Intuitionistic Fuzzy Goal Geometric Programming Problem (IFG²P²) based on Geometric Mean Method

Payel Ghosh ¹*, Tapan Kumar Roy ²

¹Department of Mathematics, Adamas Institute of Technology, Barasat, P.O.-Jagannathpur, Barbaria, 24 Parganas (N), West Bengal - 700126,

²Department of Mathematics, Bengal Engineering and Science University, Shibpur, P.O.-Botanic Garden, Howrah, West Bengal-711103,

Abstract

In this paper, a new approach of intuitionistic fuzzy goal programming is proposed. A nonlinear intuitionistic fuzzy goal programming is solved here using geometric programming technique. For this we have used geometric mean method. We have applied the proposed method on industrial waste water treatment design problem. Also there is a comparison of results on industrial waste water treatment design problem with other methods to show the benefit of this method.

Keywords: Goal programming, Geometric programming, Non-linear optimization.

1. Introduction

Intuitionistic fuzzy set (IFS) introduced by Atanassov [1], is a growing field of research in different directions. Now a day’s goal programming in fuzzy environment is common whereas intuitionistic fuzzy goal programming is rare. In this paper we have worked on intuitionistic fuzzy goal programming problem where equations are non-linear. Geometric programming gives better result than nonlinear programming (K-K-T conditions), which is already described in Ghosh, Roy [2, 3]. Therefore geometric programming is used here to solve nonlinear goal programming problem. Nan et. al. [4], Ghosh, Roy [5] discussed arithmetic mean in intuitionistic fuzzy environment. In this paper we have used geometric mean in intuitionistic fuzzy goal programming. A numerical example and an application on industrial waste water treatment design problem is taken here as an illustration. Previously Shih, Krishnan [6], Evenson [7], Ecker, McNamara [8], Beightler, Philips [9] have illustrated this design using dynamic programming, geometric programming. Later Cao [10] has discussed the same using fuzzy geometric programming. In this paper, we have compared the results of industrial wastewater treatment design problem with the results in another method.
2. Fuzzy Goal Geometric Programming Problem (FGG^2P^2)

Find \( X = (x_1, x_2 \ldots x_q)^T \) \quad (1.1)

so as to

\[ \text{Minimize } f_j(X), \text{ with target value } C_j, \text{ acceptance tolerance } a_j. \]

subject to \( f_r(X) \leq C_r, r=1, 2 \ldots m, X=(x_1, x_2 \ldots x_q)^T \geq 0. \)

Membership functions can be written as follows

\[ \mu_{f_j}(X) = \]
\[ \begin{cases} 
1, & f_j(X) \leq C_j \\
1 - \frac{f_j(X) - C_j}{a_j}, & C_j \leq f_j(X) \leq C_j + a_j \\
0, & f_j(X) \geq C_j + a_j
\end{cases} \]

Hence the crisp programming from fuzzy goal programming is

Maximize \( \mu_{f_j}(X) \).

subject to \( 0 \leq \mu_{f_j}(X) \leq 1 \)

\( f_r(X) \leq C_r, r=1, 2 \ldots m, X=(x_1, x_2 \ldots x_q)^T \geq 0. \)

Model (1.2) can be written as

Maximize \( \alpha \) \quad (1.3)

subject to \( \mu_{f_j}(X) \geq \alpha \)

\( f_r(X) \leq C_r, r=1, 2 \ldots m \)

\( 0 \leq \alpha \leq 1, X=(x_1, x_2 \ldots x_q)^T \geq 0. \)

This is equivalent to the following geometric programming problem

Minimize \( \alpha^{-1} \) \quad (1.4)

subject to \( \frac{f_j(X)}{a_j(1-\alpha) + C_j} \leq 1 \)

\( \frac{f_r(X)}{C_r} \leq 1, r=1, 2 \ldots m, \)

\( 0 \leq \alpha \leq 1, X=(x_1, x_2 \ldots x_q)^T \geq 0. \)

3. Intuitionistic Fuzzy Goal Geometric Programming Problem (IFG^2P^2)

Find \( X = (x_1, x_2 \ldots ... x_q)^T \) \quad (2.1)

so as to

\[ \text{Minimize } f_j(X), \text{ with target value } C_j, \text{ acceptance tolerance } a_j. \]

subject to \( f_r(X) \leq C_r, r=1, 2 \ldots m \)

