_t is a bounded and predictable. The cumulative claims process is $L = (L_t)_{t\epsilon[0,T]}$ by $L_t = \int_0^t \int_{R_+} zN(ds, dz)$, $L_0 = 0$. So that the jumps $\Delta L_t = z > 0$ correspond to claim payments of size z. Its predictable compensator $\bar{Z} = \int_{R_+}^{\square} zvdz$ then $E[dL_t | \mathcal{F}_{t-}] = \lambda_t \bar{Z} dt$ thedecomposition is
$$dL_t = \lambda_t \bar{Z} dt + \int_{R_+} z\tilde{N}(dt, dz)$$
  where, $\tilde{N}(dt, dz) = N(dt, dz) - \lambda_t v(dz)dt$
  dL_t is called as actuarial liabilities or liability noise.
- Surplus Dynamics: The surplus process is the difference between assets and cumulative liabilities of net claims. The surplus (reserve) dynamics under self-financing and claims outflow is

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$$dX_t = (r_t X_t + \pi_t^T(\mu_t - r_t) + P_t - C_t - \lambda_t \bar{z})dt + \pi_t^T \Sigma_t dW_t - \int_{R_+} z\tilde{N}(dt, dz) + \sigma_L dW_t^{(L)}$$

- Risk adjusted return: A portfolio strategy represented by π_t which may be an m-dimensional vector of allocation weights to assets or risky exposures. $R(\pi_t)$ is the random return of the portfolio over a period. $E[R(\pi_t)]$ is the expected return. $\rho(\pi_t)$ is a risk-measures quantifying the risk of the portfolio return. The most common form of the risk adjusted return ratio is $RAR(\pi_t) = \frac{E[R(\pi_t)] - r_t}{\rho(\pi_t)}$.
- Entropic Risk Adjustment: The entropic risk of a random terminal L is defined as the $\rho_{ent}(L)$.
$$\rho_{ent}(L) = \frac{1}{\theta}\log\left(E\left(e^{\theta L}\right)\right)$$
$$\rho_{ent}(L) = \max_{\pi_t, C_t}\left(-\frac{1}{\theta}logE\left[e^{-\theta\int_0^T u(C_t, V_t)dt}\right]\right)$$
This becomes risk sensitive of the maximization of entropic risk adjustment.
- Ruin Probability: The probability p and its asset increase S_t^i to $(S_t^i + 1)$.The ruin probability is now $P_A(S_t^i + 1)$. We can similarly reduce the ruin probability becomes $P_A(S_t^i - 1)$ if it loses at the first bet with the probability q. As there are only these two independent outcomes at the first time T_1, we see the ruin probability $P_A(S_t^i)$is given by the following:$P_A(S_t^i) = pP_A(S_t^i + 1) + qP_A(S_t^i - 1),\ \ 1 \le S_t^i \le A - 1$.
- Terminal Utility: Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[o,T]}, \mathbb{P})$ be a probability space with time horizon $T > 0$. A terminal utility is a utility function $U: \mathbb{R}_+ \longrightarrow \mathbb{R}$ applied to the terminal wealth X_T of an economic agent at time T.The agent's objective is typically to maximize the expected terminal utility $\max_{\pi\epsilon\mathcal{A}} \mathbb{E}[U(X_T^\pi)]$.

Developing our proposed model rests on a set of foundational assumptions that ensure both statistical and mathematical tractability and practical relevance. The following assumptions simplify complex and real-world dynamics, and they are consistent with prior research and provide a rigorous basis for subsequent theoretical development.

Assumption 1: Predictability and boundedness

The coefficients $r, \mu, \Sigma, p, \lambda, and\ \sigma_L$ are predictable such that for any admissible (π_t, C_t), the stochastic differential equation for X domains admit a unique strong solution and all stochastic integrals are well defined. The kernels $t \longmapsto P_t(\cdot)$ and $(t,z) \longmapsto \sigma_L(t,z)$ are $\mathcal{F}_t -$predictable. Moreover, for some constant $K > 0$,

$$|r_t| + \|\mu_t\| + \|\Sigma_t\| + \lambda_t \le K \text{ a.s. for a.e. } t\epsilon[0,T] \text{and}$$
$$\int_{R_+} (\Lambda\|\sigma_L(t,z)^2\|)P_t(dz) \le K \quad \text{a.s. for a.e. } t$$

Assumption 2: Nondegeneracy or regularity of diffusion

$\Sigma_t\Sigma_t^\top$ is uniformly bounded in the aggregator $\mathcal{F}[0,T] \times R \times R \times R \longrightarrow R$ and terminal $g: R \longrightarrow R$ satisfy the usual growth conditions ensuring there exists $K > 0$ with $\xi^\top\Sigma_t\Sigma_t^\top\xi \ge \kappa \parallel \xi \parallel^2\ for\ all\ \xi \in \mathbb{R}^d$ the drift and diffusion coefficient of X satisfy Lipschitz and linear conditions.

Assumption 3: Admissible controls.

Consider the first assumption and satisfy for some $K > 0$ and$X_t \ge 0$ a.s. for all $t\epsilon[0,T]$ then $\mathbb{E}\left[\int_0^T(\|\pi_t\|^2 + C_t)dt\right] < \infty$ and the jump integrability conditions are

$$\mathbb{E}\left[\int_0^T \int_R |\pi_t^T\sigma_L(t,z)|^2\, v_t(dz)dt\right] < \infty$$
$$\partial_t v(t,x) + \sup_{\pi\epsilon R^n, C_t}\{\mathcal{L}^{\pi,C}v(t,x) + f(t,x,C,v(t,x))\}$$
$$\text{where, } \mathcal{L}^{\pi,C}v(t,x) = (r_t x + \pi^T(\mu_t - r_t) + P_t - C_t - \lambda_t\bar{z})v_x(t,x) + \left(\frac{1}{2}\|\Sigma_t^T\pi\|^2 + \sigma_L^2\right)v_{xx}(t,x) + \int_{R_+}(v(t,x-z) - v(t,x) + zv_x(t,x))\lambda_t v(dz)$$

Assumption 4: Lipschitz and Linear growth conditions

If any coefficient depends on X_t then there exists $L > 0$ such that for all $x, x' \in \mathbb{R}$ and a.e. t,

$$|b(t,x) - b(t,x')| + \|\sigma(t,x) - \sigma(t,x')\| + \left(\int_R |\gamma(t,x,z) - \gamma(t,x',z)|^2 v_t(dz)\right)^{\frac{1}{2}} \le L|x - x'|$$
$$|b(t,x)|^2 + \|\sigma(t,x)\|^2 + \int_R |\gamma(t,x,z)|^2 v_t(dz) \le L(1 + |x|^2)$$
$$where, b(t,x) = r_t x + \pi_t^\top(\mu_t - r_t) - C_t, \sigma(t,x) = \pi_t^\top\Sigma_t, \gamma(t,x,z) = \pi_t^\top\sigma_L(t,z)$$

2.1. Some Theoretical results

To support the integrated approach SSDU - MFBSDU framework a solid and logical base, the first one sets up the surplus stochastic differential utility when we're using admissible controls, so as to make sure that the surplus process, complete with jumps and liquidity risk, has a nice recursive utility representation. The second one extends this to a multi-dimensional forward backward system because the evolution forward of the surplus and the recursive utility backward aggregation really depend on each other. We throw in an intertemporal verification theorem too, so that we can link our stochastic control set-up to the Hamilton-Jacobi-Bellman principle and thereby show that those strategies are actually optimal. Altogether, these theorems are chosen to ensure existence, uniqueness, verifying optimality, and delivering a good economic theory around the topic of solvency.

