Magnetohydrodynamic Rayleigh Problem with Hall Effect and Rotation

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Abstract

This paper gives very significant analytical and numerical results to the magnetohydrodynamic flow version of the classical Rayleigh problem including Hall effect and Rotation. An exact solution of the MHD flow of incompressible, electrically conducting, viscous fluid past a uniformly accelerated and insulated infinite plate has been presented. Numerical values for the effects of the Hall parameter N, Hartmann number M and the Rotation parameter K^2 (i.e.,) the reciprocal of Ekman number on the velocity components u and v are tabulated and their profiles are shown graphically. The numerical results show that the velocity components u and v decreases with the increase of both M and K^2, whereas the velocity component u decreases for fixed values of y and M for different values of N and t along with the increase of K^2 but v gets unstable.

Keywords: *MHD flow, Hall effect, Viscous fluid, uniformly accelerated plate*

1. Introduction

The MHD Stoke's or Rayleigh problem was first solved by Rossow [1] without taking into account the induced magnetic field. With the induced magnetic field, it was solved by Nanda and Sundaram [2],Chang and Yen[3] and Roscizewski [4]. In these papers, different aspects of the problem were considered. But in an ionized gas where the dencity is low and / or the magnetic field is very strong, the conductivity becomes as tensor. The conductivity normal to the magnetic field is reduced by the free spiraling of electrons and ions about the magnetic lines of force before they experience collisions, and a current, known as the Hall current is induced in a direction normal to both electric and magnetic fields. Steady state channels flows of ionized gases were studied by Sato [5]. The effect of Hall current on MHD Rayleigh's problem in ionized gas where studied by Mohanty [6]. Schlicting [7] has studied the unsteady flow due to an impulsive motion of an infinite plate in a fluid of an infinite extent. MHD flow past a uniformly accelerated plate under a transverse magnetic field was studied by Gupta [8]. Magnetohydrodynamic Rayleigh problem with Hall effect was studied by Haytham Sulieman[9].

The study of the MHD flow with Hall current and rotation has important engineering applications in problems of MHD generators, Hall accelerators as well as in flight Magnetohydrodynamics. The rotating flow of an electrically conducting fluid in the presence of magnetic field is encountered in cosmical and geophysical fluid dynamics. It is also important in the solar physics involved in the sunspot development, the solar cycle and the structure of rotating magnetic stars. In this study we have considered the effect of the Hall current and rotation on the magnetohydrodynamic flow version of the classical Rayleigh problem.

2. Formulation of the Problem

Consider the flow of an incompressible electrically conducting, viscous fluid past an infinite and insulated flat plate occupying the plane $y = 0$. Initially the fluid and the plate rotate in unison with a uniform angular velocity Ω about the y - axis normal to the plane. The x-axis is taken in the direction of the motion of the plate and z – axis lying on the plate normal to both x and y – axis. Relative to the rotating fluid, the plate is impulsively started from rest and set into motion with uniform acceleration in its own plane along the x - axis. A uniform magnetic field H_0, parallel to y - axis is imposed and the plate is electrically non conducting. The Physical configuration and the nature of the flow

suggest the following form of velocity vector $\bar{q}$, magnetic induction $\bar{H}$, electrostatic field $\bar{E}$, pressure P and the uniform angular velocity Ω, thus
$\bar{q}$=(u,0,v),$\bar{H}$=(H_x, H_0, H_z),$\bar{E}$=$(E_x, 0, E_z)$,
P=Constant,Ω=$(0,\Omega_y,0)$ (2.1)

The equations governing the unsteady flow and Maxwell's equations are:

Equations of continuity

$$\nabla.\bar{q}=0 \quad (2.2)$$

Equation of motion

$$\frac{\partial \bar{q}}{\partial t} + (\bar{q}.\nabla)\bar{q} + 2\Omega \times \bar{q} = -\frac{1}{\rho}\nabla P + \gamma\nabla^2\bar{q} + \frac{1}{\rho}\bar{J} \times \bar{H} \quad (2.3)$$

Equation for current

$$\nabla \times H = \mu J \quad (2.4)$$

Faraday's Law

$$\nabla \times E = -\frac{\partial H}{\partial t} \quad (2.5)$$

$$\nabla.H = 0 \quad (2.6)$$

The generalized Ohm's law, neglecting ion-slip effect but taking Hall current is,

$$\frac{J}{\sigma} = (E + \bar{q} \times H) - \frac{J \times H}{n.e} \quad (2.7)$$

where $\sigma = \frac{e^2 \tau n}{m}$ (is the electrical conductivity).