\( X=(x_1, x_2 \ldots x_q)^T \geq 0. \)

Membership and non-membership functions are

\[ \mu_{f_j}(X) = \]
\[ \begin{cases} 
1, & f_j(X) \leq C_j \\
1 - \frac{f_j(X) - C_j}{a_j}, & C_j \leq f_j(X) \leq C_j + a_j \\
0, & f_j(X) \geq C_j + a_j
\end{cases} \]

\[ \vartheta_{f_j}(X) = \]
\[ \begin{cases} 
0, & f_j(X) \leq C_j \\
f_j(X) - C_j, & C_j \leq f_j(X) \leq C_j + b_j \\
1, & f_j(X) \geq C_j + b_j
\end{cases} \]

\( \mu, \vartheta \)

\[ \text{Fig: 1 Membership and non-membership function} \]

Intuitionistic fuzzy goal programming can be transformed into crisp programming model using membership and non-membership function as

Maximize \( \mu_{f_j}(X) \), Minimize \( \vartheta_{f_j}(X) \) \quad (2.2)
subject to $0 \leq \mu f_j(X) + \vartheta f_j(X) \leq 1$

$f_r(X) \leq C_r, r = 1, 2 \ldots m$

$0 \leq \mu f_j(X), \vartheta f_j(X) \leq 1,$

$X = (x_1, x_2 \ldots x_q)^T > 0.$

This is equivalent to

Maximize $\alpha$, Minimize $\beta$ \hspace{1cm} (2.3)

subject to $\mu f_j(X) \geq \alpha, \vartheta f_j(X) \leq \beta$

$f_r(X) \leq C_r, r = 1, 2 \ldots m$

$0 \leq \alpha + \beta \leq 1, 0 \leq \alpha, \beta \leq 1,$

$X = (x_1, x_2 \ldots x_q)^T > 0.$

It is easily seen that Max $\alpha$ is equivalent to Min $(1 - \alpha)$ as $0 \leq \alpha \leq 1$. Taking geometric mean, the above model can be written as

Minimize $\beta(1 - \alpha)$ \hspace{1cm} (2.4)

subject to $f_j(X) \leq a_{j0} \times b_{j0} \beta(1 - \alpha) + c_{j0}$

$f_r(X) \leq C_r, r = 1, 2 \ldots m$

$0 \leq \alpha + \beta \leq 1, 0 \leq \alpha, \beta \leq 1,$

$X = (x_1, x_2 \ldots x_q)^T > 0.$

Let us take $\beta(1 - \alpha) = v > 0$, then the above model becomes

Minimize $v$ \hspace{1cm} (2.5)

subject to $\frac{f_j(X)}{a_{j0} \times b_{j0} v + c_{j0}} \leq 1$

$\frac{f_r(X)}{C_r} \leq 1, r = 1, 2 \ldots m$

$0 \leq \alpha + \beta \leq 1, 0 \leq \alpha, \beta \leq 1,$

$v \in (0, 1), X = (x_1, x_2 \ldots x_q)^T > 0.$

The above model (2.5) is solved by geometric programming technique with $v$ as parameter.

5. Industrial Wastewater Treatment Design

To optimize the treatment of industrial wastewater, a process flow from a paper and pulp industry has been considered. The treatment units indicate the removal of suspended solid and biological oxygen demand (BOD) from the waste water. In this paper, the design of treatment facilities is based on effluent containing BOD and essentially free from suspended solids. Wastewater treatment is consisted of primary clarification, secondary biological treatment (trickling filter followed by activated sludge or aerated lagoon), sludge disposal and tertiary treatment (coagulation, sedimentation, filtration for effluent of activated sludge and aerated lagoon;