Theorem 1: Proposed Surplus Stochastic Differential Utility (SSDU)

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Statement: Let us setup the newly synthesized on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[o,T]}, \mathbb{p})$ supporting Brownian motion W and compensated Poisson random measure $\widetilde{N}(dt, dz)$ with compensator $\lambda_t v(dz)dt$. The surplus process X_t follows

$$dX_t = (r_t X_{t-} + \pi_t^T(\mu_t - r_t) + P_t - C_t - \lambda_t \bar{z})dt + \pi_t^T \Sigma_t dW_t - \int_{R_+} z\widetilde{N}(dt, dz) + \sigma_L dW_t^{(L)}$$

With admissible controls $(\pi_t C_t)\epsilon\mathcal{A}$.

Theorem 2: Multidimensional Forward and Backward Stochastic Differential Utility (MFBSDU)

Statement: Let us consider a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[o,T]}, \mathbb{p})$ support a m-dimensional Brownian motion (W) and independent Brownian motion $\left(W^{(L)}\right)$ of a Poisson random measure $N(dt, dz)$on R_+ with compensator $\lambda_t v(dz)dt$ and its measure $\widetilde{N}(dt, dz)$. The modified forward and backward stochastic differential utility function is

$$Y_t = g(X_t) + \psi(t)$$

where, $Y_t$represents the value function process for optimal cost.

By using Super Martingale under optimal control, Equation (1) becomes

$$Y_t = E_t\left[g(X_t) + \int_t^T f(S, X_s, Y_s, Z_s, \mu_s)ds\right]$$

$$-dY_t = \sup_{\pi_t\epsilon R^n, C_t\geq 0}\left\{f(t, x_t, C_t, Y_t) + Y_t b(t, x_t, \pi_t, C_t) + Z_t^T\sigma(t, \pi_t) + \left(Z_t^{(L)}\right)^T\sigma_L - \int_{R_+} ZU_t(z)\,\lambda_t v dz\right\}dt - Z_t^T dW_t - \left(Z_t^{(L)}\right)^T dW_t^{(L)} - \int_{R_+} U_t(z)\,\widetilde{N}(dt, dz)$$

Theorem 3: Intertemporal verification theorem for MFBSDU Utility

Statement: Let us consider a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[o,T]}, \mathbb{p})$ support a m-dimensional Brownian motion (W)an optional liability Brownian motion $\left(W^{(L)}\right)$ of a Poisson random measure $N(dt, dz)$ on R_+ with compensator $\lambda_t v(dz)dt$. Fix the surplus dynamics under control (π_t, C_t).

$$dX_t = b(t, x_{t-}, \pi_t, C_t)dt + \sigma(t, \pi_t)dW_t + \sigma_L dW_t^{(L)} - \int_{R_+} z\widetilde{N}(dt, dz) \text{ and } X_0 = x$$

where, $b(t, x_{t-}, \pi_t, C_t) = r_t\pi^T(\mu_t - r_t) + P_t - C_t - \lambda_t\bar{z}$, and $\sigma(t, \pi_t) = \Sigma_t^T\pi_t$

Assume the assumptions 1, 2, 3 and 4 then for the control (π_t, C_t) the process Y_t together with

$$Z_t = v_x(t, x_{t-}) * \sigma(t, \pi_t), Z_t^{(L)} = v_x(t, x_{t-}) * \sigma_L(t, \pi_t) \text{and} U_t(Z) = v(t, x_{t-}, -z) - v_x(t, x_{t-})$$

These are satisfying the MFBSDU. The value function is $v(t, x_{t-})$ ensures the below

$$v(t, x_{t-}) = \sup_{(\pi,C)\in\mathcal{A}(t,x)}\left\{E_t\left[g(X_t) - \psi(X_t) + \int_t^T f_s(S, X_s, C_s, v(S, X_s))ds\right]\right\}$$

(π_t^*, C_t^*) is optmal control.

The Figure 1 of verification theorem really represent a strong trend toward a convergence. At more modest sample sizes, the simulated utility paths come pretty close to swinging around the theoretical line: this is pretty normal-this is sampling noise. But once you bump up the sample size from 100 up to 10k, the simulated points are almost in line with the theoretical curve, almost all the way. Basically, the regression line hits right for top on the theoretical ones in big samples, proving the recursive formulation is numerically stable and in accord with theory.

Theorem 4: Diffusive Merton Model with Additive MFBSDU Utility

Statement: Let us consider $(\pi_t^*, C_t^*)$is optmal control with the factor x is $\frac{v_t}{x^{1-\gamma}} = \frac{A'(t)}{1-\gamma}$ and$\psi(C_t^*) = \frac{k^{1-\gamma}x^{1-\gamma}}{1-\gamma}$

$$\frac{\delta}{1-\gamma}(\psi(C_t^*) - v) = \frac{\delta}{1-\gamma}[k(t)^{1-\gamma} - A(t)]$$

collecting, dividing by $x^{1-\gamma}$, The HJB yields the scalar ordinary differential equation is

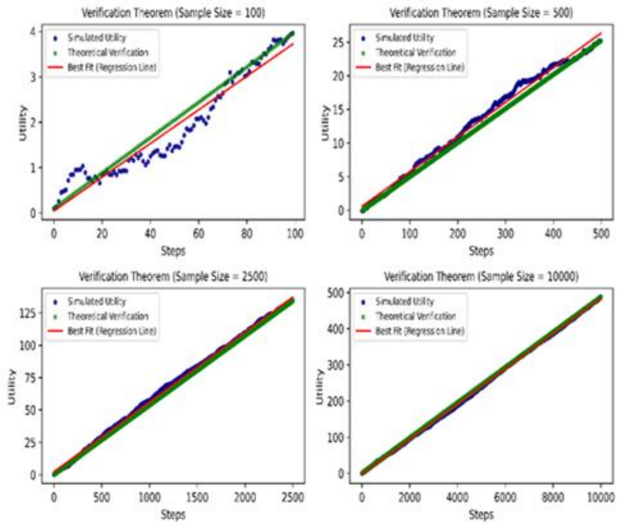

Figure 1: verification theorem illustrated: simulated utility converges toward the theoretical with regression fit confirming consistency across sample sizes.

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$$A'(t) = (\gamma - 1)A(t)\left(r_t + \frac{\eta(t)}{2\gamma} - \left(\frac{\delta}{A(t)}\right)^{\frac{1}{\gamma}}\right) + \delta\left(A(t) - \left(\frac{\delta}{A(t)}\right)^{\frac{\gamma-1}{\gamma}}\right)$$

3. RESULTS AND DISCUSSIONS

Modeling the surplus with a jump-diffusion is because the insurance claims and the policy shocks aren't smooth. We established the existence, uniqueness, and recursive structure of surplus stochastic differential utility (SDU) within a multidimensional forward–backward SDE framework, and demonstrated its connection to Epstein–Zin aggregation, risk-adjusted return optimization, and constant relative risk aversion (CRRA) terminal utility under market frictions and liquidity shocks. The recursive nature of the Epstein-Zin aggregator allows us to decouple the risk aversion from the way people trade off time, which is critical to the solvency of people over the long term. By adding CVaR, we are able to capture extreme tail risk in a coherent way to keep the model in order to comply with the regulation rule on capital. The liability driven investment structure reveals that you cannot set asset allocation without also taking into consideration liabilities. The hedge -demands breakdown reveals the change in equity exposure between policy uncertainty and the classical Merton rule. Our duality formulation is concerned with the issues of market incompleteness and non-tradable risks.

