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Linear Connections on Manifold Admitting F (2k + P, P)-Structure

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Abstract. *D. Demetropoulou [2] and others have studied linear connections in the manifold admitting f(2v+3,−1)-structure. The aim of the present paper is to study some properties of linear connections in a manifold admitting F(2K +P, P)-structure. Certain interesting results have been obtained.*

Key words. *Linear connection, projection, geodesic, parallelism.*

1. Preliminaries

Let F be a non-zero tensor field of the type (1, 1) and of class C1 on an n-dimensional manifold M^n such that [5, 8]

$$F^{2K+P} + F^P = 0, \qquad (1.1)$$

where K and P is a fixed positive integer greater than or equal to 1. The rank of (F) = r =constant.

Let us define the operators on M as follows [5, 8]

$$l = -F^{2K}, \qquad m = I + F^{2K} \qquad (1.2)$$

where I denotes the identity operator.

We will state the following two theorems [5]

Theorem 1.1. Let M^n be an F-structure manifold satisfying (1.1), then

$$\left.\begin{array}{l} a.\ l + m = I, \\ b.\ l2 = l, \\ c.\ m2 = m, \\ d.\ lm = ml = 0. \end{array}\right\} \qquad (1.3)$$

Thus for (1, 1) tensor field F(≠0) satisfying (1.1), there exist complementary distributions D_l and D_m corresponding to the projection operators l and m respectively. Then, dim D_l = r and dim D_m = (n−r).

Theorem 1.2. We have

$$\left.\begin{array}{l} a.\ lF = Fl = F, \quad mF = Fm = 0. \\ b.\ F^{2K}m = 0, \quad F^{2K}l = -l. \end{array}\right\} \qquad (1.4)$$

Thus F^K acts on D_l as an almost complex structure and on D_m as a null operator.

Let us define the operators $\bar{\nabla}$ and $\tilde{\nabla}$ on M^n in terms of an arbitrary connections ∇ as under

$$\bar{\nabla}_X Y = l\nabla_X(mY) + m\nabla_X(lY) \qquad (1.5)$$

and

$$\tilde{\nabla}_X Y = l\nabla_{lX}(mY) + m\nabla_{mX}(lY) + l[lX, mY] + m[mX, lY] \qquad (1.6)$$

Then it is easy to show that $\bar{\nabla}$ and $\tilde{\nabla}$ are linear connections on the manifold M^n [2]

2. Distributions anti-parallelism and anti-half parallelism

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In this section, first we have the following definitions:

Definition 2.1. Let us call the distribution D_L as ∇-anti parallel if for all TM^n denotes the tangent bundle of the manifold M^n.

Definition 2.2. The distribution D_L will be called ∇ anti-half parallel if for all $X \in D_L$ and $Y \in TM^n$, the vector field $\nabla_Y X \in D_M$, where

$$(\Delta F)(X, Y) = F\nabla_X Y - F\nabla_Y X - \nabla_{FX} Y + \nabla_Y FX \qquad (2.1)$$

F being a (1,1) tensor field on M^n satisfying the equation (1.1).

In a similar manner, anti-half parallelism of the distribution D_M can also be defined.

Theorem 2.3. In the F(2K+P, P)-structure manifold M^n, the distribution D_L and D_M are anti-parallel with respect to connections $\bar{\nabla}$ and $\tilde{\nabla}$.

Proof. Let $X \in TM^n$ and $Y \in D_L$, therefore mY = 0. Hence in view of equation (1.5), we get

$$\bar{\nabla}_X Y = m\nabla_X(lY) \in D_M.$$

Hence the distribution D_L is anti-half parallel with respect to the linear connection $\bar{\nabla}$. Similarly, it can also be shown that D_M is also $\bar{\nabla}$ is also anti-parallel.