<table>
<thead>
<tr>
<th>Design</th>
<th>Primary</th>
<th>Secondary</th>
<th>Tertiary</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Primary Clarifier</td>
<td>Trickling Filter &amp; Activated Sludge</td>
<td>Carbon Adsorption</td>
</tr>
<tr>
<td>2</td>
<td>Primary Clarifier</td>
<td>Trickling Filter &amp; Aerated Lagoon</td>
<td>Coagulation, Sedimentation, Filtration</td>
</tr>
<tr>
<td>3</td>
<td>Primary Clarifier</td>
<td>Activated Sludge</td>
<td>Carbon Adsorption</td>
</tr>
<tr>
<td>4</td>
<td>Primary Clarifier</td>
<td>Aerated Lagoon</td>
<td>Coagulation, Sedimentation, Filtration</td>
</tr>
<tr>
<td>5</td>
<td>Primary Clarifier</td>
<td>Trickling Filter &amp; Activated Sludge</td>
<td>Coagulation, Sedimentation, Filtration</td>
</tr>
<tr>
<td>6</td>
<td>Primary Clarifier</td>
<td>Activated Sludge</td>
<td>Coagulation, Sedimentation, Filtration</td>
</tr>
<tr>
<td>7</td>
<td>Primary Clarifier</td>
<td>Activated Sludge</td>
<td>None</td>
</tr>
<tr>
<td>8</td>
<td>Primary Clarifier</td>
<td>Trickling Filter &amp; Activated Sludge</td>
<td>None</td>
</tr>
<tr>
<td>9</td>
<td>Primary Clarifier</td>
<td>Aerated Lagoon</td>
<td>None</td>
</tr>
<tr>
<td>10</td>
<td>Primary Clarifier</td>
<td>Trickling Filter</td>
<td>None</td>
</tr>
</tbody>
</table>
carbon adsorption for effluent of aerated lagoon). There are many combination of wastewater treatment process given in Table-1 (Beightler, Philips (1976)) to remove five day BOD ($BOD_5$).

In our study, we have taken the first design. There are consecutively four processes (Primary Clarifier, Trickling Filter, Activated Sludge, and Carbon Adsorption).

Primary Clarifier $\Rightarrow$ Trickling Filter $\Rightarrow$ Carbon Adsorption $\Rightarrow$ Activated Sludge

Let $x_i$ be the percentage of remaining $BOD_5$ after each step. Then after four processes the remaining percentage of $BOD_5$ will be $x_1 x_2 x_3 x_4$. Our aim is to minimize the remaining percentage of $BOD_5$ with minimum annual cost as much as possible. The annual cost of $BOD_5$ removal by various treatments is shown in Table-2.

Table-2: List of annual costs in different treatments

<table>
<thead>
<tr>
<th>Design</th>
<th>Treatment</th>
<th>Annual Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Primary Clarifier</td>
<td>19.4$x_1^{-1.47}$</td>
</tr>
<tr>
<td>2</td>
<td>Trickling Filter</td>
<td>16.8$x_2^{-1.66}$</td>
</tr>
<tr>
<td>3</td>
<td>Activated Sludge</td>
<td>91.5$x_3^{-0.3}$</td>
</tr>
<tr>
<td>4</td>
<td>Carbon Adsorption</td>
<td>120$x_4^{-0.33}$</td>
</tr>
</tbody>
</table>

6. Fuzzy Goal Geometric Programming Problem ($FG^2P^2$)

Decision maker wants to remove about 98.5% $BOD_5$ and gives some relaxation 0.1 on this goal. Also he sets another goal as annual cost should be about 300 (thousand $) and gives flexibility 200 (thousand $) on this goal.

Then the fuzzy goal programming problem is

Minimize $f_1(x_1, x_2, x_3, x_4) = 19.4x_1^{-1.47} + 16.8x_2^{-1.66} + 91.5x_3^{-0.3} + 120x_4^{-0.33}$ with target 300, acceptance tolerance 200,

Minimize $f_2(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4$ with target 0.015, acceptance tolerance 0.1,

subject to $x_1, x_2, x_3, x_4 > 0$.

Membership and non-membership functions are given below

$\mu_{f_1}(x_1, x_2, x_3, x_4) = \begin{cases} 1, & f_1(x_1, x_2, x_3, x_4) \leq 300 \\ 1 - \frac{f_1(x_1, x_2, x_3, x_4) - 300}{200}, & 300 \leq f_1(x_1, x_2, x_3, x_4) \leq 500 \\ 0, & f_1(x_1, x_2, x_3, x_4) \geq 500 \end{cases}$

$\mu_{f_2}(x_1, x_2, x_3, x_4) = \begin{cases} 1, & f_2(x_1, x_2, x_3, x_4) \leq 0.015 \\ 1 - \frac{f_2(x_1, x_2, x_3, x_4) - 0.015}{0.1}, & 0.015 \leq f_2(x_1, x_2, x_3, x_4) \leq 0.115 \\ 0, & f_2(x_1, x_2, x_3, x_4) \geq 0.115 \end{cases}$

Following (1.2), (1.3) and (1.4), above model can be written into crisp programming problem as

Minimize $\alpha^{-1}$

subject to $19.4x_1^{-1.47} + 16.8x_2^{-1.66} + 91.5x_3^{-0.3} + 120x_4^{-0.33} \leq 200 (1-\alpha)+300$

$x_1 x_2 x_3 x_4 \leq 0.1(1-\alpha) + 0.015,$

$x_1, x_2, x_3, x_4 > 0, \alpha \in (0, 1)$.

