Integrals Involving the Product of Two Different Wright's Generalized Hypergeometric Functions

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Abstract:- **The present paper contains four interesting results with proper validity conditions, for the product of two different Wright's Generalized Hypergeometric functions. Some special cases of the general integrals lead to interesting results for Generalized Hypergeometric Functions $_pF_q$.**

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1. INTRODUCTION

The Wright's Generalized Hypergeometric function $_p\Psi_q$ (z), in contour integral form, is defined as

$$_p\Psi_q\left[\begin{matrix}(a_1,\alpha_1),....,(a_p,\alpha_p); \\ (b_1,\beta_1),....,(b_q,\beta_q);\end{matrix}\; z\right]=\frac{1}{2\pi i}\int_D \frac{\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j s\right)\Gamma(-s)(-z)^s}{\prod_{j=1}^{q}\Gamma\left(b_j+\beta_j s\right)}ds \quad .....(1.1)$$

Where D is a contour in the complex s – plane which runs from $s=\sigma-i\infty$ to $s=\sigma+i\infty$ (σ is an arbitrary real number) so that the points s = 0, 1, 2,….. resp. lie to the right of D.

Wright's generalized hypergeometric function in series form [2; p.50 (21)], denoted by $_p\Psi_q^{'}$ (z) is defined as:

$$_p\Psi_q^{'}\left[\begin{matrix}(A_1,\gamma_1),....,(A_p,\gamma_p); \\ (B_1,\delta_1),....,(B_q,\delta_q);\end{matrix}\; z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(A_j+\gamma_j n\right)z^n}{\prod_{j=1}^{q}\Gamma\left(B_j+\delta_j n\right)n!} \quad ...(1.2)$$

Here p & q are non-negative integers ; γ_1, ….., γ_p ; δ_1,….., δ_q are positive constants ; A_1,…..,A_p ; B_1,….,B_q are complex constants such that

$$A_j+\gamma_j n\neq 0,-1,-2,... \quad (j=1,....,p;\ n=0,1,2...) \qquad (1.3)$$

The series in equation (1.2) is convergent for all values of z and it defines an integral function of z if

$$\mu=1+\sum_{j=1}^{q}\delta_j-\sum_{j=1}^{p}\gamma_j \quad \text{is positive.} \qquad (1.4)$$

If μ=0 then series in (1.2) is convergent for |z|<ξ^{-1} and it defines an analytic function of z for |z|< ξ^{-1}. Here

$$\xi=\prod_{j=1}^{p}\gamma_j^{\gamma_j}\prod_{j=1}^{q}\delta_j^{-\delta_j} \qquad (1.5)$$

The generalized hypergeometric function $_pF_q$ with p numerator parameters and q denominator parameters in contour integral form [2; p.43 (6)] is defined by

$$\frac{\prod_{j=1}^{p}\Gamma(a_j)}{\prod_{j=1}^{q}\Gamma(b_j)}\,{}_pF_q\left[\begin{matrix}a_1,....,a_p; \\ b_1,....,b_q;\end{matrix}\; z\right]=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{p}\Gamma\left(a_j+s\right)\Gamma(-s)(-z)^s}{\prod_{j=1}^{q}\Gamma\left(b_j+s\right)}ds \quad .....(1.6)$$

L is a contour runs from –i∞ to +i∞, no a_j (j=1,….,p) is zero or a negative integer; $|\arg(-z)|<\pi$.

Generalized Gaussian Hypergeometric function $_pF_q$ [1; p.73 (2)] , in series form, is defined by :

$$_PF_Q^{'}\left[\begin{matrix}A_1,....,A_P; \\ B_1,....,B_Q;\end{matrix}\; z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{P}(A_j)_n\, z^n}{\prod_{j=1}^{Q}(B_j)_n\, n!}$$

(1.7)

and the pochhammer symbol $(a)_n$ [2;p.21 (14)] is defined by

$$(a)_n = \begin{cases} \dfrac{\Gamma(a+n)}{\Gamma(a)} & ; a \neq 0,-1,-2,-3.......... \\ a(a+1)(a+2).......(a+n-1) ; & n=1,2,3...... \\ 1 & ; n=0 \end{cases} \quad (1.8)$$

2. SOME AUXILIARY RESULTS

We require four integrals [3; (1), (2), (20) and (3); p. 284 – 288] in the following forms:

