International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 3 Issue 4, April - 2014

LMI based control for balancing an Inverted Pendulum Mobile Robot

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***Abstract* — Inverted pendulum on a mobile robot system is a paradigm example taken into consideration for studying the dynamics related to huge complex systems over many years. The major problems that arise with this system are to stabilize the unstable equilibrium point of it, which can be readily achieved by the moving cart underneath. This paper tries to implement an Linear Matrix Inequality (LMI) based optimal controller for this system, to ensure robust stability and optimal performance with respect to the other known techniques in the literature, being devised using convex optimization procedure.**

***Index Terms*— Inverted Pendulum, Under-actuated system, Linear Matrix Inequality (LMI).**

I. INTRODUCTION

Inverted pendulum obsoletes the benchmark problem in numerous automatic control systems since forties. It represents the basic system of the higher-order nonlinear as well as non-minimum phase systems [1]. As the system itself represents the nonlinear system, it would be beneficial to illustrate many ideas over it of non-linear control.

Wheeled mobile robot received a huge interest in recent years, as it provides high degree of efficiency and flexibility with respect to different operations, when flexible motion requires with respect to smooth grounds. Sometime lack of knowledge regarding control limits the application. Here the inverted pendulum is joined with the robot having a motor which drives it on a horizontal track. The position and velocity are the accessible parameters of the motor. The pendulum has an unstable equilibrium point and the behavior of the system is helpful for the analysis of similar systems.

Balancing an inverted pendulum on a wheeled mobile robot is inherently unstable system, and its dynamics are profoundly nonlinear. This system is under-actuated mechanical system having control inputs less than degree of freedom. This makes the controlling more challenging. Due to these properties this system is a benchmark problem for designing and evaluating different control techniques.

There are many control methods proposed in the case of an inverted pendulum. In [2] and [3] a Proportional-Integral-Derivative (PID), and Proportional-Derivative (PD) control techniques have been deployed. The execution of the dynamical frameworks being controlled is sought to be optimal. There are numerous optimization techniques which are available for linear & nonlinear dynamical frameworks like Model Predictive Control (MPC) [4] and Linear Quadratic Regulator (LQR) [5] etc are also proposed for the same system.

The method proposed in this paper is for balancing an inverted pendulum attached on the wheeled mobile robot by using LMI [6] based state feedback controller [7] method. This type of control provides quick stabilization by state feedback controller. The search for the desired controller is casted as a problem in the convex frame. LMI updates parameters of the control law according to the Lyapunov Stability Theorem[8]. The basic LQR controller can also get the optimal results but is highly sensitive to disturbance. An LMI design technique overcomes this problem and hence stabilizes any unpredictive behavior of the system due to uncertainties. The implementation of the controller with more analytical approach will be tested in MATLAB environment.

The aim of this paper is designing a controller which meets the following requirements:

- T_s i.e. settling time < 5 seconds
- Overshoot < 10 degees
- T_r i.e. rise time < 0.5 seconds

This paper contain mathematical modeling of the Wheeled Mobile Robot which is the Inverted Pendulum mounted on a wheeled robot in Section II, Section III explains the fundamental designing of the LMI based controller for the stabilization on the given under-actuated system, in Section IV, the simulation and results of the given system are shown and we conclude the paper in Section V.

II. MATHEMATICAL MODEL

The purpose of this section is to turn up with a credible model to be used as the base for control design. The block diagram depicts the system. The mass of robot is given as M, the mass of pendulum is by m, applied force by F and angle by θ related to the vertical axis.

While modeling the system, many conditions needs to be considered within which some are vital than others. Without loss of generality, the following assumptions are taken into account for simplification of the model.

- Frictionless hinge between pendulum and robot
- Frictionless contact between wheel and horizontal plane

ISSN: 2278-0181
Vol. 3 Issue 4, April - 2014

- Little angle approximation i.e. movement of pendulum is restricted within few degrees
- Measurement of states are taken by sensors assuming the availability of LQR controller

Applying Lagrange's Equation corresponds to the position of the robot i.e. x and deflection angle of pendulum θ and considering moments around the center of mass, the non linear dynamic equation of the pendulum is as given below,

$$(I+ml^2)\ddot{\theta} - mglsin\theta = -ml\ddot{x}cos\theta \quad (1)$$

Where I represents moment of inertia of pendulum, θ represent the angle with respect to the vertical line and L represents length of the pendulum with distance to the center of mass $L/2 = l$, as shown in Figure 1.