3.1. Surplus Stochastic Differential Utility

Let us consider $x_1, x_2, x_3, \dots, x_n$ be a random variables X be a vector of random variables. Suppose X_t is a process influenced by several random variables of the stochastic differential equation with vector randomness is dX_t.

$$dX_t = f(t, x_t, x_1, x_2, \dots x_n)dt + g(t, x_t, x_1, x_2, \dots x_n)dW_t$$

where, f and $g$are suitable drift and diffusion functionsuppose $X_t$becomes multidimensional SDE then$X_t = \left(x_t^{(1)}, x_t^{(2)}, \dots, x_t^{(n)}\right)$ then $dX_t = f(t, X_t)dt + \Sigma(t, X_t)dW_t$,where Σ is an $n \times m$ matrix and W_t is a m-dimensional Brownian motion . Let us consider risk free asset dB_t and risky asset S_t^i are $dB_t = r_t B_t dt$ therefore $B_0 = 1$, where, B_t is a bank account.

$$dS_t^i = diag(S_t^i)(\mu_t dt + \Sigma_t dW_t)$$

where, $(\mu_t \in \mathbb{R}^n)$ is an expected return, $(\Sigma_t \in \mathbb{R}^{n\times m})$ is a volatility matrix and $(W_t \in \mathbb{R}^m)$ is a Brownian motion. Consider self-financing portfolio dynamics $dX_t = r_t x_t + \pi_t^T(\mu_t - r_t)dt + \pi_t^T \Sigma_t dW_t$. Apply the consumption and exogenous income of this equation and subtract compensated Poisson measure.

$$dX_t = (r_t x_t + \pi_t^T(\mu_t - r_t) + P_t - C_t)dt + \pi_t^T \Sigma_t dW_t) - \int_{R_+} \tilde{N}\,(dt, dz)$$

$$dX_t = (r_t x_t + \pi_t^T(\mu_t - r_t) + P_t - C_t)dt + \pi_t^T \Sigma_t dW_t) - \int_{R_+} zN\,(dt, dz) + \int_{R_+} \lambda_t z v(dz)dt$$

$$dX_t = (r_t x_t + \pi_t^T(\mu_t - r_t) + P_t - C_t)dt + \pi_t^T \Sigma_t dW_t) - \int_{R_+} zN\,(dt, dz) + \int_{R_+} \lambda_t \bar{z}(dz)dt$$

Adding liquidity risk σ_L to the above equation, we get

$$dX_t = (r_t x_t + \pi_t^T(\mu_t - r_t) + P_t - C_t)dt + \pi_t^T \Sigma_t dW_t) - \int_{R_+} zN\,(dt, dz) + \int_{R_+} \lambda_t \bar{z}(dz)dt + \sigma_L dW_t^{(L)}$$

The above equation becomes the wealth dynamics with risky investments, consumption, Laboure income, jumps and liquidity shocks.

3.2. Relationship between Surplus SDE and Epstein-Zin Aggregator

By using surplus dynamics consumption ratio of habit adjusted surplus of $S_t^i$is $S_t^i = \frac{C_t - H_t}{H_t}$

Consider Epsten-Zin recursive utility aggregator of v_t

$$v_t = \left((1-\beta)C_t^{1-\frac{1}{\psi}} + \beta\left(E_t\left[v_{t+1}^{1-\gamma}\right]\right)^{\frac{\left(1-\frac{1}{\psi}\right)}{1-\gamma}}\right)^{\frac{1}{1-\frac{1}{\psi}}}$$

In case consider continuous time, the aggregator takes the form of a MFBSDU

$$dv_t = \left(\frac{\delta}{1-\frac{1}{\psi}} v_t\left(\left(\frac{C_t}{v_t}\right)^{1-\frac{1}{\psi}} - 1\right)\right)dt + z_t dW_t$$

If H_t grows deterministically at rate η then $dH_t = \eta H_t dt$ and substitute into Epstein-Zin aggregator

$$dv_t = \frac{\delta}{1-\frac{1}{\psi}} v_t\left(\left(1+S_t^i\right)^{1-\frac{1}{\psi}} H_t^{1-\frac{1}{\psi}} v_t^{-\left(1-\frac{1}{\psi}\right)} - 1\right)dt + z_t dW_t$$

$$dv_t = \frac{\delta}{1-\frac{1}{\psi}} v_t\left(\left(\frac{\left(1+S_t^i\right)H_t}{v_t}\right)^{1-\frac{1}{\psi}} - 1\right)dt + z_t dW_t$$

$$dv_t = \frac{\delta}{1-\frac{1}{\psi}} v_t\left(\left(\frac{C_t}{v_t}\right)^{1-\frac{1}{\psi}} - 1\right)dt + z_t dW_t$$

The relationship between surplus dynamics and Epstein-Zin aggregator. It is also called as the standard continuous time EZ aggregator of MFBSDU.

3.3. Risk Adjusted return with constraints

A portfolio strategy of the risk adjusted return ratio is $RAR(\pi_t)$is

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$$RAR(\pi_t) = \frac{E[R(\pi_t)] - r_t}{\rho(\pi_t)}$$

The available investments strategies π_t are restricted by a feasible set $\mathcal{A}$ reflecting at market limitation is $\pi_t = \mathcal{A} = \{\pi_t \epsilon \mathbb{R}^n | g_i(\pi_t) \leq 0 \text{: for all } i = 1(1)m\}$ to maximize the risk adjusted return subjected to constraints $\max_{\pi_t \in \mathcal{A}} (E[R(\pi_t)] - r_t)$subjected to$\rho(\pi_t) \leq \rho_0$ for some risk free threshold ρ_0. $\min_{\pi_t \in \mathcal{A}} \rho(\pi_t)$ subjected to $(E[R(\pi_t)] - r_t) \geq 0$for minimum return target r_0.

Let X_t be the surplus under strategy π_t and T be the risk adjusted return over horizon time as

$$RAR(\pi_t) = \frac{E[R(X_t - X_0)] - r_t T}{\rho(X_t - X_0)}$$

$RAR(\pi_t)$ is represents the risk measure and constraints interact in forming optimized risk adjusted returns.

3.4. Ruin Probability Constraints

Let us consider drift and diffusion of the SDE and the probability before horizon time T and Exponential Super-Martingale inequality.

$$b_t(X_t, \pi_t, C_t) = r_t X_t + \pi_t^T(\mu_t - r_t) - C_t$$

$$\tau = \inf\{t \geq 0; X_t \leq 0\} \text{ and } P(\tau < T) = P\left(\inf_{0 \leq t \leq T} X_t \leq 0\right)$$

$$M_t(\eta) = \exp\left\{-\eta X_t - \frac{1}{2}\eta^2 \int_0^T \|\sigma_s\|^2 ds\right\}$$

Apply Ito's lemma for Martingale equation and differentiate w.r.t. X_t we get

$$d(f(X_t)) = -\eta e^{-\eta X_t} dX_t + \frac{1}{2}\eta^2 e^{-\eta X_t} d\langle X \rangle_t$$

$$d(f(X_t)) = -\eta e^{-\eta X_t}(b_t dt + \sigma_t^T dW_t) + \frac{1}{2}\eta^2 e^{-\eta X_t} \|\sigma_s\|^2$$

Now, construct the bounding the ruin probability for

$$P\left(\inf_{0 \leq t \leq T} X_t \leq 0\right)$$

$$P(\tau < T) = P\left(\sup_{0 \leq t \leq T} (-X_t) \leq X_0\right)$$

By using exponential Markov Inequality for $\eta > 0$

$$P(\tau < T) = P\left(\sup_{0 \leq t \leq T} (-X_t) \leq X_0\right) \geq X_0 = P(\tau < T)$$

$$= P\left(\sup_{0 \leq t \leq T} e^{-\eta X_t} \geq e^{-\eta X_0}\right)$$

Apply the Doob's maximal inequality for above equation

$$P\left(\sup_{0 \leq t \leq T} M_t(\eta) \geq e^{-\eta X_0}\right) \leq \frac{E[M_\tau(\eta)]}{e^{-\eta X_0}}$$

$$P\left(\sup_{0 \leq t \leq T} M_t(\eta) \geq e^{-\eta X_0}\right) \leq \frac{e^{-\eta X_0}}{e^{-\eta X_0}} = 1$$

This ruin probability constraint is nontrivial exponential bound.