Again in view of the equation (1.5), taking mY = 0, we obtain

$$\tilde{\nabla}_X Y = m\nabla_{mX}(lY) + m[mX, lY] \in D_M. \qquad (2.2)$$

Thus the distribution D_L is anti-parallel with respect to the linear connection $\tilde{\nabla}$. A similar result for D_M can also be proved in a similar manner.

Theorem 2.4. In the F(2K+P, P)-structure manifold M^n, the distribution D_L and D_M are anti-parallel with respect to connection ∇ if and only if ∇ and $\bar{\nabla}$ are equal.

Proof. Since the distributions D_L and D_M are anti-parallel with respect

to the ∇, hence

$$l\nabla_X(lY) = m\nabla_X(mY) = 0, \qquad (2.3)$$

for the vector fields X, Y $\in TM^n$.

Since $l + m = I$, hence in view of equation (2.3), it follows that

$$\nabla_X(lY) = m\nabla_X(lY),$$

$$\nabla_X(mY) = l\nabla_X(mY) \qquad (2.4)$$

Thus in view of the equations (1.5) and (2.4), it follows that

$$\bar{\nabla}_X Y = \nabla_X Y.$$

Hence, the connections ∇ and $\bar{\nabla}$ are equal.

The converse can also be proved easily.

Theorem 2.5. In a F(2K+P, P)-structure manifold M^n, the distribution D_M is anti-half parallel with respect to connection $\bar{\nabla}$ if

$$m\nabla_{FX}(lY) = m\bar{\nabla}_Y(FX), \qquad (2.5)$$

for arbitrary X $\in D_M$ and Y $\in TM^n$.

Proof.

Since mF = Fm = 0, hence in view of the equation (2.1), we get for the connection $\bar{\nabla}$

$$m(\Delta)(X, Y) = m\bar{\nabla}_Y(FX) - m\nabla_{FX}(Y). \qquad (2.6)$$

By virtue of the equation (1.5), the above equation (2.6) takes the form

$$m(\Delta F)(X, Y) = m\{l\bar{\nabla}_Y(mFX) + m\bar{\nabla}_Y(lFX)\}$$
$$- m\{l\nabla_{FX}(mY) + m\nabla_{FX}(lY)\}. \qquad (2.7)$$

Since, $ml = lm = 0$; $Fl = lF = F$ and m is the projection operator, the above equation (2.7) takes the form,

$$m(\Delta F)(X, Y) = m\bar{\nabla}_Y(FX) - m\nabla_{FX}(lY). \qquad (2.8)$$

Since the distribution D_M is $\bar{\nabla}$ anti-half parallel so far all X$\in D_M$, Y $\in TM^n$,

$$m(\Delta F)(X, Y) \in D_L.$$

Thus,

$$m\bar{\nabla}_Y(FX) = m\nabla_{FX}(lY).$$

Hence, the theorem is proved.

Theorem 2.6. In the manifold M^n equipped with F(2K+P, P)-structure, the distribution D_L is anti-half parallel with respect to the connection $\bar{\nabla}$ if

$$F\nabla_X(lY) = l\nabla_{FX}(mY),$$

for arbitrary X $\in D_L$ and Y $\in TM^n$.

Proof. Proof follows easily in a way similar to that of the theorem 2.5.

Theorem 2.7. In the F(2K+P, P)-structure manifold M^n, the distribution D_M is anti-half parallel with respect to the connection $\tilde{\nabla}$ if for X $\in$ DM and Y$\in TM^n$ the equation

$$m\nabla_{mY}(FX) + m[mY, FX] = 0$$

is satisfied.

