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Implementation of Conservation principles for Runner conduit in Plastic Injection Mould Design

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Abstract

Conservation principles are used to represent all physical transformations occurring in the universe, accordingly are also adopted to design runner conduit for thermoplastic melt injection. Conservation principles for thermoplastic melt injection through runner conduit are implemented by considering cylindrical co-ordinates system relevant to its geometrical configuration for deriving governing equations. While the continuity equation ensures volumetric conservation of thermoplastic melt, the momentum equation represents equilibrium of forces on thermoplastic melt injection through runner conduit. During an injection moulding cycle heat and work done energy transformations are balanced by implementing first and second law of thermodynamics. Thermoplastic melt state change through the runner conduit for a particular cycle is appreciated by heat conduction equation. Traditionally inertia and entropy contribution is neglected to skip rigorousness, nevertheless they continue to prevail. Especially in very frequently used non-circular runner cross section conduits, their influence is highly significant. Hence the current endeavour attempts to computationally model, continuousness, equilibrium, energy balance and phase transformative runner design criteria by implementing conservation principles.

Keywords: Runner Design, Plastic Injection Mould, Conservation Principles

1. Introduction

Designing moulds necessitate fundamental insight into physical injection moulding behaviour of chosen melt relative to desired component characteristics as well as complementary classical principles of mechanics. Off mould design runner system is the focussed subject of investigation. Runner conduit is one of the integral and influential elements of injection mould design to process plastic parts [1]. Specific spatial region defining runner conduit is considered in isolation from everything else as a well-defined control surface or system boundary [2]. Non-newtonian thermoplastic melt is then injected through runner system conduit to contrive component impression space. Since runner conduit involves continuous melt injection through its inlet and exit boundary surfaces, it will be an open system often referred here afterwards as control volume [3]. Here the runner conduit is considered as a Euclidian flow field and thus thermoplastic melt properties are defined as spatiotemporal functions [4]. Hence here we consider an elemental volume of thermoplastic melt inside runner conduit and conservation laws are implemented by differential approach [5].

The condition of thermoplastic melt at any instant of time is typically referred as its state and expressed as a function of melt characteristics [6], which are basically a non-newtonian. Though it experiences the runner conduit for fraction of cycle, it directly influences the ideal feed system design. Consequent to injection action, exclusive melt state metrics and their extents are used to design effective runner parameters by implementing mass, momentum and energy principles [7]. Thus its control volume formulation is critically important to represent overall balance [8].

The mass conservation principle implements continuity of thermoplastic melt injection rate. The momentum conservation principle appreciates forces acting on thermoplastic melt over the entire runner conduit [9]. Similarly energy conservation principle apprehends energy change of thermoplastic melt throughout the process. Here two simultaneous modes of energy transformations are recognised, injection work and solidification heat transfer. The primary objective of mould designer is to realize maximum injection power (rate of injection work transfer) from runner conduit design criteria. The energy conservation principle articulates energy is conserved during a

Nomenclature

m	Mass	Kg
V	Volume	m^3
P	Pressure	Kgf / m^2
T	Temperature	K
k	Thermal conductivity	W/m
A	Cross-section area	m^2
$\vec{U}$	Linear velocity	m/s
$\vec{U}_r$	Velocity in radial direction	m/s
$\vec{U}_\theta$	Velocity in tangential direction	m/s
$\vec{U}_\xi$	Velocity in arbitrary direction	m/s
a	Acceleration	m/s^2
$\vec{M}$	Linear momentum	$Kg-m/s$
$\vec{H}$	Angular momentum	$Kg-m/s$
I	Moment of inertia	$Kg-m^2$
e	Specific total energy	KJ/Kg
$\hat{u}$	Specific internal energy	KJ/Kg
$\sum T$	Resultant torque	$N-m$
$\sum M$	Resultant moment	$N-m$
$\sum F$	Resultant force	N/m^2
F_r	Force acting in radial direction	N/m^2
F_θ	Force acting in tangential direction	N/m^2
F_ξ	Force acting in arbitrary direction	N/m^2
$\dot{Q}$	Rate of heat transfer	KW
q	Rate of heat transfer per unit mass	KW
$\dot{W}_v$	Rate of work done by viscous forces	KW
$\dot{W}_p$	Rate of work done by pressure forces	KW
dS	Entropy change	KJ/Kg
C_v	Specific heat at constant volume	KJ/KgK
C_p	Specific heat at constant pressure	KJ/KgK
$\vec{n}$	Unit normal vector	
$\vec{r}$	Position vector	

Greek symbols

ρ	Density	Kg/m^3
υ	Specific volume	m^3/Kg
$\vec{\omega}$	Angular Velocity	m/s
α	Angular acceleration	m/s^2
σ	Surface force	N/m^2
τ	Shear stress	N/m^2
ϕ	Viscous dissipation function	

energy exchange is reversible or not. However in most circumstances, melt injection through runner conduit for a particular cycle is an irreversible process. So an additional quantity called entropy is required to balance energy transformations. To gain better understanding of realizable energy transformations the concept of entropy is implemented [10].

Hence fundamental conservation of injection mechanics from mass, momentum and energy perspective is essential to realize specific relationships among chosen thermoplastic, desired component and available machine. All these relationships along with processing objectives facilitate designing an idealistic runner system.

2. Conservation laws

Plastic injection phenomena through runner conduit can be explained by using conservation laws. The conservation equations fundamentally seek to tell us how three important field variables are distributed in space and time. According to conservation laws any measurable property of the system does not change as the system evolves/ undergoes a change. Thus during any process the quantities of the system will be conserved. This concept of conservation can be applied on thermoplastic melt diffusing through runner conduit of injection mould to determine the properties of thermoplastic melt throughout runner conduit.

Let us begin by considering a three dimensional runner conduit of a thermoplastic injection mould in cylindrical co-ordinates system having a representative melt element between ' r ' and ' $r+dr$ ' as shown in figure 1, subtending an angle ' $d\theta$ ' at the centre. Since there are no sources or sinks in the conduit, we can conveniently postulate that the mass of representative melt element does not change in position and time i.e., mass is conserved spatiotemporally. If ' $\vec{U}$ ' is melt velocity then ' $\vec{U}_r$ ' is velocity in radial direction, ' $\vec{U}_\theta$ ' is velocity in tangential direction and ' $\vec{U}_\xi$ ' in arbitrary injection direction respectively. Here ' ξ ' is linear arbitrary path function in the YZ plane on machine.