(1) If a > 0, b ≠ -1, Re(α) > -1, Re (σ)> -1, then

$$\int_0^a \frac{t^\alpha (a-t)^\sigma}{(a+bt)^{2+\alpha+\sigma}} P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right)dt$$

$$= \frac{\Gamma(\alpha+u+1)}{a(b+1)^{\alpha+1}\, u!} \frac{\Gamma(1+\sigma)\Gamma(1\text{-}\beta+\sigma)}{\Gamma(2+u+\alpha+\sigma)\Gamma(1\text{-}u\text{-}\beta+\sigma)} \quad (2.1)$$

(2) If a > 0, b ≠ –1, Re(ρ) > –1, Re (β) > –1, then

$$\int_0^a \frac{t^\rho (a-t)^\beta}{(a+bt)^{2+\rho+\beta}} P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right)dt$$

$$= \frac{(-1)^u \Gamma(\beta+u+1)}{a(b+1)^{\rho+1}\, u!} \frac{\Gamma(1+\rho)\Gamma(1\text{-}\alpha+\rho)}{\Gamma(1\text{-}u\text{-}\alpha+\rho)\Gamma(2+u+\beta+\rho)} \quad (2.2)$$

(3) $$\int_0^a \frac{t^\rho (a-t)^\beta}{(a+bt)^{2+\rho+\beta}} P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right) P_v^{(\gamma,\delta)}\left(\frac{a-bt-2t}{a+bt}\right)dt$$

$$= \frac{\Gamma(\beta+u+1)}{a(b+1)^{\rho+1}\, u!} \sum_{r=0}^{v} \frac{\Gamma(1+r+\rho)\Gamma(1+r\text{-}\alpha+\rho)}{\Gamma(1+r\text{-}u\text{-}\alpha+\rho)\Gamma(2+r+u+\beta+\rho)} J_{\gamma,\delta}^{v,r} \quad (2.3)$$

Where

$$J_{\gamma,\delta}^{v,r} = \frac{(-1)^u \Gamma(\gamma+v+1)(-v)_r(\gamma+\delta+v+1)_r}{v!\,r!\,\Gamma(\gamma+r+1)}$$

(4) If a > 0, b ≠ –1, Re (ρ) > –1, Re(σ) > –1, then

$$\int_0^a \frac{t^\rho (a-t)^\sigma}{(a+bt)^{2+\rho+\sigma}} P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right)dt$$

$$= \frac{\Gamma(\alpha+u+1)}{a(b+1)^{\rho+1}\, u!\,\Gamma(\alpha+\beta+u+1)} \sum_{r=0}^{u} \frac{\Gamma(1+r+\rho)\Gamma(1+\sigma)}{\Gamma(2+r+\rho+\sigma)} I_{\alpha,\beta}^{u,r} \quad (2.4)$$

Where

$$I_{\alpha,\beta}^{u,r} = \frac{(-u)_r\, \Gamma(\alpha+\beta+u+r+1)}{r!\,\Gamma(\alpha+r+1)}$$

3. MAIN INTEGRALS

Theorem (3.1)

(i) If a>o, b ≠ -1, Re (α)>-1, d >o, n=0,1,2,.... and Re (σ) + n Re (c) > -1

(ii) Each of Ψ' & Ψ - functions occurring in (3.1) satisfy convergence conditions (1.4), (1.5) and (1.1) then

$$\int_o^a \frac{t^\alpha (a-t)^\sigma}{(a+bt)^{2+\alpha+\sigma}} P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right). \; {}_P\Psi'_Q\left[\frac{z(a-t)^c}{(a+bt)^c}\right] {}_p\Psi_q\left[\frac{x(a-t)^d}{(a+bt)^d}\right]dt$$

$$= \sum_{n=0}^{\infty} \frac{z^n \prod_{j=1}^{P} \Gamma(A_j+\gamma_j n)\Gamma(\alpha+u+1)}{n!\prod_{j=1}^{Q}\Gamma(B_j+\delta_j n)\; u!\; a(b+1)^{\alpha+1}}\,.$$

$${}_{p+2}\Psi_{q+2}\left[\begin{matrix}((a_p,\alpha_p)), & (1+\sigma+cn,d), & (1-\beta+\sigma+cn,d) & ; \\ ((b_q,\beta_q)), & (1\text{-}u\text{-}\beta+\sigma+cn,d), & (2+u+\alpha+\sigma+cn,d); & \end{matrix} x\right]$$