3.5. Existence and Uniqueness of MFBSDU

Under the all above assumptions is well defined and unique value at (t, x) by $v(t, x)$. Which is monotone in y but non-Lipschitz and induces quadratic growth in the control channel when coupled with optimal portfolios.

$$dY_t = \widehat{H}_t(t, x_t, z_t, z_t^T, U_t)dt - z_t^T dW_t - \left(z_t^{(L)} dW_t^{(L)}\right) - \int_{R_+} U_t(z)\widetilde{N}(dt, dz)$$

It admits a unique solution $(Y, z, z^L, U) \in S_t^2 * H_t^2 * J$moreover $z * W + z^L W^{(L)} + \int U\widetilde{N}$. It is called a BMO Martingale.

$$\frac{A'}{1-\gamma} + A\left(r + \frac{\eta}{\gamma} - k\right) - \frac{1}{2\gamma}A\eta + \frac{\delta}{1-\gamma}(k^{1-\gamma} - A) = 0$$

$$A'(t) = (\gamma - 1)A(t)\left(r + \frac{\eta(t)}{2\gamma} - \left(\frac{\delta}{A(t)}\right)^{\frac{1}{\gamma}}\right) + \delta\left(A(t) - \left(\frac{\delta}{A(t)}\right)^{\frac{1-\gamma}{\gamma}}\right)$$

$A'(t)$are the existence and uniqueness of MFBSDU.

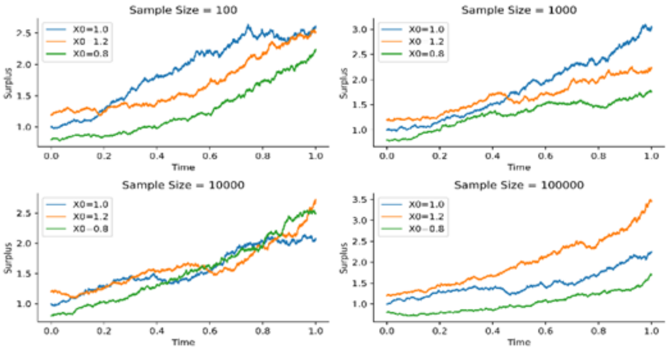

Figure 2: simulated surplus dynamics for different initial reserves across sample sizes, illustrating existence and uniqueness of SSDU solutions.

The Figure 2 gives the capacity of the surplus dynamics plots to show a similar stabilization effect is also shown. When the sample size is small, there are clearly more volatile and sensitive to initial reserve levels of surplus paths. As the number of simulations is increased the trajectories in the simulations become smoother and more structured. The order in which paths are placed using initial surplus levels is the same, so the claims of existence of the SSDU solution and uniqueness are supported. The model is predictable even with stochastic disturbances.

3.6. Epstein-Zin with CRRA Terminal Utility

A CRRA is$U(C_t) = \begin{cases} \frac{C_t^{1-\gamma}}{1-\gamma} & if\ \gamma \neq 1 \\ \log(C_t) & if\ \gamma = 1 \end{cases}$, A terminal wealth utility at X_T is $U_T(X_T) = \frac{(X_T)^{1-\gamma}}{1-\gamma}$, The surplus of our MFBSDU is $U_t = X_t - L_t$ thus the terminal surplus is $U_T = X_T - L_T$ then CRRA terminal surplus utility is

$$U_T(S_T^i) = \frac{(S_T^i)^{1-\gamma}}{1-\gamma}$$

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The recursive utility can be represented as an MFBSDU with risk adjustment process is

$$Y_T = U_T + \int_t^T f(S, X_s, Y_s, Z_s, \mu_s)ds - \int_t^T Z_s dW_s$$

$$Y_T = \frac{(S_T^i)^{1-\gamma}}{1-\gamma}$$

The fair of actuarial premium for a contingent claim C_t under SDU is obtained by solving

$$\pi = \inf\{p \in \mathbb{R}: U_0(S_0 - P_t + C_t) \geq U_0(S_0)\}$$

It leads to certainly equivalent price through the MFBSDU solution. Thus, the CRRA terminal utility for surplus SDU under MFBSDU dynamics is

$$Y_T = E_T\left[\frac{(S_T^i)^{1-\gamma}}{1-\gamma} + \int_t^T f(S, X_s, Y_s, Z_s, \mu_s)ds\right] - \int_t^T Z_s dW_s$$

This recursive equation quantifies how policy uncertainty shock influences portfolio volatility, systemic risk and actuarial fair pricing through the endogenous utility adjustments captured by MFBSDU.

3.7. Incorporating CVaR via State Augmentation

For loss variables L and confidence interval $\alpha \in [0,1]$, to incorporate CVaR into recursive utility, we augment the terminal condition is

$$CVaR_\alpha(L) = \inf\left\{\eta + \frac{1}{1-\alpha}E[(L-\eta)^+]\right\}$$

$$CVaR_\alpha(-S_t^i) = \inf\left\{\eta + \frac{1}{1-\alpha}E\left[\left(-S_t^i - \eta\right)^+\right]\right\}$$

$$Y_t^{CVaR} = \frac{(S_t^i)^{1-\gamma}}{1-\gamma} - \lambda \inf_\eta\left\{\eta + \frac{1}{1-\alpha}E\left[-S_t^i - \eta\right]^+\right\}$$

MFBSDU with CVaR augmented terminal utility then the our MFBSDU becomes

$$Y_t = \frac{(S_t^i)^{1-\gamma}}{1-\gamma} - \lambda \inf_\eta\left\{\eta + \frac{1}{1-\alpha}E\left[-S_t^i - \eta\right]^+\right\} + \int_t^T f(S, X_s, Y_s, Z_s, \mu_s)ds - \int_t^T Z_s dW_s$$

This is the actuarial fair pricing of a claim under SDU preferences. The augmented MFBSDU with CVaR.

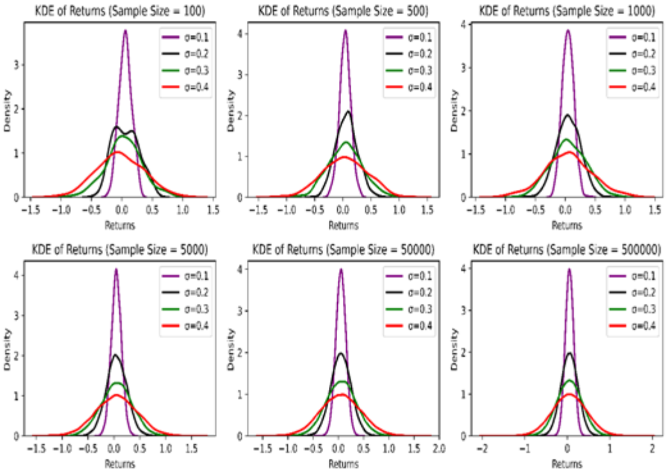

Figure 3: the wider distributions with higher volatility (σ) and convergence to the theoretical shape as sample size increases.

The Figure 3 demonstrated the KDE return distributions illustrate the relationship between the return and the risk exposure caused by volatility. Reduced volatility makes the distributions very peaked and concentrated and increased volatility makes the distribution spread and with thick tails. As the sample size increases, the distribution of the density curves become smoother and symmetric indicating that there is convergence to the theoretical distribution. The behavior of the tail becomes more stable which reinforces the strength of the simulation framework.