2.1 Conservation of Mass

The continuity equation comes from the basic principle that matter can neither be created nor be destroyed. This principle is then applied to a small volume of thermoplastic melt under injection resulting in equation representing continuity.

Lemma

"Melt state change at any instant within runner conduit is always equal to influx and efflux melt state change rates across the runner conduit inlet and outlet respectively"

$$\int_{rcv} \frac{\partial \rho}{\partial t} dV + \sum_{out}\left(\rho A\vec{U}\right) - \sum_{in}\left(\rho A\vec{U}\right) = 0 \qquad (1)$$

Proof

Consider injection in arbitrary direction,
Mass of melt entering/unit time

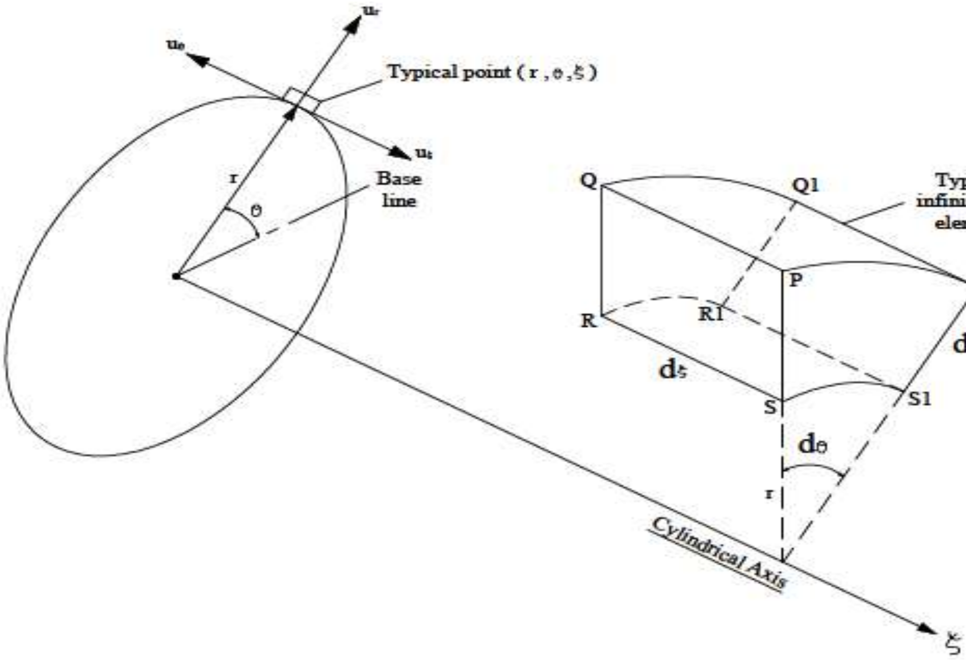

Figure 1: Three dimensional runner conduit in cylindrical co-ordinates [5]

= density × velocity in ξ-direction × inlet cross section area

$$= \rho \overrightarrow{U_\xi}\left(r d\theta dr\right)$$

Similarly mass of melt leaving/unit time

$$= \rho\left(\overrightarrow{U_\xi} + \frac{\partial \overrightarrow{U_\xi}}{\partial \xi} d\xi\right)\left(r d\theta dr\right)$$

Thus, total change of mass in ξ-direction/unit time

= mass entering - mass leaving

$$= -\rho\left(\frac{\partial \overrightarrow{U_\xi}}{\partial \xi}\right)\left(r dr d\theta d\xi\right)$$

$$= -\rho\left(\frac{\partial \overrightarrow{U_\xi}}{\partial \xi}\right) dV$$

Correspondingly,

Total change of mass in θ-direction/unit time

$$= -\rho \frac{1}{r}\left(\frac{\partial \overrightarrow{U_\theta}}{\partial \theta}\right) dV$$

Total change of mass in r-direction/unit time

$$= -\rho\left(\frac{\overrightarrow{U_r}}{r} + \frac{\partial \overrightarrow{U_r}}{\partial r}\right) dV$$

Therefore net change of melt mass in element/unit time is

$$= -\rho\left(\frac{\overrightarrow{U_r}}{r} + \frac{\partial \overrightarrow{U_r}}{\partial r}\right) dV - \rho \frac{1}{r}\left(\frac{\partial \overrightarrow{U_\theta}}{\partial \theta}\right) dV - \rho\left(\frac{\partial \overrightarrow{U_\xi}}{\partial \xi}\right) dV$$

$$= -\rho\left(\frac{\overrightarrow{U_r}}{r} + \frac{\partial \overrightarrow{U_r}}{\partial r} + \frac{1}{r}\frac{\partial \overrightarrow{U_\theta}}{\partial \theta} + \frac{\partial \overrightarrow{U_\xi}}{\partial \xi}\right) dV$$

$$= -\rho\left(\frac{\partial (r\overrightarrow{U_r})}{\partial r} + \frac{\partial \overrightarrow{U_\theta}}{\partial \theta} + r\frac{\partial \overrightarrow{U_\xi}}{\partial \xi}\right) dV \qquad (2)$$

Instantaneous mass of melt within the runner conduit element = Density × Volume

$$= \rho dV$$

So rate of change of melt in element $= \frac{\partial (\rho dV)}{\partial t}$

Since runner element volume 'dV' is stationary and invariable with respect to time or independent of time change it becomes $= dv\frac{\partial \rho}{\partial t}$