Here J is the current density, t is the time, ρ is density, γ is kinematic viscosity, e is electric charge, m is mass of an electron, n is the electron number density, τ is the mean collision time and μ is magnetic permeability.

The initial and boundary conditions are

$t \le 0$: $u = 0, v = 0$ for $y \ge 0$,

$t > 0$: $u = U_0, v = 0$ for $y = 0$,

$u \to 0$: $v = 0$ as $y \to \infty$

$H_x \to 0, H_y = H_0, H_z \to 0$ as $y \to \infty$ (2.8)

At infinity the magnetic induction is uniform with components $(0, H_0, 0)$, and hence the current density in (2.4) vanishes. And since the free stream is at rest, it follows from generalized Ohm's law that $E = 0$ as $y \to \infty$. Assuming small magnetic Reynolds number for the flow, the induced magnetic field is neglected in comparison to the applied constant field H_0. Now introducing the non-dimensional quantities

$$y^* = \frac{U_0.y}{\gamma}, \quad u^* = \frac{u}{U_0}, \quad v^* = \frac{v}{U_0}, \quad t^* = \frac{U_0^2 t}{\gamma} \quad (2.9)$$

The equation of motion (2.3) in component term becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} - \frac{\sigma H_0^2 \gamma}{\rho U_0^2(1+\omega^2\tau^2)}(u + \omega\tau v) - \frac{2\gamma}{U_0^2} v\Omega_y \quad (2.10)$$

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial y^2} - \frac{\sigma H_0^2 \gamma}{\rho U_0^2(1+\omega^2\tau^2)}(\omega\tau u - v) + \frac{2\gamma}{U_0^2} u\Omega_y \quad (2.11)$$

Now let $M^2 = \frac{\sigma H_0^2 \gamma}{\rho U_0^2}$ is the Hartman number, $N = \omega\tau$ is the Hall Parameter and $K^2 = \frac{\gamma \Omega_y}{\rho U_0^2}$ is the Rotation parameter i.e.,the reciprocal of Ekmann number. The initial and boundary conditions are

$$u(0, y) = v(0, y) = 0; \ u(t, 0) = 1, v(t, 0) = 0$$

$$u(t, y) \text{ and } v(t, y) \to 0 \text{ as } y \to \infty \quad (2.12)$$

Now multiplying both sides of equation (2.10) and (2.11) by e^{-st} and integrating from 0 to ∞ with respect to t we get

$$\frac{d^2\hat{u}}{dy^2} - \left(\frac{M^2}{1+N^2} + s\right)\hat{u} = \left(\frac{NM^2}{1+N^2} + 2K^2\right)\hat{v} \quad (2.13)$$

$$\frac{d^2\hat{v}}{dy^2} - \left(\frac{M^2}{1+N^2} + s\right)\hat{v} = -\left(\frac{NM^2}{1+N^2} + 2K^2\right)\hat{u} \quad (2.14)$$

where $\hat{u}(s, y) = L\{u(t, y)\} = \int_0^\infty u(t, y)e^{-st}\, dt$,
$\hat{v}(s, y) = L\{v(t, y)\} = \int_0^\infty v(t, y)e^{-st}\, dt$

By introducing the complex function $\hat{q} = \hat{u} + i\hat{v}$, then equation (2.13) and (2.14) can be combined into the single equation

$$\frac{d^2\hat{q}}{dy^2} - \left(\frac{M^2}{1+N^2} + s\right)\hat{q} = -i\left(\frac{NM^2}{1+N^2} + 2K^2\right)\hat{q} \quad (2.15)$$

3. Analytical Solution

By introducing the complex function $\hat{q} = \hat{u} + i\hat{v}$, then equation (2.10) and (2.11) becomes

$$\frac{\partial q}{\partial t} = \frac{\partial^2 q}{\partial y^2} - \left[\left(\frac{M^2}{1+N^2}\right)(1 - iN) - 2iK^2\right] q \quad (3.1)$$

The initial and boundary conditions take the form

$$q(0, y) = 0, \ q(t, 0) = 1, \ q(t, y) \to 0 \text{ as } y \to \infty \quad (3.2)$$

Using the abbreviation

$$\alpha = -\left[\left(\frac{M^2}{1+N^2}\right)(1 - iN) - 2iK^2\right]$$

Equation (3.1) can be written as

$$\frac{\partial q}{\partial t} = \frac{\partial^2 q}{\partial y^2} + \alpha q \quad (3.3)$$