3.8. Liability Driven Investment (LDI) and Ruin control

The asset allocation depends explicitly on liabilities with portfolio weights π_t is

$$dA_t = [rA_t + \pi_t^T(\mu_t - r_t)] - l_t dt + \pi_t^T \sigma_t dW_t$$

The liabilities process with drift l_t and liability volatilityv_t and the ruin probability under policy shocks is

$$dL_t = l_t dt + v_t dB_t$$

$$dS_t^i = (r_t A_t + \pi_t^T(\mu_t - r_t) - l_t)dt + \pi_t^T \sigma_t dW_t - v_t dB_t$$

$P(\tau \leq T)$, We introduce a ruin penalty into utility is $\phi(S_t^i) = 1_{\{S_t^i \leq 0\}}$ so the augmented terminal utility becomes$Y_T^{LDI+Ruin} = \frac{(s_t^i)^{1-\gamma}}{1-\gamma} - \lambda_t P(\tau \leq T)$

$$Y_t = \frac{(S_t^i)^{1-\gamma}}{1-\gamma} - \lambda_t P(\tau \leq T) + \int_t^T f(S, X_s, Y_s, Z_s, \mu_s)ds - \int_t^T Z_s dW_s$$

The optimal portfolio under LDI with ruin control the yields a stochastic control problem embedded in the MFBSDU is dS_t^i with control variable π_t chosen to balance. These two equations adjust actuarial fair pricing to account for systematic fragility.

$$Y_t = \frac{(S_t^i)^{1-\gamma}}{1-\gamma} - \lambda_t 1_{\{S_t^i \leq 0\}} + \int_t^T f(S, X_s, Y_s, Z_s, \mu_s)ds - \int_t^T Z_s dW_s$$

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The Figure 4 explains the LDI and non - LDI strategies comparison reveals a stumbling structural difference. At all sample sizes the non LDI strategy has higher expected surplus, but more variable. The LDI strategy instead generates a lower but more stable expected surplus path. As simulations improve and the number increases, the difference between the two becomes more evident and less noisy and points to the fact that the trade off between return and stability is systematic, rather than random. This points out the economic significance of liability sensitivity, which is to limit aggressive growth in return for controlled solvency risk.

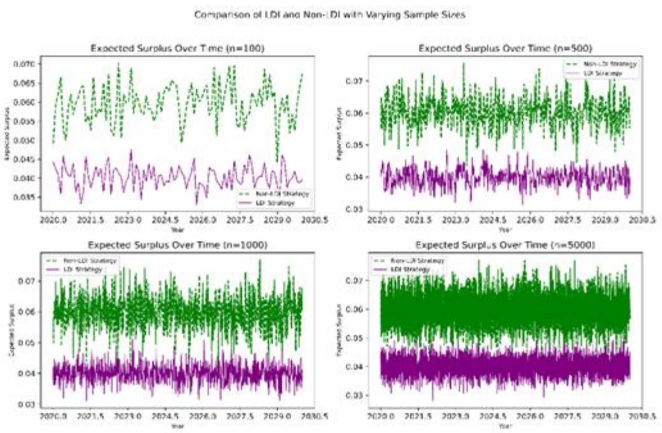

Figure 4: comparison of expected surplus over time for LDI and Non-LDI strategies with varying sample sizes

3.9. Duality in Incomplete Markets

We have risky assets S_t^i driven by Brownian motion W_t, possible non-tradable risks drawn by a Brownian motion B_t. The market is incomplete because not all risk sources can be hedged.

The duality Martingale method in incomplete markets for surplus SDU is as

$U(x) = \sup_{\pi\epsilon\mathcal{A}} E\left[U\left(S_t^i\right)\right] = \inf_{y\geq 0}\left\{\inf_{Y\in M} E[v(yY_T)] + xy\right\}$ where, $v(yY_T) = \frac{\gamma}{1-\gamma} y^{\frac{(\gamma-1)}{\gamma}}$ it is dual conjugate. This decomposition shows how policy uncertainty shocks propagate into systemic risk and actuarial pricing via the enlarged dual set of Martingale measures.

3.10. Comparison with Existing Findings

The results are consistent with previous studies showing recursive preferences and solvency limits cause people to invest more cautiously, which is pretty different from the former expected utility theory. As compared to other recursive portfolio studies, our hedge-demand component reduces exposure to equity relative to the standard portfolio model from Merton and convinced us that forward looking risk tweaks slow down risky moves when things are uncertain. Meanwhile, the addition of the ruin probability limits in the optimization keeps in line with other stochastic control papers finding hard solvency constraints change the surplus path a lot. But unlike most models where ruin, CVaR or the liability effects are added on top of one another, the proposed methodology demonstrates that if you fold all those bits into one forward/backward recursive framework, suddenly all the premium, allocations and risk exposures adjustments are lined up much nicer. The comparison implies a need to treat the dynamics of surplus, recursive utility and solvency control together, as this will yield stronger and more structurally aligned risk moderation than when each of them is broken up.

4. PRACTICAL IMPLICATIONS

All data used and illustrations presented in the paper are primarily generated within the theoretical framework. Some applications of the proposed framework are listed below.

Problem 1: Actuarial Fair Premium with Policy Shock Loading

An insurer promises a 10-year claim of \$120.Using the pragmatic MFBSDU, the actuarially fair premium must incorporate not only the expected claim but also additional loadings for volatility and systemic uncertainty shocks. By using $P = E[C_t] + \lambda_t\sigma_t^2 + \theta_t \cdot ShockRisk$, In this case we will consider volatility loading = 5%, shock loading = 3%. compute the numerical value explicitly.

Given: Expected claim: $E[C_t] = 120$

Volatility loading (5% of claim) is $\lambda_t\sigma_t^2$: $\lambda_t\sigma_t^2 = 0.05 \times 120 = 6.00$

Shock loading (3% of claim) is $\theta_t \cdot ShockRisk$: $\theta_t \cdot ShockRisk = 0.03 \times 120 = 3.60$

Total premium is P_t: $P_t = 120 + 6 + 3.6 = 129.6$

So, the insurer should charge \$129.60.

Suppose the insurer has CRRA terminal utility with risk aversion $\alpha > 0$. Then the volatilityloading is$\lambda_t\sigma_t^2 = \frac{\alpha}{2}Var(C) = \frac{\alpha}{2}\sigma_C^2$

$$\alpha = \frac{2\times\lambda_t\sigma_t^2}{\sigma_C^2} = \frac{2\times 6}{\sigma_C^2} = \frac{12}{\sigma_C^2}$$

In the shock term is given by an expected shock loss m_S (in dollars), then $\theta_t = \frac{0.03\times E[C_t]}{m_S}$

$m_S = 10$ consider as the expected shock then $\theta_t$is$\theta_t = \frac{0.03\times 120}{10} = \frac{3.6}{10} = 0.36$

Elasticity of premium w.r.t expected claim:

$$\frac{\partial logP_t}{\partial logE[C_t]} = \frac{E[C_t] + (loadings)}{P_t} \approx \frac{(120+9.6)}{129.6} \approx 0.926.$$

Thus, the premium is almost proportional to the claim but slightly amplified by the loadings.

This shows the consistent with the certainty equivalent obtained from a second-order Taylor expansion of the insurer's expected utility; MFBSDU produces the same structure when one linearizes the backward component and aggregates variance and shock moments into the premium.

Problem 2: Ruin Probability Bound via SSDU

Consider a surplus process X_t governed by the Surplus Stochastic Differential Utility (SSDU) framework, where the

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insurer invests in a risky portfolio. The initial surplus is 100, Drift of liabilities-adjusted surplus is 4% per annum for the 10-year investment. Using the exponential Martingale inequality of the ruin probability satisfies $P(Ruin) \leq exp\left(-\frac{2x_0\mu}{\sigma^2 T}\right)$. Compute the numerical bound for the ruin probability over the 10-year horizon and interpret its practical implications for insurer solvency under SSDU dynamics.