Hence by applying conservation law we get,

$$dV\frac{\partial \rho}{\partial t} + \rho\left(\frac{\partial (r\overrightarrow{U_r})}{\partial r} + \frac{\partial \overrightarrow{U_\theta}}{\partial \theta} + r\frac{\partial \overrightarrow{U_\xi}}{\partial \xi}\right) dV = 0 \qquad (3)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho r\overrightarrow{U_r})}{\partial r} + \frac{\partial (\rho \overrightarrow{U_\theta})}{\partial \theta} + r\frac{\partial (\rho \overrightarrow{U_\xi})}{\partial \xi} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\rho \overrightarrow{U_r}}{r} + \frac{\partial (\rho \overrightarrow{U_r})}{\partial r} + \frac{1}{r}\frac{\partial (\rho \overrightarrow{U_\theta})}{\partial \theta} + \frac{\partial (\rho \overrightarrow{U_\xi})}{\partial \xi} = 0 \qquad (4)$$

Now by substituting vector gradient operator,

$$\nabla.\vec{U} = \frac{\overrightarrow{U_r}}{r} + \frac{\partial \overrightarrow{U_r}}{\partial r} + \frac{1}{r}\frac{\partial \overrightarrow{U_\theta}}{\partial \theta} + \frac{\partial \overrightarrow{U_\xi}}{\partial \xi} \qquad (5)$$

Hence equation reduces (4) to

$$\frac{\partial \rho}{\partial t} + \nabla.\left(\rho \vec{U}\right) = 0 \qquad (6)$$

The above equation is called continuity equation.

The above equation describes that the thermoplastic melt injection is compressible and melt is conserved through the runner conduit [11], during complete injection moulding cycle.

2.2 Conservation of Momentum

We hereby develop governing equations for runner conduit design considering dynamic forces consequent to thermoplastic melt injection. According to Newtonian mechanics thermoplastic melt injection through runner conduit should obey Newton's Second Law of Motion for Conservation of Momentum. The implied momentum could be linear or angular and its corresponding actions are force $\left(\sum F\right)$ and moment $\left(\sum M\right)$ respectively.

2.2.1 Conservation of Linear Momentum

Lemma

"At any moment resultant force acting on thermoplastic melt is proportional to its instantaneous injection momentum change rates in that direction and momentum change rates across the runner conduit inlet and outlet"

$$\text{i.e.,} \sum F = \frac{\partial\left(m\vec{U}\right)}{\partial t} + \sum_{out}\left(m\vec{U}\right) - \sum_{in}\left(m\vec{U}\right) \qquad (7)$$

We have,

$$\frac{\partial\left(m\vec{U}\right)}{\partial t} = \frac{\partial}{\partial t}\int_{rcv} \rho \vec{U} dV \quad \text{and}$$

$$\sum_{out}\left(m\vec{U}\right) - \sum_{in}\left(m\vec{U}\right) = \int_{in}^{out} \rho \vec{U}\left(\vec{U}\right) dA$$

$$\therefore \sum F = \frac{\partial}{\partial t}\int_{rcv} \rho \vec{U} dV + \int_{in}^{out} \rho \vec{U}\left(\vec{U}\right) dA \qquad (8)$$

The term $\rho\vec{U}\left(\vec{U}\right)dA$ represents momentum change rates through mass transfer across runner conduit inlet and outlet.

Proof

The element is so small that volume integral simply reduces to a derivative term

i.e., $$\frac{\partial}{\partial t}\int_{rcv}\rho\vec{U}dV = \frac{\partial}{\partial t}\rho\vec{U}dV \quad (9)$$

Consider rate of momentum change in r-direction

Rate of momentum change at inlet of runner conduit

$$= \rho\vec{U_r}\vec{U}\left(rd\theta d\xi\right)$$

Rate of momentum change at exit of runner conduit

$$= \left(\rho\vec{U_r}\vec{U} + \frac{\partial\left(\rho\vec{U_r}\vec{U}\right)}{\partial r}dr\right)\left(r+dr\right)d\theta d\xi$$

Rate of momentum change in r-direction

$$= \left(\rho\vec{U_r}\vec{U} + \frac{\partial\left(\rho\vec{U_r}\vec{U}\right)}{\partial r}dr\right)\hat{n}.\left(r+dr\right)d\theta d\xi - \rho\vec{U_r}\vec{U}\left(rd\theta d\xi\right)$$

$$= \left(\frac{\rho\vec{U_r}\vec{U}}{r} + \frac{\partial\left(\rho\vec{U_r}\vec{U}\right)}{\partial r}\right)dV \qquad = \left(\frac{\partial\left(r.\rho\vec{U_r}\vec{U}\right)}{\partial r}\right)dV$$

Similarly in 'θ' and 'ξ' -directions

Rate of momentum change in θ - direction

$$= \left(\frac{1}{r}\frac{\partial\left(\rho\vec{U_\theta}\vec{U}\right)}{\partial\theta}\right)dV$$

Rate of momentum change in ξ - direction

$$= \left(\frac{\partial\left(\rho\vec{U_\xi}\vec{U}\right)}{\partial\xi}\right)dV$$

Therefore net momentum change rate through mass transfer

$$= \left(\frac{\partial\left(r.\rho\vec{U_r}\vec{U}\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\rho\vec{U_\theta}\vec{U}\right)}{\partial\theta} + \frac{\partial\left(\rho\vec{U_\xi}\vec{U}\right)}{\partial\xi}\right)dV \quad (10)$$

Forces acting on thermoplastic melt

The forces acting on melt mass are body forces and surface forces. The body forces are mainly due to gravity and their contribution in momentum conservation would be very less because plastic melt within the runner conduit is a small fraction of shot volume (approximately 50 times), melt injection forces are predominantly larger (more than 100 times) than inertial forces and melt density itself being very less. Therefore body forces can be neglected for runner conduit design.