Let $\phi(t, y) = e^{-\alpha t}\, q(t, y)$ (3.4)

From (3.3) we get $\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial y^2}$ (3.5)

From equations (3.2) and (3.4) we conclude that

$$\phi(0, y) = 0,\ \phi(t, 0) = e^{-\alpha t},\ \phi(t, y) \to 0 \text{ as } y \to \infty \quad (3.6)$$

To solve (3.5) subject to the initial and boundary conditions (3.6) we apply the Laplace transform method and obtain the solution as

$$q(t, y) = e^{at} \cos bt\ erfc\left(\frac{y}{2\sqrt{t}}\right) - \int_0^t e^{a\tau}\ erfc\left(\frac{y}{2\sqrt{\tau}}\right) [a \cos b\tau - b \sin b\tau] d\tau$$

$$+i\left[e^{at} \sin bt\ erfc\left(\frac{y}{2\sqrt{t}}\right)\right] - \int_0^t e^{a\tau}\ erfc\left(\frac{y}{2\sqrt{\tau}}\right) [a \sin b\tau - b \cos b\tau] d\tau$$

where $\alpha = a + ib$ with $a = -\frac{M^2}{1+N^2}$, $b = \frac{NM^2}{1+N^2} + 2K^2$ and $erfc(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$

4. Numerical solution for the second order BVP

In order to get a clear understanding of the flow fluid we have carried out numerical calculations of equation (2.15). The boundary value problem can be stated as

$$\frac{d^2 \hat{q}}{dy^2} - \omega \hat{q} = 0 \quad (4.1)$$

$$\hat{q}(0, s) = \frac{1}{s},\ \hat{q}(\infty, s) = 0$$

$$\text{where } \omega = \left(\frac{M^2}{1+N^2} + s\right) - i\left(\frac{NM^2}{1+N^2} + 2K^2\right) \quad (4.2)$$

To ensure that the Laplace transforms are well-defined, it should be assumed that $s > 0$. This implies $Re(\omega) = \frac{M^2}{1+N^2} + s > 0$. Hence there exists η in the complex number such that $\eta^2 = \omega$ with $Re(\eta) < 0$. Furthermore $\hat{q}(y, s) = \frac{e^{\eta y}}{s}$ (4.3)

satisfy the boundary value problem (4.1) and (4.2). For $y = 0$ we have

$$\hat{q}(0, s) = \frac{1}{s} = \int_0^\infty 1.e^{-st}\, dt = \int_0^\infty (1 + 0i).e^{-st}\, dt$$

Thus $u(0, t) \equiv 1$ and $v(0, t) \equiv 0$ for all t.

Recall that the inverse Laplace transform is

$$q(y, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \hat{q}\,(y, s) e^{st} ds$$

Where γ > 0 is chosen so that all the singularities of $\hat{q}(y, s)$ are to the left of γ. The above integral is over the vertical line z=γ in the complex plane. Since $\hat{q}(y, s) = \frac{e^{\eta y}}{s}$, we can choose γ to be any positive number. In the calculations below we choose γ=0.25.

We will define q strictly as a function of t using Mathematic's NIntegrate command. We will approximate the integral above by integrating from 0.25 – 500i to 0.25 + 500i.

The effect of the Hall parameter N, the Hartmann number M and the rotation parameter K^2 in the velocity components u and v is illustrated in the following figures