...(3.1)

Theorem (3.2)

If the conditions of theorem (3.1) with α and σ replaced by ρ & β resp. hold good then

$$\int_o^a \frac{t^\rho (a-t)^\beta}{(a+bt)^{2+\rho+\beta}} P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right). \; {}_P\Psi'_Q\left[\frac{z(b+1)^c\, t^c}{(a+bt)^c}\right] {}_p\Psi_q\left[\frac{x(b+1)^d\, t^d}{(a+bt)^d}\right] dt$$

$$= \frac{\Gamma(\beta+u+1)(-1)^u}{a(b+1)^{1+\rho}\, u!} \sum_{n=0}^{\infty} \frac{\prod_1^P \Gamma(A_j+\gamma_j n)\, z^n}{\prod_1^Q \Gamma(B_j+\delta_j n)\; n!}\,.$$

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$$\cdot\, {}_{p+2}\Psi_{q+2}\left[\begin{matrix}((a_p,\alpha_p)),(1+\rho+cn,d),(1-\alpha+\rho+cn,d) & ; \\ ((b_q,\beta_q)),(1-u-\alpha+\rho+cn,d),(2+u+\beta+\rho+cn,d); & \end{matrix} x\right] \quad ....(3.2)$$

Theorem (3.3)

If the conditions of theorem (3.1) with α and σ replaced by $\rho\ \&\ \beta$ resp. hold good then

$$\int_o^a \frac{t^\rho (a-t)^\beta}{(a+bt)^{2+\rho+\beta}} P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right) P_\upsilon^{(\gamma,\delta)}\left(\frac{a-bt-2t}{a+bt}\right).$$

$$\cdot\, {}_P\Psi'_Q\left[\frac{z(b+1)^c t^c}{(a+bt)^c}\right] {}_p\Psi_q\left[\frac{x(b+1)^d t^d}{(a+bt)^d}\right] dt$$

$$.= \frac{\Gamma(\beta+u+1)}{a(b+1)^{1+\rho}\ u!} \cdot \sum_{n=0}^{\infty} \frac{\prod_1^P \Gamma(A_j+\gamma_j n) z^n}{\prod_1^Q \Gamma(B_j+\delta_j n)\ n!} \sum_{r=0}^{\upsilon} J_{\gamma,\delta}^{\upsilon,r}.$$

$$\cdot\, {}_{p+2}\Psi_{q+2}\left[\begin{matrix}((a_p,\alpha_p)),(1+r+\rho+cn,d),(1+r-\alpha+\rho+cn,d) & ; \\ ((b_q,\beta_q)),(1+r-u-\alpha+\rho+cn,d),(2+r+u+\beta+\rho+cn,d); & \end{matrix} x\right] \quad ....(3.3)$$

Where $J_{\gamma,\delta}^{\upsilon,r}$ is defined in (2.3).

Theorem (3.4)

If

(i) a>o, b $\neq$ -1, k > 0, l > 0,

(ii) Re (ρ)+ n Re (c) > -1 and Re (σ) + n Re (d) > -1,

(iii) Each of Ψ' & Ψ - functions occurring in (3.4) satisfy (1.4) & (1.5) then

$$\int_o^a \frac{t^\rho (a-t)^\sigma}{(a+bt)^{2+\rho+\sigma}} {}_P\Psi'_Q\left[\frac{z(b+1)^c t^c (a-t)^d}{(a+bt)^{c+d}}\right].$$

$$\cdot\, {}_p\Psi_q\left[\frac{x(b+1)^k t^k (a-t)^l}{(a+bt)^{k+l}}\right] .P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right) dt$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^P \Gamma(A_j+\gamma_j n) z^n\, \Gamma(\alpha+u+1)}{\prod_{j=1}^Q \Gamma(B_j+\delta_j n)\ n!\, a(b+1)^{1+\rho}\ u!\,\Gamma(\alpha+\beta+u+1)} \cdot \sum_{r=0}^{u} I_{\alpha,\beta}^{u,r}.$$

$${}_{p+2}\Psi_{q+1}\left[\begin{matrix}((a_p,\alpha_p)),(1+r+\rho+cn,k),(1+\sigma+dn,l); \\ ((b_q,\beta_q)),(2+r+\rho+\sigma+cn+dn,k+l)\ ; \end{matrix} x\right] \quad ....(3.4)$$

Where $I_{\alpha,\beta}^{u,r}$ is defined in (2.4).