Consider stresses in r-direction,

Resultant force acting at runner conduit entrance (boundary)/unit time $= \sigma_{rr}n.\left(rd\theta d\xi\right)$

Since mass is entering into the runner control volume, unit normal vector to exit boundary $n = -1$

Therefore resultant force acting at runner conduit entrance/unit time $= -\sigma_{rr}\left(rd\theta d\xi\right)$

Resultant force acting at runner conduit exit/unit time

$$= \left(\sigma_{rr} + \frac{\partial\sigma_{rr}}{\partial r}\partial dr\right)\hat{n}.\left(r+dr\right)d\theta d\xi$$

Unit normal vector to entrance boundary, n = 1

Thus Resultant force

$$= \left(\sigma_{rr} + \frac{\partial\sigma_{rr}}{\partial r}dr\right)\left(r+dr\right)d\theta d\xi - \sigma_{rr}\left(rd\theta d\xi\right)$$

$$= \left(\frac{\sigma_{rr}}{r} + \frac{\partial\sigma_{rr}}{\partial r}\right)rdrd\theta d\xi$$

$$= \left(\frac{\sigma_{rr}}{r} + \frac{\partial\sigma_{rr}}{\partial r}\right)dV$$

So that the net resultant force in r-direction is given by

$$dF_{r,resultant} = \left(\frac{\partial(\sigma_{rr})}{\partial r} + \frac{1}{r}\frac{\partial(\sigma_{\theta r})}{\partial\theta} + \frac{\partial(\sigma_{\xi r})}{\partial\xi} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}\right)dV$$

Similarly in other directions,

$$dF_{\theta,resultant} = \left(\frac{\partial\left(\sigma_{r\theta}\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\sigma_{\theta\theta}\right)}{\partial\theta} + \frac{\partial\left(\sigma_{\xi\theta}\right)}{\partial\xi} + \frac{2\sigma_{r\theta}}{r}\right)dV$$

$$dF_{\xi,resultant} = \left(\frac{\partial\left(\sigma_{r\xi}\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\sigma_{\theta\xi}\right)}{\partial\theta} + \frac{\partial\left(\sigma_{\xi\xi}\right)}{\partial\xi} + \frac{\sigma_{r\xi}}{r}\right)dV$$

Net resultant surface force due to injection/melt convection exerts stress on runner conduit boundary.

$$\sigma_{ij} = \begin{vmatrix} -P+\tau_{rr} & \tau_{\theta r} & \tau_{\xi r} \\ \tau_{r\theta} & -P+\tau_{\theta\theta} & \tau_{\xi\theta} \\ \tau_{r\xi} & \tau_{\theta\xi} & -P+\tau_{\xi\xi} \end{vmatrix}$$

Splitting this into pressure and viscous stresses, we can rewrite it as

$$\frac{dF_r}{dV} = -\frac{\partial P}{\partial r} + \left(\frac{1}{r}\frac{\partial(r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial(\tau_{\theta r})}{\partial\theta} + \frac{\partial(\tau_{\xi r})}{\partial\xi} - \frac{\tau_{\theta\theta}}{r}\right)$$

Similarly in other directions,

$$\frac{dF_\theta}{dV} = -\frac{1}{r}\frac{\partial P}{\partial\theta} + \left(\frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial(\tau_{\theta\theta})}{\partial\theta} + \frac{\partial(\tau_{\xi\theta})}{\partial\xi} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r}\right)$$

$$\frac{dF_\xi}{dV} = -\frac{\partial P}{\partial\xi} + \left(\frac{1}{r}\frac{\partial(r\tau_{r\xi})}{\partial r} + \frac{1}{r}\frac{\partial(\tau_{\theta\xi})}{\partial\theta} + \frac{\partial(\tau_{\xi\xi})}{\partial\xi}\right)$$

Hence net resultant forces is given by

$$\left(\frac{dF}{dV}\right)_{resultant} = -\left(\frac{\partial P}{\partial r}+\frac{1}{r}\frac{\partial P}{\partial \theta}+\frac{\partial P}{\partial \xi}\right) +\left(\frac{1}{r}\frac{\partial(r\tau_{rr})}{\partial r}+\frac{1}{r}\frac{\partial(\tau_{\theta r})}{\partial \theta}+\frac{\partial(\tau_{\xi r})}{\partial \xi}-\frac{\tau_{\theta\theta}}{r}\right) +\left(\frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r}+\frac{1}{r}\frac{\partial(\tau_{\theta\theta})}{\partial \theta}+\frac{\partial(\tau_{\xi\theta})}{\partial \xi}+\frac{\tau_{\theta r}-\tau_{r\theta}}{r}\right) +\left(\frac{1}{r}\frac{\partial(r\tau_{r\xi})}{\partial r}+\frac{1}{r}\frac{\partial(\tau_{\theta\xi})}{\partial \theta}+\frac{\partial(\tau_{\xi\xi})}{\partial \xi}\right) \quad (11)$$

That is,

$$\left(\frac{dF}{dV}\right)_{resultant} = -\nabla P + \left(\frac{dF}{dV}\right)_{viscous} \quad (12)$$

$$\left(\frac{dF}{dV}\right)_{viscous} = \nabla.\overleftrightarrow{\tau}$$

$$\therefore \left(\frac{dF}{dV}\right)_{resultant} = -\nabla P + \nabla.\overleftrightarrow{\tau} \quad (13)$$

The resultant force is thus the sum of the pressure gradient vector and the divergence of the viscous stress tensor.

According to conservation law,

Substituting (9), (10) & (13) in equation (8) we get

$$-\nabla P + \nabla.\overleftrightarrow{\tau} = \left(\frac{\partial(\rho\vec{U})}{\partial t}+\frac{\partial(r.\rho\vec{U_r}\vec{U})}{\partial r}+\frac{1}{r}\frac{\partial(\rho\vec{U_\theta}\vec{U})}{\partial \theta}+\frac{\partial(\rho\vec{U_\xi}\vec{U})}{\partial \xi}\right) \quad (14)$$

Equation on the right side can be split into,

$$\left(\frac{\partial(\rho\vec{U})}{\partial t}+\frac{\partial(r.\rho\vec{U_r}\vec{U})}{\partial r}+\frac{1}{r}\frac{\partial(\rho\vec{U_\theta}\vec{U})}{\partial \theta}+\frac{\partial(\rho\vec{U_\xi}\vec{U})}{\partial \xi}\right)$$

$$=\vec{U}\left(\frac{\partial\rho}{\partial t}+\nabla.(\rho\vec{U})\right)+\rho\frac{d\vec{U}}{dt}$$