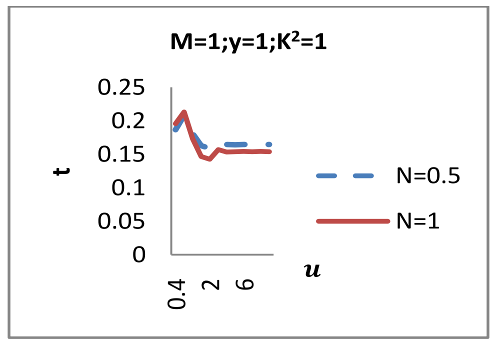

Figure 1. Variation of Hall parameter in u

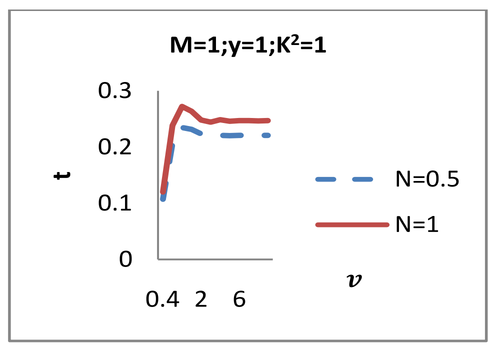

Figure 2. Variation of Hall parameter in v

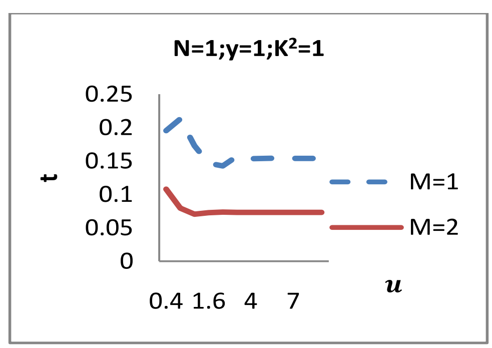

Figure 3. Variation of Hartmann number in u

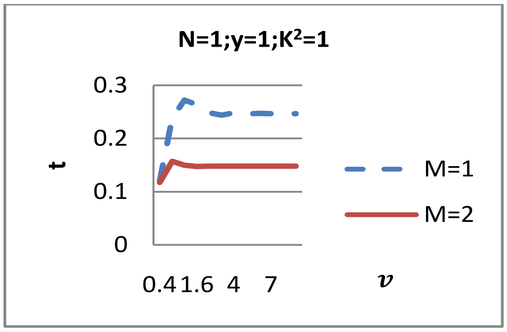

Figure 4.Variation of Hartmann number in v

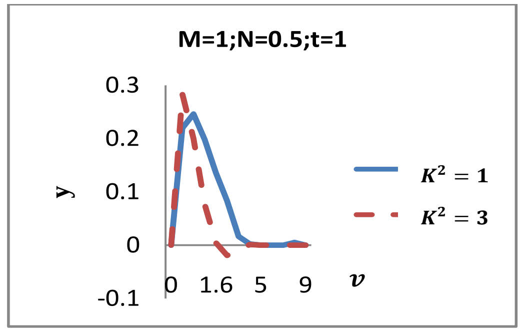

Figure 8. Variation of Ekman number in v

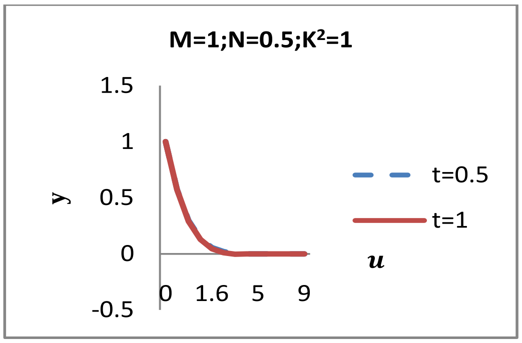

Figure 5. Variation of t in u

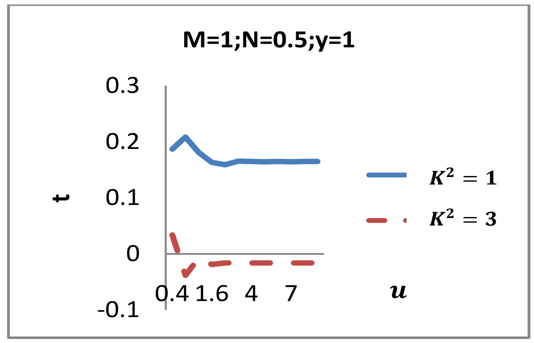

Figure 9. Variation of K^2 in u

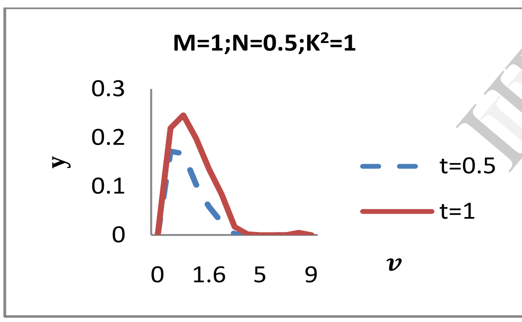

Figure 6. Variation of t in v

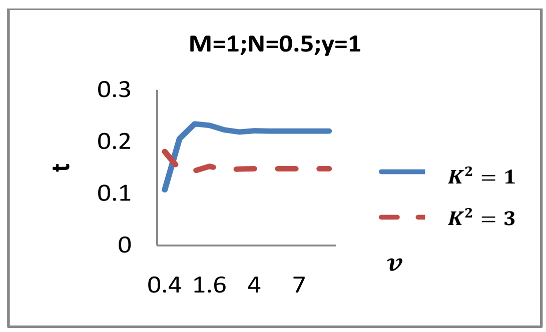

Figure 10. Variation of Ekman number in v

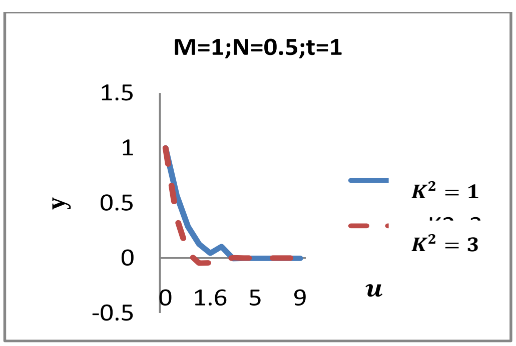

Figure 7. Variation of Ekman number in u

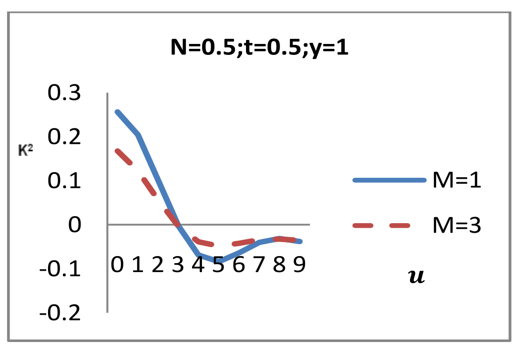

Figure 11. Variation of Hartmann number in u

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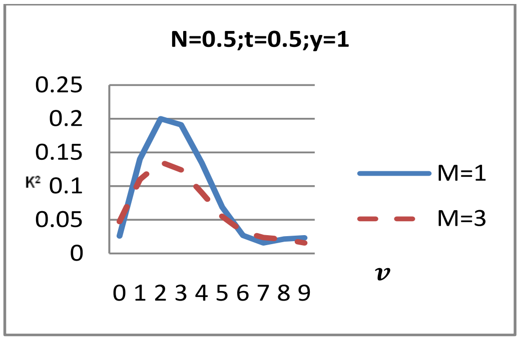

Figure 12 Variation of Hartmann number in v

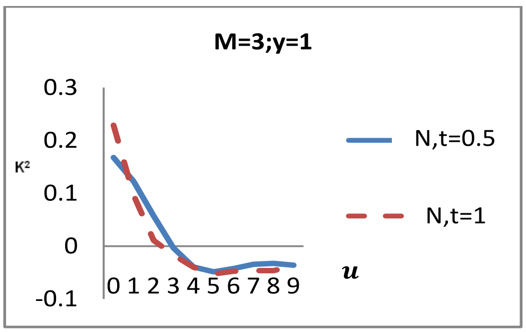

Figure 13. Effect of N, t in u