Proof. For X$\in D_M$ and Y$\in TM^n$, we have for the connection $\tilde{\nabla}$

$$(\Delta F)(X, Y) = F\tilde{\nabla}_X Y - F\tilde{\nabla}_Y X - \tilde{\nabla}_{FX} Y + \tilde{\nabla}_Y FX. \qquad (2.9)$$

As Fm = mF = 0, hence from the above equation (2.9), it follows that

$$m(\Delta F)(X, Y) = m\tilde{\nabla}_Y FX - m\tilde{\nabla}_{FX} Y. \qquad (2.10)$$

In view of the equation (1.4) and (1.6), it is easy to show that

$$m\tilde{\nabla}_{FX} Y = 0 \qquad (2.11)$$

and

$$m\tilde{\nabla}_Y FX = m\nabla_{mY}(FX) + m[mY, FX]. \qquad (2.12)$$

Thus, we get

$$m(\Delta F)(X,Y) = m\nabla_{mY}(FX)+m[mY, FX]. \qquad (2.13)$$

The distribution D_M will be $\tilde{\nabla}$ anti-half parallel if X$\in D_M$, Y $\in TM^n$, the vector field $(\Delta F)(X,Y) \in D_L$. Thus,

$$m(\Delta F)(X, Y) = 0$$

i.e.,

$$m\nabla_{mY}(FX) + m[mY, FX] = 0.$$

Hence, the theorem is proved.

3. Geodesic in the manifold M^n

Let C be a curve in M^n, T a tangent field and ∇ arbitrary connection on M^n. Then, we have

Definition 3.1. The curve C is a geodesic with respect to the connection ∇ if $\nabla_T T = 0$ along C.

Applying the definition for the connection $\bar{\nabla}$ and $\tilde{\nabla}$, we have the following results in the F(2K + P, P)-structure manifold M^n.

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Theorem 3.2. A curve C is a geodesic in the manifold M^n with respect to the connection ∇ if the vector fields

$$\nabla_T T - \nabla_T (lT) \in D_M \text{ and } \nabla_T (lT) \in D_L.$$

Proof. The curve C will be $\overline{\nabla}$ geodesic if $\overline{\nabla}_T T = 0$.

In view of the equation (1.5), the above equation takes the form

$$l\nabla_T (I - l)T + m\nabla_T (lT) = 0$$

or equivalently

$$l\nabla_T T - l\nabla_T (lT) + m\nabla_T (lT) = 0,$$

which implies that

$$\nabla_T T - \nabla_T (lT) \in D_M \text{ and } \nabla_T (lT) \in D_L.$$

This proves the theorem.

Theorem 3.3. A curve C is a geodesic in the manifold M^n with respect to the connection ∇ if

$$\nabla_{lT} T - \nabla_{lT} (lT) + [lT, mT] \in D_M$$

And $\nabla_{mT} (lT) + [mT, lT] \in D_L$.

Proof. Using definition of ∇ from the equation (1.6), proof follows easily as of theorem 3.2.

Theorem 3.4. The (1, 1) tensor field l is covariant constant with respect to the connection $\overline{\nabla}$ if

$$m\nabla_X(lY) = l\nabla_X(mY) \qquad (3.1)$$

but the tensor field m is always covariant constant.

Proof. We have

$$(\overline{\nabla}_X l)Y = \overline{\nabla}_X(lY) - l\overline{\nabla}_X Y \qquad (3.2)$$

In view of equation (1.5), the above equation takes the form

$$(\overline{\nabla}_X l)Y = l\nabla_X(mlY) + m\nabla_X(lY) - l\{l\nabla_X(mY) + m\nabla_X(lY)\}. \qquad (3.3)$$

Since $l^2 = l$ and $lm = ml = 0$, the equation (3.3) takes the form

$$(\overline{\nabla}_X l)Y = m\nabla_X(lY) - l\nabla_X(mY). \qquad (3.4)$$

The (1, 1) tensor field l is covariant with respect to the connection $\overline{\nabla}$ if

$$(\overline{\nabla}_X l)Y = 0. \qquad (3.5)$$

Hence in view of the equation (3.4) and (3.5), we get

$$m\nabla_X(lY) = l\nabla_X(mY).$$

This proves the first part of the theorem.

Again it can be easily shown that

$$(\overline{\nabla}_X m)Y = 0,$$

for all vector fields $X, Y \in TM^n$. Thus, the tensor field m is always covariant constant.

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