We know that, from continuity equation number (6)

$$\frac{\partial\rho}{\partial t}+\nabla.(\rho\vec{U})=0$$

Hence,

$$\left(\frac{\partial(\rho\vec{U})}{\partial t}+\frac{\partial(r.\rho\vec{U_r}\vec{U})}{\partial r}+\frac{1}{r}\frac{\partial(\rho\vec{U_\theta}\vec{U})}{\partial \theta}+\frac{\partial(\rho\vec{U_\xi}\vec{U})}{\partial \xi}\right)=\rho\frac{d\vec{U}}{dt}$$

$$\text{i.e.,}\left(\frac{dF}{dV}\right)_{resultant} = \rho\frac{d\vec{U}}{dt} \quad (15)$$

Hence equation becomes,

$$-\nabla P + \nabla.\overleftrightarrow{\tau} = \rho\frac{d\vec{U}}{dt} \quad (16)$$

The above equation is also called Cauchy equation

This can be expanded as follows

Momentum in r-direction

$$-\frac{\partial P}{\partial r}+\left(\frac{1}{r}\frac{\partial(r\tau_{rr})}{\partial r}+\frac{1}{r}\frac{\partial(\tau_{\theta r})}{\partial \theta}+\frac{\partial(\tau_{\xi r})}{\partial \xi}-\frac{\tau_{\theta\theta}}{r}\right)$$

$$=\rho\left(\frac{\partial\vec{U_r}}{\partial t}+\vec{U_r}\frac{\partial\vec{U_r}}{\partial r}+\frac{\vec{U_\theta}}{r}\frac{\partial\vec{U_r}}{\partial\theta}+\vec{U_\xi}\frac{\partial\vec{U_r}}{\partial\xi}-\frac{U_\theta^2}{r}\right)$$

Momentum in θ-direction

$$-\frac{1}{r}\frac{\partial P}{\partial\theta}+\left(\frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r}+\frac{1}{r}\frac{\partial(\tau_{\theta\theta})}{\partial \theta}+\frac{\partial(\tau_{\xi\theta})}{\partial \xi}+\frac{\tau_{\theta r}-\tau_{r\theta}}{r}\right)$$

$$=\rho\left(\frac{\partial(\vec{U_\theta})}{\partial t}+\vec{U_r}\frac{\partial\vec{U_\theta}}{\partial r}+\frac{\vec{U_\theta}}{r}\frac{\partial\vec{U_\theta}}{\partial\theta}+\vec{U_\xi}\frac{\partial\vec{U_\theta}}{\partial\xi}+\frac{\vec{U_r}\vec{U_\theta}}{r}\right)$$

Momentum in ξ-direction

$$-\frac{\partial P}{\partial\xi}+\left(\frac{1}{r}\frac{\partial(r\tau_{r\xi})}{\partial r}+\frac{1}{r}\frac{\partial(\tau_{\theta\xi})}{\partial \theta}+\frac{\partial(\tau_{\xi\xi})}{\partial \xi}\right)$$

$$=\rho\left(\frac{\partial\vec{U_\xi}}{\partial t}+\vec{U_r}\frac{\partial\vec{U_\xi}}{\partial r}+\frac{\vec{U_\theta}}{r}\frac{\partial\vec{U_\xi}}{\partial\theta}+\vec{U_\xi}\frac{\partial\vec{U_\xi}}{\partial\xi}\right)$$

Since the contribution of body forces is negligible, the rate of change of momentum of thermoplastic melt inside runner conduit is completely depended on surface forces which is a combination of pressure and viscous forces

2.2.2 Conservation of Angular Momentum

Lemma

"Resultant torque acting on thermoplastic melt at any instant is equal to instantaneous angular momentum change within the conduit and angular momentum change rates at runner conduit inlet & exit"

$$\text{i.e.,} \sum T = \frac{\partial\vec{H}}{\partial t}+\sum_{out}\vec{H}-\sum_{in}\vec{H} \quad (17)$$

Where, $\vec{H}$ = Angular momentum.

Also we have

$$\frac{\partial\vec{H}}{\partial t}=\frac{\partial}{\partial t}\int_{rcv}(\vec{r}\times\vec{U})\rho dV$$

$$\sum_{out}\vec{H}-\sum_{in}\vec{H}=\int_{in}^{out}\vec{r}\times\rho\vec{U}(\vec{U})dS$$

The vector cross product $(\vec{r}\times\vec{U})$ represents angular momentum per unit mass. Where '$\vec{r}$' is position vector from origin or fixed central axis and $\vec{U}$ is linear velocity. Thus we can designate $(\vec{r}\times\vec{U})\rho dV$ as angular momentum acting on representative melt element. So the right hand side of equation (17) becomes

$$\frac{\partial \vec{H}}{\partial t}+\sum_{out}\vec{H}-\sum_{in}\vec{H} = \frac{\partial}{\partial t}\int_{rcv}\left(\vec{r}\times\vec{U}\right)\rho dV+\int_{in}^{out}\vec{r}\times\rho\vec{U}\left(\vec{U}\right)dS \quad (18)$$

Since body torque is negligible as stated in section 2.2.1 only surface torque acting on runner conduit accounts to net moment. Since contribution of pressure forces on angular momentum is negligible, the surface torque will mainly due to viscous torques. So left hand side of equation (17) becomes,

$$\sum T=\int_{rcs}\left(\vec{r}\times\vec{\tau}\right)dS \quad (19)$$

Substituting equation (18) and (19) in (17), we get

$$\therefore \int_{rcs}\left(\vec{r}\times\vec{\tau}\right)dS=\frac{\partial}{\partial t}\int_{rcv}\left(\vec{r}\times\vec{U}\right)\rho dV+\int_{in}^{out}\vec{r}\times\rho\vec{U}\left(\vec{U}\right)dS \quad (20)$$

For surface integral,

$$\vec{r}\times\vec{\tau}=-\vec{\tau}\times\vec{r}=-\left(\hat{n}.\overleftrightarrow{\tau}\right)\times\vec{r}=-\hat{n}.\left(\overleftrightarrow{\tau}\times\vec{r}\right)$$