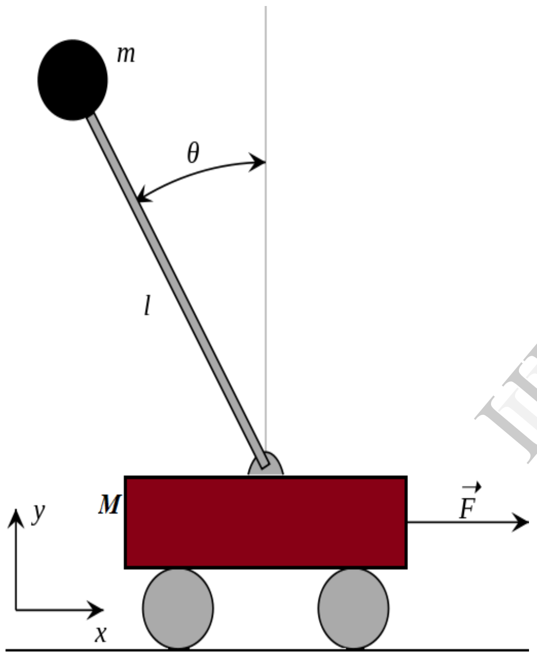

Figure1: Inverted Pendulum mounted on mobile robot

The robotic system governs the equation of motion corresponds to the applied forces from pendulum to robot. Which yield the Newton's law of motion and the motion of robot in horizontal direction is as,

$$(M+m)\ddot{x} + ml\ddot{\theta}cos\theta - ml\dot{\theta}^2 sin\theta = F \quad (2)$$

Where F is the physical quantity i.e. applied force of the motor [11].

$$F = \frac{\eta_t K_m K_t}{Rr} V - \frac{Km^2 K_g^2}{Rr^2}\dot{x} \quad (3)$$

Taking K_g, K_m, η_t, K_t, R and r as the coefficients which depends on the physical properties of gear and motor. Inferring from above derivation states can be defined as,

$$x = [x_1\ x_2\ x_3\ x_4]^T = \left[x\ \dot{x}\ \theta\ \dot{\theta}\ \right]^T \quad (4)$$

After linearization the state space model of the system is as,

$$\dot{x} = Ax + BV \quad (5)$$

$$y = Cx \quad (6)$$

Where A, B and C are the matrices of relevant dimension which are required to design the controller. V represents the input voltage applied to the motor and the interested linearization point is unstable equilibrium of the system.

$$x = [0\ 0\ 0\ 0\ ]^T$$

Approximating the angle of approximation very small gives $sin\theta = \theta, cos\theta = 1$ and $\dot{\theta} = 0$ the following linearization equations occurred.

$$\ddot{\theta} = \frac{(mgl\theta - ml\ddot{x})}{(I+ml^2)} \quad (7)$$

$$\ddot{x} = \frac{(F - ml\ddot{\theta})}{(M+m)} \quad (8)$$

The more convenient way for state space representation of the above equations is,

$$\ddot{\theta} = \frac{K_m^2 K_g^2}{Rr^2(M_t L - ml)}\dot{x} + \frac{g}{(L-ml)}\theta - \frac{K_m K_g}{Rr(M_t L - ml)}V \quad (9)$$

$$\ddot{x} = -\frac{LK_m^2 K_g^2}{Rr^2(LM_t - ml)}\dot{x} - \frac{mgl}{(LM_t - ml)}\theta + \frac{LK_m K_g}{Rr(LM_t - ml)}V \quad (10)$$

where $L = \frac{I+ml^2}{ml}$ and $M_t = M + m$.

as above the state vector,

$$\dot{x} = [x_2\ \ddot{x}\ x_4\ \ddot{\theta}]^T$$

represent the first order system from (9) and (10) as the state space form,

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{K_m^2 K_g^2}{Rr^2(M_t L - ml)} & \frac{g}{(L-ml)} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{LK_m^2 K_g^2}{Rr^2(LM_t - ml)} & -\frac{mgl}{(LM_t - ml)} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{LK_m K_g}{Rr(LM_t - ml)} \\ 0 \\ -\frac{K_m K_g}{Rr(M_t L - ml)} \end{bmatrix} V \quad (11)$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (12)$$

III. DESIGN OF CONTROLLER: AN LMI APPROACH

In this section, it is portrayed how a state feedback controller for the given system explained in section II can be mounted as an LMI based convex problem [6].

For the linearized model of the system explained in equation (11) and (12) input V can be formulated as a linear function of state vector i.e. $V(t) = Kx(t)$ where $K$is gain of state feedback controller. Now the state space equations can be formulated as follows,

$$\dot{x}(t) = (A + BK)x(t), \qquad x(0) = x_0 \quad (13)$$

By applying LMI on the system family of controller design problem such that the stabilized state is attained and control effort is reduced relating to a measure of mean square deviation.