Proof of (3.1) to (3.4) :

Expressing the Ψ'- function in the L.H.S. of (3.1) in series form by (1.2) and the Ψ-function in contour integral form by (1.1), we get

$$\int_o^a \frac{t^\alpha (a-t)^\sigma}{(a+bt)^{2+\alpha+\sigma}} P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right).$$

$$.\sum_{n=0}^{\infty} \frac{\prod_{j=1}^P \Gamma(A_j+\gamma_j n) z^n (a-t)^{nc}}{\prod_{j=1}^Q \Gamma(B_j+\delta_j n)\ n!(a+bt)^{nc}}.$$

$$.\frac{1}{2\pi\iota}\int_D \frac{\prod_{j=1}^p \Gamma(a_j+\alpha_j s)\,\Gamma(-s)(-x)^s (a-t)^{sd}}{\prod_{j=1}^q \Gamma(b_j+\beta_j s)\,(a+bt)^{sd}} .ds.dt \quad ....(3.5)$$

Changing the order of t-integral and summation and again of the t-integral & contour integral and then evaluating the inner t-integral by (2.1), then on interpreting the contour integral into Ψ- function by (1.1), the RHS of (3.1) follows immediately. This completes the proof of (3.1).

The remaining theorems (3.2), (3.3) and (3.4) can also be proved in a similar way, as above, by using (2.2), (2.3) and (2.4) respectively instead of (2.1).

Our results (3.1), (3.2), (3.3) and (3.4) seem to be new.

4. SPECIAL CASES

If we replace ${}_p\Psi_q(z)$ by ${}_pF_q(z)$; ${}_P\Psi'_Q(z)$ by ${}_PF'_Q(z)$ and take **d=1** in (3.1), (3.2),(3.3), we get the cases (4.1), (4.2),(4.3) respectively. If we take **k=1, *l*=0** in (3.4), We get (4.4) & taking **k=0, *l*=1** in (3.4), We get the case (4.5).

Cor. (4.1)

$$\int_o^a \frac{t^\alpha (a-t)^\sigma}{(a+bt)^{2+\alpha+\sigma}} P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right).$$

$$\cdot\, {}_PF'_Q\left[\frac{z(a-t)^c}{(a+bt)^c}\right] {}_pF_q\left[\frac{x(a-t)}{(a+bt)}\right] dt$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{P}(A_j)_n\, z^n\Gamma(\alpha+u+1)\,\Gamma(1+\sigma+nc)\,\Gamma(1-\beta+\sigma+nc)}{\prod_{j=1}^{Q}(B_j)_n\; n!\,a(b+1)^{1+\alpha}\; u!\,\Gamma(2+u+\alpha+\sigma+nc)\Gamma(1-u-\beta+\sigma+nc)}$$

$$ {}_{p+2}F_{q+2}\left[\begin{matrix}a_1,a_2,....,a_p,\ 1+\sigma+cn,1-\beta+\sigma+cn & ; \\ b_1,b_2,....,b_q,2+u+\alpha+\sigma+cn,\ 1-u-\beta+\sigma+cn; \end{matrix} x\right]$$

..... (4.1)

Cor. (4.2)

$$\int_o^a \frac{t^{\rho}(a-t)^{\beta}}{(a+bt)^{2+\rho+\beta}}\,P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right).$$

$$\cdot\,{}_pF_Q^{'}\left[\frac{z(b+1)^c t^c}{(a+bt)^c}\right]\cdot{}_pF_q\left[\frac{x(b+1)t}{(a+bt)}\right]dt$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{P}(A_j)_n\, z^n\Gamma(\beta+u+1)(-1)^u\,\Gamma(1-\alpha+\rho+nc)\,\Gamma(1+\rho+cn)}{\prod_{j=1}^{Q}(B_j)_n\; n!\,a(b+1)^{1+\rho}\; u!\,\Gamma(2+\rho+cn+u+\beta)\Gamma(1-u-\alpha+\rho+nc)}.$$

$$ {}_{p+2}F_{q+2}\left[\begin{matrix}a_1,.....,a_p,\ 1+\rho+cn,1-\alpha+\rho+cn & ; \\ b_1,......,b_q,1-u-\alpha+\rho+cn,\ 2+u+\beta+\rho+cn; \end{matrix} x\right]$$