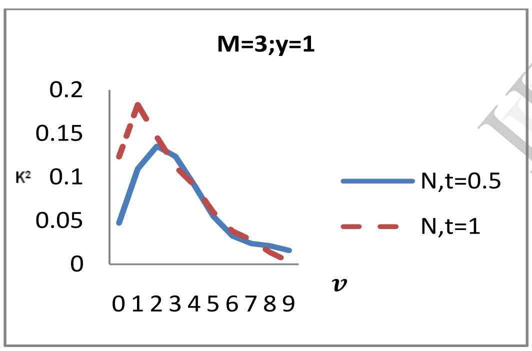

Figure 14 Effect of N, t in v

5. Conclusion

1. From Fig 1 and 2 the velocity component u decreases and v increases with the increase of N at equal heights of y and attains a steady state earlier with the increase of N.
2. From Fig 3 and 4, the velocity component u and v decreases with the increase of M.
3. From Fig 5 and 6, we conclude that when y increases at different values of t, u decreases and v increases for fixed values of M and N.
4. From Fig 7 and 8, if rotation parameter K^2 is increased then u and v decreases for fixed values of N, M and t with increase in y.
5. From Fig 9 and 10, If the rotation parameter K^2 is increased then u and v decreases for fixed values of N, M and y with increase in t.
6. From Fig 11 and 12, the velocity components u and v decreases with the increase of M for various values of K^2.
7. From Fig 13 the velocity components u decreases for fixed values of y and M, for different values of N and t along with the increase of K^2.
8. From Fig 14, the velocity component v gets unstable.

6. References

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