Given: Initial surplus, $x_0 = 100$

liabilities-adjusted drift of the surplus, $\mu_t = 0.04$

Variance and volatility, $\sigma^2 = 0.04$

Time horizon, $T = 10$

The exponential martingale inequality states that the ruin probability is bounded above as:

$$P(Ruin) \leq exp\left(-\frac{2x_0\mu}{\sigma^2 T}\right)$$

$$P(Ruin) \leq exp\left(-\frac{2 \times 100 \times 0.04}{10 \times 0.04}\right)$$

$$P(Ruin) \leq exp\left(-\frac{8}{0.4}\right)$$

$$P(Ruin) \leq exp(-20)$$

$$P(Ruin) \leq 2.06 \times 10^{-9} \approx 0$$

The computed upper bound on the ruin probability is extremely small, essentially converging to zero. This outcome implies that, under the SSDU framework with the given parameters namely, a reasonable drift of surplus, moderate portfolio volatility, and a sufficiently high initial surplus the insurer faces virtually no risk of insolvency within the 10-year horizon. From an actuarial solvency perspective, this result highlights the strong capital adequacy and robust resilience of the portfolio, demonstrating its ability to withstand potential shocks while maintaining financial stability over the planning period.

Problem 3: Optimal Equity Weight with Hedge Demand (Policy Shock Factor)

Consider an insurer operating under the SSDU-MFBSDU with CRRA utility aversion $\gamma = 5$. The insurer invests in a single risky asset with the following parameters: The risk-free rate is 1% per annum, expected excess return on risky asset is 4% per annum, volatility of risky asset is 0.20 and factor loading of the risky asset is 2%. Compute separately the Merton (myopic) component and the hedge demand correction. And interpret the final allocation π^*in terms of investment policy: does the hedge component increase or reduce equity exposure.

Given,$r = 0.01, \mu - r = 0.04,\ \sigma^2 = 0.04\ (\sigma = 0.20), \gamma = 5, k = 1, \Sigma_f = 1, B = 0.02$

And $\beta_S = +1$. The Merton (myopic) allocation is

$$\pi^{Merton} = \frac{\mu - r}{\gamma\sigma^2}$$

$$\pi^{Merton} = \frac{0.04}{5 \times 0.04} = \frac{0.04}{0.20} = 0.20$$

Therefore, the myopic allocation is 20% of wealth in the risky asset. And the hedge demand of policy shock factor is

$$\pi^{hedge} = -\left(\frac{B\Sigma_f^{-1}\beta_S}{\gamma\sigma^2}\right)$$

$$\pi^{hedge} = -\frac{1}{5} \times \left(\frac{(0.02 \times 1)}{0.04}\right) = -0.2 \times \frac{0.02}{0.04}$$

$$\pi^{hedge} = -0.2 \times 0.5$$

$$\pi^{hedge} = -0.10$$

The hedge correction equals to the 10%. It reduces risky exposure by 10%.

Total approximate allocation is$\pi^* = \pi^{Merton} + \pi^{hedge}$

$$\pi^* = 0.20 + (-0.10)$$

$$\pi^* = 0.10 \approx 10\%$$

The classical Merton rule prescribes a 20% equity exposure, but once policy-uncertainty shocks are recognized, the optimal allocation is slashed to just 10%. This sharp reduction reflects the dual role of policy shocks: while they elevate asset returns, they simultaneously magnify surplus losses, creating a systemic vulnerability. The result is a markedly more conservative stance, where hedge demand compels the insurer to sacrifice return in exchange for resilience.

4.1. Limitations

- Theoretical model is basically set up in a continuous time framework but not to calibrated it using any real insurer balance sheet data.
- Stylized Market Assumptions: The set-up assumes that the coefficients are well-behaved and the level of the markets remain stable - there is no regime-switching in macro into the market set-up.
- Computational Complexity: Working out these forward-backward stochastic systems get pretty heavy when you decide to add more and more risk factors which can slow things down a lot.
- Limited Regulatory Frictions: All specific regulation of capital, accounting limits etc. have been left out except basic solvency boundaries.
- Behavioral Scope: There is appearance of recursive preferences, need to explore broader behavioral issues such as ambiguity aversion beyond the Epstein-Zin set-up.

5. Conclusions

In this study, a combined framework to fuse the dynamics of surplus, recursive utility and solvency control within a multidimensional forward-backward stochasticity contraption is given. By combining SSDU and MFBSDU, the model helps to attract important characteristics of the real world such as jumps in claims, liquidity shocks, liability-driven investment, probability of ruin, and CVaR. Unlike the typical models that separate these components, this approach goes and works together on one coherent structure. The verification theorem supports the optimality of the development of control strategies that we have derived. These findings ensure that both the process of surplus and recursive utility remain consistent and stable, even when the unpredictable nature of the market. From the economic perspective, the work shows that recursive

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preferences and constraints from solvency really affect portfolio allocation and premium pricing. When tail risks and policy uncertainty come into play, insurers reduce taking risks as much as possible compared to classic portfolio rules. Premiums end up incorporating the volatility and shock loadings, and cause more realistic and solvency-consistent pricing. Overall, this framework provides a way for insurers to handle profitability and financial stability in an uncertain and incomplete market.

Future work should be aimed at empirical calibration work, regime switching extensions and the integration of regulatory capital and the development of advanced numerical methods for many insurance applications in the real world.

Conflict of interest statement: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Author Contributions: All authors contributed to the conceptualization, methodology development, data analysis, and manuscript preparation, and have approved the final version of the paper.

Data availability: The data presented in this research are fully displayed in the article.

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Appendices

Proof of Theorem 1: Consider the $(W_1, W_2, W_3, \dots, W_m) = W$ a multidimensional standard Brownian motion with respect to probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[o,T]}, \mathbb{p})$ and $N(dt, dz)$ Poisson random measure on $[0, T]$ at X with predictable compensator $\Lambda(dt, dz) = \lambda_t \gamma(dz)dt. W^{(L)}$is a Brownian component for liability diffusion, may be one of the components of W.We used primitive processes for model are $r_t, \mu_t, \Sigma_t, \lambda_t, v \ and \ \pi_t$.
$\mu_t \epsilon \mathbb{R}^d, \Sigma_t \epsilon \mathbb{R}^{d\times m}$
$\Longrightarrow$ Asset drift and volatility $\Sigma_t \Sigma_t^T$is a positive definite and bounded $v = \int_{R_+}(1 \wedge z^2)vdz < \infty$ $\pi_t$is risky asset, P_t is premium inflow and $C_t \geq 0$ is payouts. An admissible control $(\pi, C) \in \mathcal{A}$means $E\int_0^T (\|\Sigma_t^T \pi_t\|^2 + |P_t|^2 + |C_t|^2)dt < \infty$and L_t is

$$L_t = \int_0^t \int_{R_+} zN(ds, dz), \quad L_0 = 0$$

We assume $\mathcal{F}_0$ is measurable then X_0 with $E|X_0|^2 < \infty$. The assumptions are below

i. Probability space and filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[o,T]}, \mathbb{p})$
ii. Market Assets (r_t), $dS_t^i = diag(S_t^i)(\mu_t dt + \Sigma_t dW_t)$ with drift $\mu_t$andΣ_t both are measurable and satisfying integrability conditions $\int_0^T \|\mu_t\| dt < \infty$, $\int_0^T \|\Sigma_t\|^2 dt < \infty$.
iii. Portfolio strategy (π_t), consumption (C_t), Insurance Liability (L_t) and jumps (P_t, λ_t).