Table 3: Optimal values of decision variables and objective functions of model (3)

<table>
<thead>
<tr>
<th>Dual Variables</th>
<th>Primal Variables</th>
<th>Optimal objective functions</th>
<th>Membership and non-membership functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_{11}^* = 1,$ $\delta_{12}^* = 0.088967278$</td>
<td>$x_1^* = 0.7059559$</td>
<td>$f_1^*(x_1, x_2, x_3, x_4) = 363.8048$</td>
<td>$\mu_{f_1}(x_1, x_2, x_3, x_4) = 0.680976$</td>
</tr>
<tr>
<td>$\delta_{12}^* = 0.078784276$</td>
<td>$x_2^* = 0.7248393$</td>
<td>$f_2^*(x_1, x_2, x_3, x_4) = 0.0469024$</td>
<td>$\mu_{f_2}(x_1, x_2, x_3, x_4) = 0.680976$</td>
</tr>
<tr>
<td>$\delta_{13}^* = 0.435939662$</td>
<td>$x_3^* = 0.1598653$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\delta_{14}^* = 0.396308784$</td>
<td>$x_4^* = 0.5733523$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\delta_{21}^* = 0.130781899$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Solving it using Cao’s geometric programming method taking \( \alpha \) as a parameter, where degree of difficulty is \((4+1) = 0\), we have the results given in Table-3. The table shows that, here 100-0.0469024×100= 95.30976% \( BOD_5 \) removes with the cost of 363.8048 (thousand $).

### 7. Intuitionistic Fuzzy Goal Geometric Programming Problem (IFG^2P^2)

Let decision maker wants to remove about 98.5% \( BOD_5 \) and the tolerances of acceptance and rejection on this goal are 0.1 and 0.2 respectively. Also he wants to remove the said amount of \( BOD_5 \) within 300 (thousand dollar) $ tolerances of acceptance and rejection on this goal are 200 and 300 respectively. Hence the intuitionistic fuzzy goal programming problem is

Minimize \( f_1(x_1, x_2, x_3, x_4) = 19.4x_1^{-1.47} + 16.8x_2^{-1.66} + 91.5 x_3^{-0.3} + 120 x_4^{-0.33} \) with target 300, acceptance tolerance 200 and rejection tolerance 300

Minimize \( f_2(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4 \) with target 0.015, acceptance tolerance 0.1 and rejection tolerance 0.2

subject to \( x_1, x_2, x_3, x_4 > 0 \).

Membership and non-membership functions are given below

\[
\mu_{f_1}(x_1, x_2, x_3, x_4) = \begin{cases}
1, & f_1(x_1, x_2, x_3, x_4) \leq 300 \\
\frac{1 - f_1(x_1, x_2, x_3, x_4) - 300}{200}, & 300 \leq f_1(x_1, x_2, x_3, x_4) \leq 500 \\
0, & f_1(x_1, x_2, x_3, x_4) \geq 500
\end{cases}
\]

\[
\theta_{f_1}(x_1, x_2, x_3, x_4) = \begin{cases}
0, & f_1(x_1, x_2, x_3, x_4) \leq 300 \\
\frac{300 - f_1(x_1, x_2, x_3, x_4)}{300}, & 300 \leq f_1(x_1, x_2, x_3, x_4) \leq 600 \\
1, & f_1(x_1, x_2, x_3, x_4) \geq 600
\end{cases}
\]

\[
\mu_{f_2}(x_1, x_2, x_3, x_4) = \begin{cases}
1, & f_2(x_1, x_2, x_3, x_4) \leq 0.015 \\
1 - \frac{f_2(x_1, x_2, x_3, x_4)}{0.015}, & 0.015 \leq f_2(x_1, x_2, x_3, x_4) \leq 0.115 \\
0, & f_2(x_1, x_2, x_3, x_4) \geq 0.115
\end{cases}
\]

\[
\theta_{f_2}(x_1, x_2, x_3, x_4) = \begin{cases}
0, & f_2(x_1, x_2, x_3, x_4) \leq 0.015 \\
\frac{f_2(x_1, x_2, x_3, x_4) - 0.015}{0.2}, & 0.015 \leq f_2(x_1, x_2, x_3, x_4) \leq 0.215 \\
1, & f_2(x_1, x_2, x_3, x_4) \geq 0.215
\end{cases}
\]