The global asymptotic stability of (13) can be attained by the Lyapunov function,

$$V(x) = x^T P x \quad (14)$$

For stability the derivative of $V(x)$ to be negative for the solution of (13). Here Lyapunov function P must be symmetric positive definite $(P > 0)$. The control problem is to find a Lyapunov function P and a controller gain K that minimizes the bound $x_0^T P x_0$. The solution of this problem can be determined by getting a Lyapunov function and drawing K which leads to a satisfied performance bound. From which the optimization problem can be formulated as,

$$\begin{gathered} \min x_0^T P x_0 \\ \text{s.t.}\, P > 0 \\ (A + BK)^T + P(A + BK) + Q + K^T R K \leq 0 \end{gathered} \quad (15)$$

This optimization problem taken as SDP unlikely Semi-Definite Programming (SDP) [6], as the constrained contained by this problem are bilinear term P and quadratic term K. This problem can be taken as SDP with some translations. Introducing new matrices Y and W such that,

$$Y = P^{-1}, W = KP^{-1}$$

Where P is greater than zero and Y is also greater than zero. This leads us,

$$P = Y^{-1}, K = WY^{-1}$$

The substitution of Y and W in the place of P and K in equation (15) with the pre-multiplication and post-multiplication by Y, we get the inequality as,

$$YA^T + W^T B^T + AY + BW + YQY + W^T RW \leq 0 \quad (16)$$

The LMI representation of equation (15) can be formulated as,

$$\begin{bmatrix} -Z & Y & W^T \\ Y & Q^{-1} & 0 \\ W & 0 & R \end{bmatrix} \geq 0 \quad (17)$$

where the value of Z is,

$$Z = (YA^T + W^T B^T + AY + BW)$$

and the terms R and Q are invertible with Schur compliment hence the cost can be formulated as LMI,

$$\begin{bmatrix} \gamma & x_0^T \\ x_0 & Y \end{bmatrix} \geq 0 \quad (18)$$

The resulting LMI formulation of the convex problem is as follows,

$$\begin{gathered} \min\ \gamma \\ \text{s.t.} \begin{bmatrix} -Z & Y & W^T \\ Y & Q^{-1} & 0 \\ W & 0 & R \end{bmatrix} \geq 0 \\ \begin{bmatrix} \gamma & x_0^T \\ x_0 & Y \end{bmatrix} \geq 0 \end{gathered} \quad (19)$$

IV. SIMULATION RESULTS

The simulation substantiation of the given under-actuated system after applying LMI proposed in section II. The response of the system is as shown in Figure 2. For solving the LMI CVX Toolbox [9] is used in MTLAB.

The under-actuated system gives lower settling time and lower damping over the values provided in [10] based on the derivation (3) and putting the values in (11) and (12), in the matrix equation the values of A and B are as follows,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -15.14 & -3.04 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 37.23 & 31.61 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 3.39 \\ 0 \\ -8.33 \end{bmatrix}$$

After solving the stabilization problem with LMI based control, the value K is [9.9998, 16.5573, 64.5656, 9.2759], by taking the value of Q as diag[100, 1, 2000, 1] and R as 1 we reach the required specifications.

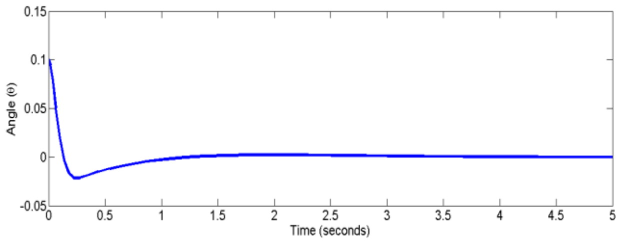

Figure 2: Response of the system after applying LMI

designed for stabilizing the highly unstable inverted pendulum mounted on mobile robot. This controller is obtained using convex optimization hence the controlled system shows faster yet accurate responses, which were the desired system requirement. Stability of the controller is guaranteed by using Lyapunov stability approach.

As seen from the simulation results, the controller stabilizes the pendulum in less than 2 *seconds*. While the LQR and other techniques does not give global stability and takes more time to stabilizing and shows more oscillations.

VI. ACKNOWLEDGMENT

The authors would like to acknowledge the support of Mr. Pratik Bajaria and VJTI, Matunga, Mumbai, India for their valuable guidance and support for this work.

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