The compensated measure and liability process are $\tilde{N}(dt, dz)$ and L_t

$$\tilde{N}(dt, dz) = N(dt, dz) - \lambda_t v(dz)dt$$

$$L_t = \int_0^t \int_{R_+} zN(ds, dz)$$

Apply Ito decomposition of integer- valued random measure

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$$L_t = \int_0^t \int_{R_+} f(z)\lambda_s v(ds)dz + \int_0^t \int_{R_+} f(z)\tilde{N}(dt,dz)$$

The Martingale decomposition for Poisson random measure. The first term is compensator predictable; second one is purely random fluctuation. Let us consider $f(z) = z$ then

$$L_t = \int_0^t \int_{R_+} z\lambda_s v(ds)dz + \int_0^t \int_{R_+} z\tilde{N}(dt,dz)$$

Differentiate with respect to dsabove equation

$$\frac{dL_t}{ds} = \lambda_s \bar{z}(ds)dz + \int_{R_+} z\tilde{N}(ds,dz)$$

Put s=t

$$\frac{dL_t}{dt} = \lambda_s \bar{z}(dt)dz + \int_{R_+} z\tilde{N}(dt,dz)$$

Taking conditional expectation on both side

$$E(L_{t+dt} - L_t|\mathcal{F}_t) = E\left[\int_{R_+} z\tilde{N}((t,t+dt),dz)|\mathcal{F}_t\right]$$

$$E(L_{t+dt} - L_t|\mathcal{F}_t) = \lambda_t \int_{R_+} zv(dz)dt$$

$$E(L_{t+dt} - L_t|\mathcal{F}_t) = \lambda_t \bar{z}dt$$

Subtract the compensator to get a zero mean

$$\int_{R_+} z\tilde{N}((t,t+dt),dz) - \lambda_t \bar{z}dt$$

$$= \int_{R_+} z\tilde{N}((t,t+dt),dz) - \lambda_t \int_{R_+} zv(dz)dt$$

$\int_{R_+} z\tilde{N}((t,t+dt),dz) - \lambda_t \bar{z}dt = \int_{R_+} z\tilde{N}(dt,dz) + \int_{R_+} zv(dz)dt - \lambda_t \int_{R_+} zv(dz)dt$

$\int_{R_+} z\tilde{N}((t,t+dt),dz) - \lambda_t \bar{z}dt = L_{t+dt} - L_t$

$$\int_{R_+} z\tilde{N}((t,t+dt),dz) - \lambda_t \bar{z}dt = dL_t$$

This is called predictable compensator of stochastic integration with respect to jump processes.

Let us consider traded assets and self-financing portfolio $\frac{dS_t^i}{S_t^i}$and bank account B_t

$$\frac{dS_t^i}{S_t^i} = \mu_t dt + \Sigma_t dW_t \text{and } dB_t = r_t B_t dt$$

$$dX_t = (X_t - \pi_t)r_t dt + \pi_t^T(\mu_t dt + \Sigma_t dW_t) + P_t dt - C_t dt - dL_t + \sigma_L dW_t^{(L)}$$

$$dX_t = (X_t - \pi_t)r_t dt + \pi_t^T(\mu_t dt + \Sigma_t dW_t + P_t - C_t) - dL_t + \sigma_L dW_t^{(L)}$$

Suppose wealth is subject to jumps by a Poisson random measure $N(dt,dz)$ with compensator. Wealth is also subject to liquidity risk modeled by an idiosyncratic Brownian motion $W_t^{(L)}$ independent of W_t.The volatility of this component is$\sigma_L$then liquidity risk is $\sigma_L dW_t^{(L)}$.

$$dX_t = (r_t X_t + \pi_t^T(\mu_t - r_t) + P_t - C_t)dt + \pi_t^T \Sigma_t dW_t - \int_{R_+} z\tilde{N}(dt,dz)$$

$$dX_t = (r_t X_t + \pi_t^T(\mu_t - r_t) + P_t - C_t)dt + \pi_t^T \Sigma_t dW_t - \int_{R_+} z(N - \lambda_t v)(dt,dz)$$

$$dX_t = (r_t X_t + \pi_t^T(\mu_t - r_t) + P_t - C_t)dt + \pi_t^T \Sigma_t dW_t - \int_{R_+} zN(dt,dz) - \int_{R_+} \lambda_t zv\,(dt,dz)$$

$$dX_t = (r_t X_t + \pi_t^T(\mu_t - r_t) + P_t - C_t)dt + \pi_t^T \Sigma_t dW_t - \int_{R_+} zN(dt,dz) - \int_{R_+} \lambda_t\, \bar{z}dt$$

$$dX_t = (r_t X_t + \pi_t^T(\mu_t - r_t) + P_t - C_t)dt + \pi_t^T \Sigma_t dW_t - \int_{R_+} zN(dt,dz) - \int_{R_+} \lambda_t\, \bar{z}dt + \sigma_L dW_t^{(L)}$$

The wealth dynamics equation with risky investments, consumption, Laboure income, jumps and liquidity risk.

Proof of Theorem 2: The adjoint equation of stochastic control of the objective function is Y_t for MFBSDU.

$$Y_t = E_t\left[g(X_T) + \int_t^T f(t,X_t,C_t,Y_t)dt\right]$$

To optimize the above equation to the maximize

$$Y_t = \int_t^T f(t,X_t,C_t,Y_t)dt + g(X_t)$$

Suppose we have control (π_t, C_t) be predictable and admissible the surplus X_t satisfy

$$dX_t = b(t,X_t,C_t,\pi_t)dt + \sigma(t,\pi_t)dW_t + \sigma_L dW_t^{(L)} - \int_{R_+} z\tilde{N}(dt,dz)$$

where, $b(t,X_t,C_t,\pi_t) = r_t X_t + \pi_t^T(\mu_t - r_t) + P_t - C_t - \lambda_t \bar{z} \text{ and } \sigma(t,\pi_t) = \pi_t \Sigma_t^T$

The dynamic programming principle (DPP) for any stopping $\tau\epsilon[t,T]$ then

$$Y_t = \operatorname*{ess\,sup}_{(\pi_t,C_t)}\left\{E_t\left[\int_t^T f(S,X_s,C_s,Y_s)ds + Y_\tau\right]\right\}$$

Apply Doob-Mayer decomposition for any optimal control (π_t^*, C_t^*) a predictable process becomes

$$Y_t = \operatorname*{ess\,sup}_{(\pi_t,C_t)}\left\{E_t\left[\int_t^T f(S,X_s,C_s,Y_s)ds + g(X_t)|\mathcal{F}_t\right]\right\}$$

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$$Y_t = Y_\tau + \int_t^T \Gamma_{S_t}^{\pi,C} ds - M_{t,\tau} \tag{1}$$

where, $M_{t,\tau} = \int_t^T z_s^T dW_S + \int_t^T \left(z_s^{(L)}\right)^T dW_S^{(L)} + \int_t^T \int_{R_+} U_S(z)\tilde{N}(ds,dz)$

Equation (1) becomes

$Y_t - Y_\tau = \int_t^T \Gamma_{S_t}^{\pi,C} ds - \left(\int_t^T z_s^T dW_S + \int_t^T \left(z_s^{(L)}\right)^T dW_S^{(L)} + \int_t^T \int_{R_+} U_S(z)\tilde{N}(ds,dz)\right)$

Take conditional expectation of above equation, we get

$E_t(Y_t - Y_\tau) = E_t\left(\int_t^T \Gamma_{S_t}^{\pi,C} ds - \left(\int_t^T z_s^T dW_S + \int_t^T (z_s^{(L)})^T dW_S^{(L)} + \int_t^T \int_{R_+} U_S(z)\tilde{N}(ds,dz)\right)\right)$

$E_t(Y_t) = E_t(Y_\tau) + E_t\left(\int_t^T \Gamma_{S_t}^{\pi,C} ds - \left(\int_t^T z_s^T dW_S + \int_t^T \left(z_s^{(L)}\right)^T dW_S^{(L)} + \int_t^T \int_{R_+} U_S(z)\tilde{N}(ds,dz)\right)\right)$

Differentiate with respect to t and in Martingale increment is equated to zero.