Following (2.1), (2.2), (2.3) and (2.4), the crisp programming problem is

Minimize \( v \)

subject to

\[
\frac{19.4x_1^{-1.47} + 16.8x_2^{-1.66} + 91.5 x_3^{-0.3} + 120 x_4^{-0.33}}{200 \times 300 + 300} \leq 1
\]

\[
\frac{x_1 x_2 x_3 x_4}{0.1 \times 0.2 \times 0.015} \leq 1,
\]

\( x_1, x_2, x_3, x_4 > 0 \), \( v \in (0, 1) \)

Solving it using Cao’s geometric programming method having degree of difficulty \((4+1) = 0\), we have the results given in Table 4.

<table>
<thead>
<tr>
<th>Dual Variables</th>
<th>Primal Variables</th>
<th>Optimal objective functions</th>
<th>Membership and non-membership functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta_{11} = 1 )</td>
<td>( x_1' = 0.6380199 )</td>
<td>( f_1'(x_1, x_2, x_3, x_4) = 422.1483 )</td>
<td>( \mu_{f_1}(x_1, x_2, x_3, x_4) = 0.3892583 )</td>
</tr>
<tr>
<td>( \delta_{12} = 0.088967278 )</td>
<td>( x_2' = 0.6627170 )</td>
<td>( \theta_{f_1}(x_1, x_2, x_3, x_4) = 0.4071612 )</td>
<td></td>
</tr>
<tr>
<td>( \delta_{13} = 0.078784276 )</td>
<td>( x_3' = 0.09737155 )</td>
<td>( \mu_{f_2}(x_1, x_2, x_3, x_4) = 0.9995928 )</td>
<td></td>
</tr>
<tr>
<td>( \delta_{14} = 0.45939662 )</td>
<td>( x_4' = 0.3653206 )</td>
<td>( \theta_{f_2}(x_1, x_2, x_3, x_4) = 0.00020358 )</td>
<td></td>
</tr>
<tr>
<td>( \delta_{21} = 0.130781899 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The table shows that membership and non-membership functions satisfy all the restrictions as in model (2.2). The percentage of $BOD_5$ removed from the wastewater is $(100 - 0.01504072 \times 100) = 98.495928\%$ which attains the set quota by the national standard and the annual total cost is 422.1483 (thousands $).

8. Comparison

Here is a comparison of results between other method and our proposed method.

<table>
<thead>
<tr>
<th>Method</th>
<th>Total Annual Cost (Thousand $)</th>
<th>Remaining $BOD_5$ in wastewater</th>
<th>Removed $BOD_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$FG^2P^2$ Arithmetic mean (Ghosh, Roy (2013b))</td>
<td>359.2533</td>
<td>0.04919147</td>
<td>95.080853%</td>
</tr>
<tr>
<td>IF$G^2P^2$ Geometric mean</td>
<td>422.1483</td>
<td>0.01504072</td>
<td>98.495928%</td>
</tr>
</tbody>
</table>

9. Conclusion

We have applied fuzzy and intuitionistic fuzzy goal geometric programming on industrial waste water treatment design. We have compared the results of various methods on industrial waste water treatment design. We have seen that in fuzzy goal geometric programming and intuitionistic fuzzy goal geometric programming with arithmetic mean, percentage of $BOD_5$ removal is almost same. But 4551.5 $ is saved in intuitionistic fuzzy goal geometric programming with arithmetic mean. In the proposed method, intuitionistic fuzzy goal geometric programming with geometric mean, 98.495928% $BOD_5$ is removed. Hence for better purification intuitionistic fuzzy goal geometric programming with geometric mean is more appropriate.

Acknowledgement

It’s my privilege to thank my respected guide for his enormous support and encourage for the preparation of this research paper. The great support of my family members makes me possible to do it. Last but not the least I am thankful to Prof. (Dr.) Chanchal Majumder, Department of Civil Engineering, Bengal Engineering and Science University, Shibpur.

References