By using Gauss Divergence Theorem, the left hand side of equation is converted from surface integral to volume integral

$$\int_{rcs}\left(\vec{r}\times\vec{\tau}\right)dS=-\int_{rcs}\hat{n}.\left(\overleftrightarrow{\tau}\times\vec{r}\right)dS=-\int_{rcv}\nabla.\left(\overleftrightarrow{\tau}\times\vec{r}\right)dV$$

Similarly, $\int_{in}^{out}\vec{r}\times\rho\vec{U}\left(\vec{U}\right)dS$

becomes $\int_{rcv}\nabla.\left(\vec{r}\times\rho\vec{U}\left(\vec{U}\right)dV\right)$

Thus equation (20) becomes,

$$-\int_{rcv}\nabla.\left(\overleftrightarrow{\tau}\times\vec{r}\right)dV=\frac{\partial}{\partial t}\int_{rcv}\left(\vec{r}\times\vec{U}\right)\rho dV+\int_{rcv}\nabla.\left(\vec{r}\times\rho\vec{U}\left(\vec{U}\right)dV\right)$$

$$-\nabla.\left(\overleftrightarrow{\tau}\times\vec{r}\right)=\frac{\partial}{\partial t}\rho\left(\vec{r}\times\vec{U}\right)+\nabla.\left(\vec{r}\times\rho\vec{U}\left(\vec{U}\right)\right) \quad (21)$$

But we have

$$\frac{\partial\rho\left(\vec{r}\times\vec{U}\right)}{\partial t}+\nabla.\left(\vec{r}\times\rho\vec{U}\left(\vec{U}\right)\right) = \rho\frac{d\left(\vec{r}\times\vec{U}\right)}{dt}+\left(\vec{r}\times\vec{U}\right)\left(\frac{\partial\rho}{\partial t}+\nabla.\left(\rho\vec{U}\right)\right)$$

Hence equation (21) becomes,

$$\rho\frac{d\left(\vec{r}\times\vec{U}\right)}{dt}+\left(\vec{r}\times\vec{U}\right)\left(\frac{\partial\rho}{\partial t}+\nabla.\left(\rho\vec{U}\right)\right)=-\nabla.\left(\overleftrightarrow{\tau}\times\vec{r}\right) \quad (22)$$

From equation (6) we have

$$\left(\frac{\partial\rho}{\partial t}+\nabla.\left(\rho\vec{U}\right)\right)=0$$

Thus above equation (22) reduces to

$$\rho\frac{d\left(\vec{r}\times\vec{U}\right)}{dt}=-\nabla.\left(\overleftrightarrow{\tau}\times\vec{r}\right) \quad (23)$$

This is called angular momentum equation

The above equation describes that rate of change of angular momentum of the thermoplastic melt inside the runner conduit is proportional to viscous forces which causes the angular motion. Angular momentum being negligible in circular or axis-symmetric runner conduits has a significant influence particularly on parametric distribution in non-circular cross section runner conduits.

2.3 Conservation of Energy

Here we seek to adopt Joule's energy principle to mathematically express conservation on the basis of First of law of thermodynamics by explicitly quantifying various energy entities and balancing them in accordance with the conservation notion. So in runner design we recognise melt heat, injection momentum work and polymer's internal energy as different forms of energy. In physical mould design sense, heat is thought of an energy exchange by melt in runner conduit to the surrounding in-deformable mould runner insert.

2.3.1 First Law of Thermodynamics

Lemma

"The algebraic sum of thermoplastic melt heat transfer rates across runner conduit inlet and exit plus rate of injection force acting on it throughout runner conduit is proportional to rate of internal energy change within runner conduit and rate of internal energy changes across the runner conduit inlet and exit"

$$\dot{Q}+\dot{W}_v+\dot{W}_p=\frac{\partial}{\partial t}\left(\int_{rcv}e\rho dV\right)+\int_{rcs}e\rho\left(\vec{U}.\vec{n}\right)dA \quad (24)$$

Here 'e' is total energy and it is sum of internal and kinetic energy.

i.e., $e = \hat{u} + \frac{\vec{U}^2}{2}$

The product $\rho\left(\vec{U}.\vec{n}\right)dA$ represents melt change rate at runner conduit inlet and exit

Here rate of work done by pressure forces $\left(\dot{W}_p\right)$

$$\left(\dot{W}_p\right) = -\int P\left(\vec{U}.\vec{n}\right)dA = -\int \frac{P}{\rho}\rho\left(\vec{U}.\vec{n}\right)dA \qquad (25)$$

Hence the equation (24) becomes

$$\dot{Q} + \dot{W}_v = \frac{\partial}{\partial t}\left(\int e\rho dV\right) + \int\left(e + \frac{P}{\rho}\right)\rho\left(\vec{U}.\vec{n}\right)dA \qquad (26)$$

This is very convenient form of energy equation since pressure work is now combined with energy of thermoplastic melt leaving at runner conduit outlet; we no longer have to deal with pressure work.

$$\dot{Q} + \dot{W}_v = \frac{\partial}{\partial t}\left(\int e\rho dV\right) + \sum_{out} m\left(\frac{P}{\rho} + e\right) - \sum_{in} m\left(\frac{P}{\rho} + e\right) \quad (27)$$

Proof

Consider rate of energy leaving at runner conduit exit by mass transfer in r-direction,

Let $\lambda = \left(\frac{p}{\rho} + e\right)$

Rate of energy change at runner conduit inlet

$$= \rho\vec{U}_r\lambda\left(rd\theta d\xi\right)$$

Rate of energy change at runner conduit outlet

$$= \left(\rho\vec{U}_r\lambda + \frac{\partial\left(\rho\vec{U}_r\lambda\right)}{\partial r}dr\right)\left(r + dr\right)d\theta d\xi$$

Therefore rate of energy change in r-direction

$$= \left(\rho\vec{U}_r\lambda + \frac{\partial\left(\rho\vec{U}_r\lambda\right)}{\partial r}dr\right)\left(r + dr\right)d\theta d\xi - \rho\vec{U}_r\lambda\left(rd\theta d\xi\right)$$

$$= \left(\frac{\rho\vec{U}_r\lambda}{r} + \frac{\partial\left(\rho\vec{U}_r\lambda\right)}{\partial r}\right)dV = \left(\frac{\partial\left(r.\rho\vec{U}_r\lambda\right)}{\partial r}\right)dV$$