$\frac{d(E_t(Y_t))}{dt} = \frac{d}{dt}(E_t(Y_\tau)) + \frac{d}{dt}\left[E_t\left(\int_t^T \Gamma_{S_t}^{\pi,C} ds - \left(\int_t^T z_s^T dW_S + \int_t^T (z_s^{(L)})^T dW_S^{(L)} + \int_t^T \int_{R_+} U_S(z)\tilde{N}(ds,dz)\right)\right)\right] = 0$

$$dY_\tau + \Gamma_t^{\pi,C} dt - z_t^T dW_t - \left(z_t^{(L)}\right)^T dW_t^{(L)} + \left(z_t^{(L)}\right)^T dW_t^{(L)} = 0$$

To compute $d(X_t, Y_t)$ by using above Ito product rule for jump diffusion

$$d(X_t, Y_t) = X_t dY_t + Y_t dX_t + d\langle X^c, Y^c\rangle + \int_{R_+} (\Delta X_t(z))(\Delta Y_t(z)) N(dt,dz)$$

$$\frac{d}{dt}(X_t Y_t) = Y_t b(t, X_t, \pi_t, C_t) + z_t^T \sigma(t,\pi_t) + \sigma_L \left(z_t^{(L)}\right)^T + \int_{R_+} U_t(z)(-z)\lambda_t v dz \tag{2}$$

Martingale optimality becomes Hamiltonian-Jacobi-Bellmen equivalently drift of S_t^i is

$$\frac{d}{dt}(X_t Y_t) = -\frac{d}{dt}(Y_t) - f(t, X_t, C_t, Y_t)$$

Put the above equation in equation (2)

$$-\frac{d}{dt}(Y_t) - f(t, X_t, C_t, Y_t) = Y_t b(t, X_t, \pi_t, C_t) + z_t^T \sigma(t,\pi_t) + \sigma_L \left(z_t^{(L)}\right)^T + \int_{R_+} U_t(z)(-z)\lambda_t v dz$$

$$-\frac{d}{dt}(Y_t) \geq -f(t, X_t, C_t, Y_t) + Y_t b(t, X_t, \pi_t, C_t) + z_t^T \sigma(t,\pi_t) + \sigma_L \left(z_t^{(L)}\right)^T + \int_{R_+} U_t(z)(-z)\lambda_t v dz$$

$$-d(Y_t) \geq \left\{-f(t, X_t, C_t, Y_t) + Y_t b(t, X_t, \pi_t, C_t) + z_t^T \sigma(t,\pi_t) + \sigma_L \left(z_t^{(L)}\right)^T + \int_{R_+} U_t(z)(-z)\lambda_t v dz\right\} dt$$

This must be hold for every admissible (π_t, C_t) and be the optimizer (π_t^*, C_t^*) then we take the pointwise supremum over controls in the drift.

Proof of Theorem 3: Assume the assumption 1, 2, 3 and 4. Fix for jump of SDE under arbitrary control (π_t^*, C_t^*) of X_t then solve the function of $v(t, X_t)$.

$$v(t, X_t) = b(t, X_{t-}, \pi_t, C_t) v_x(t,x) + \frac{1}{2}(\|\Sigma_t^T \pi_t\|^2 + \sigma_L^2) v_{xx}(t,x) + \int_{R_+} (v(t,x,z) - v(t,x) + z v_x(t,x)) \lambda_t v dz$$

Apply Ito-Doeblin to $v(t,x)$ then above equation becomes

$$dv(t, X_t) = b_t v_t dt + v_x dX_t + \frac{1}{2} v_{xx}\, d\langle X^c\rangle_t + \int_{R_+} (v(t,x,z) - v(t,x) + z v_x(t,x)) N(dt,dz)$$

Substitute dX_t and replace the jump integral by its compensator to obtain the integral decomposition

$$dv(t, X_t) = (\partial_t v + L^{\pi_t, C_t} v)(t, x_{t-}) dt + v_x(t, X_{t-}) * \sigma(t,\pi_t) dW_t + v_x(t, x_{t-}) * \sigma_L dW_t^{(L)} + \int_{R_+} (v(t, x_{t-}, -z) - v(t, x_{t-})) \tilde{N}(dt,dz)$$

Use the MFBSDU inequality from HJB equation

$$\partial_t v(t,x) + L^{\pi_t, C_t} v(t,x) + f(t, x, C, v(t,x)) \leq 0$$

Evaluating at $x = x_{t-}$ and (π_t, C_t) combine with equation of $\partial_t v(t, x_t)$ yield is

$$dv(t, X_t) \leq -f(t, x_t, C_t, v(t, x_{t-})) dt + dM_t$$

By the Martingale property is $E_t[M_T - m_t] = 0$

$$v(t,x) \geq E_t\left[v(T, x_T) + \int_t^T f(S, x_S, C_S, v(S, x_S)) ds\right]$$

Taking the essential supremum over admissible controls to obtain

$$v(t,x) \geq \operatorname*{ess\,sup}_{(\pi_t, C_t)} E_t\left[v(T, x_T) + \int_t^T f(S, x_S, C_S, v(S, x_S)) ds\right] \tag{3}$$

Taking conditional expectation ψ integrals above equation gives exact equality

$$v(t,x) = E_t\left[g(X_T^*) - \psi(X_T^*) + \int_t^T f(S, x_S^*, C_S^*, v(S, x_S^*)) ds\right] \quad 4)$$

From equation (3) and (4), we obtain

$$v(t,x) = \operatorname*{ess\,sup}_{(\pi_t, C_t)} \left\{E_t\left[g(X_T) - \psi(X_T) + \int_t^T f(S, x_S, C_S, v(S, x_S)) ds\right]\right\}$$

Proof of Theorem 4: Suppose $v(t,x) = \frac{A(t)}{1-\gamma}$, $A(t) > 01$, $x > 0$, The derivatives are

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$$v_x = A(t)x^{-\gamma}, v_{xx} = -\gamma A(t)x^{-\gamma-1}, v(t) = \frac{A'(t)}{1-\gamma}x^{1-\gamma}$$

Consider diffusion with no jumps or no Poisson limits of MFBSDU is

$$v_t + \sup_{\pi\in R^n, C\geq 0}\left\{(r_t x + \pi^T\theta - C)v_x + \frac{1}{2}\pi^T\Sigma\Sigma^T\pi v_{xx} + \delta(\psi(C) - v)\right\}$$

$$\frac{A'}{1-\gamma} + A\left(r_t + \frac{\eta}{\gamma} - K\right) - \frac{1}{2\gamma}A\eta + \frac{\delta}{1-\gamma}(K^{1-\gamma} - A) = 0$$

Multiplied by $(1-\gamma)$ and simplify

$$\frac{A'}{1-\gamma}(1-\gamma) + A(1-\gamma)\left[\left(r_t + \frac{\eta}{\gamma} - K\right) - \frac{1}{2\gamma}\eta\right] + \frac{\delta}{1-\gamma}(1-\gamma)(K^{1-\gamma} - A) = 0$$

$$A' + A(1-\gamma)\left[\left(r_t + \frac{\eta}{\gamma} - K\right) - \frac{1}{2\gamma}\eta\right] + \delta(K^{1-\gamma} - A) = 0$$

Noting $(1-\gamma)\frac{\eta}{\gamma} - \frac{1-\gamma}{2\gamma}\eta = \frac{(1-\gamma)\eta}{2\gamma}$ we obtain

$$A'(t) = (\gamma - 1)A(t)\left(r_t + \frac{\eta(t)}{2\gamma} - K(t)\right) + \delta(A(t) - K(t)^{1-\gamma})$$

Therefore, $K(t) = \left(\frac{\delta}{A(t)}\right)^{\frac{1}{\gamma}}$

$$A'(t) = (\gamma - 1)A(t)\left(r_t + \frac{\eta(t)}{2\gamma} - \left(\frac{\delta}{A(t)}\right)^{\frac{1}{\gamma}}\right) + \delta(A(t) - K(t)^{1-\gamma})$$

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