.....(4.2)

Cor.(4.3)

$$\int_o^a \frac{t^{\rho}(a-t)^{\beta}}{(a+bt)^{2+\rho+\beta}}\,P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right)P_v^{(\gamma,\delta)}\left(\frac{a-bt-2t}{a+bt}\right).$$

$$\cdot\,{}_pF_Q^{'}\left[\frac{z(b+1)^c t^c}{(a+bt)^c}\right]{}_pF_q\left[\frac{x(b+1)t}{(a+bt)}\right]dt$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{P}(A_j)_n\; z^n\;\Gamma(\beta+u+1)\;\Gamma(1+r+\rho+cn)}{\prod_{j=1}^{Q}(B_j)_n\; n!\,a(b+1)^{1+\rho}\; u!\;\Gamma(1+r+\rho+cn-u-\alpha)}.$$

$$.\frac{\Gamma(1+r-\alpha+\rho+cn)}{\Gamma(2+r+u+\beta+\rho+cn)}\sum_{r=0}^{\upsilon}J_{\gamma,\delta}^{\upsilon,r}.$$

$$\cdot\,{}_{p+2}F_{q+2}\left[\begin{matrix}a_1,...,a_p,\ 1+r+\rho+cn,1+r-\alpha+\rho+cn & ; \\ b_1,....,b_q,1+r+\rho+cn-u-\alpha,\ 2+r+u+\beta+\rho+cn; \end{matrix} x\right]$$

....(4.3)

$J_{\gamma,\delta}^{\upsilon,r}$ is defined in (2.3).

Cor. (4.4)

$$\int_o^a \frac{t^{\rho}(a-t)^{\sigma}}{(a+bt)^{2+\rho+\sigma}}\,P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right){}_pF_Q^{'}\left[\frac{z(b+1)^c t^c(a-t)^d}{(a+bt)^{c+d}}\right].$$

$$\cdot\,{}_pF_q\left[\frac{x(b+1)t}{(a+bt)}\right]dt$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{P}(A_j)_n\; z^n\;\Gamma(\alpha+u+1)\Gamma(1+\sigma+dn)}{\prod_{j=1}^{Q}(B_j)_n\; n!\,a(b+1)^{1+\rho}\;\; u!}.$$

$$.\sum_{r=0}^{u}\frac{(-u)_r\,(\alpha+\beta+u+1)_r\,\Gamma(1+r+\rho+cn)}{r!\,\Gamma(\alpha+r+1)\;\Gamma(2+r+\rho+\sigma+cn+dn)}.$$

$$\cdot\,{}_{p+1}F_{q+1}\left[\begin{matrix}a_1,...,a_p,1+r+\rho+cn & ; \\ b_1,...,b_q,2+r+\rho+\sigma+cn+dn; \end{matrix} x\right]$$

... (4.4)

Cor. (4.5)

$$\int_o^a \frac{t^{\rho}(a-t)^{\sigma}}{(a+bt)^{2+\rho+\sigma}}\,P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right){}_pF_Q^{'}\left[\frac{z(b+1)^c t^c(a-t)^d}{(a+bt)^{c+d}}\right].$$

$$\cdot\,{}_pF_q\left[\frac{x(a-t)}{(a+bt)}\right]dt$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{P}\Gamma(A_j)_n\; z^n\;\Gamma(\alpha+u+1)\Gamma(1+\sigma+dn)}{\prod_{j=1}^{Q}(B_j)_n\;\; n!\,a(b+1)^{1+\rho}\;\; u!}.$$

$$.\sum_{r=0}^{u}\frac{(-u)_r\,(\alpha+\beta+u+1)_r\,\Gamma(1+r+\rho+cn)}{r!\,\Gamma(\alpha+r+1)\;\Gamma(2+r+\rho+\sigma+cn+dn)}.$$

$$\cdot\,{}_{p+1}F_{q+1}\left[\begin{matrix}a_1,...,a_p,1+\sigma+dn & ; \\ b_1,...,b_q,2+r+\rho+\sigma+cn+dn; \end{matrix} x\right]$$

...(4.5)

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