Similarly in "θ" and "ξ" directions

Rate of energy change in "θ" direction

$$= \left(\frac{1}{r}\frac{\partial\left(\rho\vec{U}_\theta\lambda\right)}{\partial\theta}\right)dV$$

Rate of energy change in ξ - direction

$$= \left(\frac{\partial\left(\rho\vec{U}_\xi\lambda\right)}{\partial\xi}\right)dV$$

Therefore net rate of energy change

$$= \left(\frac{\partial\left(r.\rho\vec{U}_r\lambda\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\rho\vec{U}_\theta\lambda\right)}{\partial\theta} + \frac{\partial\left(\rho\vec{U}_\xi\lambda\right)}{\partial\xi}\right)dV \qquad (28)$$

Net rate of heat change of thermoplastic melt in runner conduit

To evaluate net heat change $\left(\dot{Q}\right)$, we neglect radiation and consider only heat conduction through the thermoplastic melt.

According to Fourier law of heat conduction,

"Rate of heat change per unit area is proportional to gradience of temperature"

$$q = -k\nabla T \qquad (29)$$

Where k= thermal conductivity

Consider rate of heat change through radial direction

Rate of heat change at conduit inlet

$$= q_r\left(rd\theta d\xi\right)$$

Rate of heat change at conduit outlet

$$= \left(q_r + \frac{\partial q_r}{\partial r}dr\right)\left(r + \partial dr\right)d\theta d\xi$$

By subtracting inlet term with outlet term, we obtain heat change in that direction

Hence rate of heat change in radial direction

$$= q_r\left(rd\theta d\xi\right) - \left(q_r + \frac{\partial q_r}{\partial r}dr\right)\left(r + dr\right)d\theta d\xi$$

$$= -q_r drd\theta d\xi - \left(\frac{\partial q_r}{\partial r}\right)rdrd\theta d\xi$$

$$= -\left(\frac{\partial\left(rq_r\right)}{\partial r}\right)dV$$

Similarly in other directions,

Rate of heat change in tangential direction

$$= -\left(\frac{1}{r}\frac{\partial q_\theta}{\partial\theta}\right)dV$$

Rate of heat change in longitudinal direction

$$= -\left(\frac{\partial q_\xi}{\partial\xi}\right)dV$$

Therefore net rate of heat change on thermoplastic melt

$$= -\left(\frac{\partial\left(rq_r\right)}{\partial r} + \frac{1}{r}\frac{\partial q_\theta}{\partial\theta} + \frac{\partial q_\xi}{\partial\xi}\right)dV$$

$$\dot{Q} = -\nabla.qdV$$

From above, rate of heat change is proportional to runner conduit volume. Thus Introducing Fourier law of heat conduction, we have net rate of heat conducted

$$\left(\dot{Q}\right) = \nabla.\left(k\nabla T\right)dV \qquad (30)$$

Rate of work done on thermoplastic melt due to viscous forces

Net rate of work done by viscous forces on thermoplastic melt in r-direction is given by

$$\left(\frac{\partial\left(r\tau_{rr}\overrightarrow{U_r}\right)}{\partial r} + \frac{1}{r}\frac{\partial(\tau_{\theta r}\overrightarrow{U_r})}{\partial\theta} + \frac{\partial\left(\tau_{\xi r}\overrightarrow{U_r}\right)}{\partial\xi}\right) drd\theta d\xi$$

Similarly in 'θ' and 'ξ'- directions

$$\left(\frac{\partial\left(r\tau_{r\theta}\overrightarrow{U_\theta}\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\tau_{\theta\theta}\overrightarrow{U_\theta}\right)}{\partial\theta} + \frac{\partial\left(\tau_{\xi\theta}\overrightarrow{U_\theta}\right)}{\partial\xi}\right) rdrd\theta d\xi$$

$$\left(\frac{\partial\left(r\tau_{r\xi}\overrightarrow{U_\xi}\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\tau_{\theta\xi}\overrightarrow{U_\xi}\right)}{\partial\theta} + \frac{\partial\left(\tau_{\xi\xi}\overrightarrow{U_\xi}\right)}{\partial\xi}\right) rdrd\theta d\xi$$

Therefore, net rate of work done by surface forces is given by

$$= \left(\begin{array}{l}\left(\frac{\partial\left(r\tau_{rr}\overrightarrow{U_r}\right)}{\partial r} + \frac{1}{r}\frac{\partial(\tau_{\theta r}\overrightarrow{U_r})}{\partial\theta} + \frac{\partial\left(\tau_{\xi r}\overrightarrow{U_r}\right)}{\partial\xi}\right) + \\ \left(\frac{\partial\left(r\tau_{r\theta}\overrightarrow{U_\theta}\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\tau_{\theta\theta}\overrightarrow{U_\theta}\right)}{\partial\theta} + \frac{\partial\left(\tau_{\xi\theta}\overrightarrow{U_\theta}\right)}{\partial\xi}\right) + \\ \left(\frac{\partial\left(r\tau_{r\xi}\overrightarrow{U_\xi}\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\tau_{\theta\xi}\overrightarrow{U_\xi}\right)}{\partial\theta} + \frac{\partial\left(\tau_{\xi\xi}\overrightarrow{U_\xi}\right)}{\partial\xi}\right)\end{array}\right) rdrd\theta d\xi$$

$$= \nabla.\left(\vec{U}.\overleftrightarrow{\tau}\right) rdrd\theta d\xi$$

$$= \nabla.\left(\vec{U}.\overleftrightarrow{\tau}\right) dV \qquad (31)$$

Hence from conservation law, Substituting (28), (30) & (31) in equation (27), we get

$$\nabla.\left(k\nabla T\right)dV + \nabla.\left(\vec{U}.\overleftrightarrow{\tau}\right)dV = \left(\frac{\partial(\rho e)}{\partial t} + \frac{\partial\left(r.\rho\overrightarrow{U_r}\lambda\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\rho\overrightarrow{U_\theta}\lambda\right)}{\partial\theta} + \frac{\partial\left(\rho\overrightarrow{U_\xi}\lambda\right)}{\partial\xi}\right) dV$$

$$\nabla.\left(k\nabla T\right)dV + \nabla.\left(\vec{U}.\overleftrightarrow{\tau}\right)dV = \left(\frac{\partial(\rho e)}{\partial t} + \frac{\partial\left(r.\rho\overrightarrow{U_r}\lambda\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\rho\overrightarrow{U_\theta}\lambda\right)}{\partial\theta} + \frac{\partial\left(\rho\overrightarrow{U_\xi}\lambda\right)}{\partial\xi}\right) dV \qquad (32)$$

But the right hand side of the equation can be expanded as,

$$\left(\frac{\partial(\rho e)}{\partial t} + \frac{\partial\left(r.\rho\overrightarrow{U_r}\lambda\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\rho\overrightarrow{U_\theta}\lambda\right)}{\partial\theta} + \frac{\partial\left(\rho\overrightarrow{U_\xi}\lambda\right)}{\partial\xi}\right)$$

$$= \rho\left(\frac{\partial e}{\partial t} + \overrightarrow{U_r}\frac{\partial e}{\partial r} + \frac{1}{r}\overrightarrow{U_\theta}\frac{\partial e}{\partial\theta} + \overrightarrow{U_\xi}\frac{\partial e}{\partial\xi}\right)$$

$$+P\left(\frac{\overrightarrow{U_r}}{r} + \frac{\partial\overrightarrow{U_r}}{\partial r} + \frac{1}{r}\frac{\partial\overrightarrow{U_\theta}}{\partial\theta} + \frac{\partial\overrightarrow{U_\xi}}{\partial\xi}\right)$$

$$+e\left(\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial\left(r.\rho\overrightarrow{U_r}\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\rho\overrightarrow{U_\theta}\right)}{\partial\theta} + \frac{\partial\left(\rho\overrightarrow{U_\xi}\right)}{\partial\xi}\right)$$

$$+\left(\overrightarrow{U_r}\frac{\partial P}{\partial r} + \overrightarrow{U_\theta}\frac{1}{r}\frac{\partial P}{\partial\theta} + \overrightarrow{U_\xi}\frac{\partial P}{\partial\xi}\right)$$

Further,

$$\left(\frac{\partial(\rho e)}{\partial t} + \frac{\partial\left(r.\rho\overrightarrow{U_r}\lambda\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\rho\overrightarrow{U_\theta}\lambda\right)}{\partial\theta} + \frac{\partial\left(\rho\overrightarrow{U_\xi}\lambda\right)}{\partial\xi}\right)$$

$$= \rho\frac{de}{dt} + P\nabla.\vec{U} + e\left(\frac{\partial\rho}{\partial t} + \nabla.\rho\vec{U}\right) + \vec{U}.\nabla P$$

$$\text{But}\left(\frac{\partial\rho}{\partial t} + \nabla.\rho\vec{U}\right) = 0$$

Hence right hand side of equation reduces to

$$\left(\frac{\partial(\rho e)}{\partial t} + \frac{\partial\left(r.\rho\overrightarrow{U_r}\lambda\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(\rho\overrightarrow{U_\theta}\lambda\right)}{\partial\theta} + \frac{\partial\left(\rho\overrightarrow{U_\xi}\lambda\right)}{\partial\xi}\right)$$

$$= \rho\frac{de}{dt} + P\nabla.\vec{U} + \vec{U}.\nabla P \qquad (33)$$

Substituting (33) in equation (32), we get

$$\nabla.\left(k\nabla T\right) + \nabla.\left(\vec{U}.\overleftrightarrow{\tau}\right) = \rho\frac{de}{dt} + \nabla.\left(P\vec{U}\right)$$

$$\rho\frac{d\left(\hat{u} + \frac{\vec{U}^2}{2}\right)}{dt} = \nabla.\left(k\nabla T\right) + \nabla.\left(\vec{U}.\overleftrightarrow{\tau}\right) - \nabla.\left(P\vec{U}\right) \qquad (34)$$

From equation of mechanical energy,

$$\rho\frac{d\left(\frac{\vec{U}^2}{2}\right)}{dt} = -\nabla.\left(P\vec{U}\right) + P\left(\nabla.\vec{U}\right) + \nabla.\left(\vec{U}.\overleftrightarrow{\tau}\right) - \overleftrightarrow{\tau}:\nabla\vec{U} \qquad (35)$$

Subtracting equation (35) from equation (34) we get,

$$\rho\frac{d\hat{u}}{dt} = \nabla.\left(k\nabla T\right) - P\left(\nabla.\vec{U}\right) + \overleftrightarrow{\tau}:\nabla\vec{U} \qquad (36)$$

The above equation describes that rate of change of internal energy of the thermoplastic melt inside runner conduit is proportional to amount of energy added to the thermoplastic melt via heat and work done.

2.3.2 Second Law of Thermodynamics

Lemma

"Rate of injective work done on thermoplastic melt across the runner conduit inlet and outlet is proportional to thermoplastic melt energy change rates within the runner conduit and rate of entropy change across the runner conduit inlet and outlet"

Entropy change (dS) gives us,

$$TdS = de + Pd\upsilon$$

Substituting specific volume $\upsilon = \frac{1}{\rho}$ we get,

$$TdS = de - \frac{P}{\rho^2}d\rho \qquad (37)$$

Since injection moulding is a dynamic process, we now differentiate above equation (37) for a small representative interval to get,

$$T\frac{dS}{dt} = \frac{de}{dt} - \frac{P}{\rho^2}\frac{d\rho}{dt} \qquad (38)$$

Equation of internal energy is given by,

$$\rho\frac{de}{dt} = -\nabla.q - P\left(\nabla.\vec{U}\right) + \phi$$

where ϕ = Viscous dissipation function

$$\therefore \quad \frac{de}{dt} = -\frac{\nabla.q}{\rho} - \frac{P\left(\nabla.\vec{U}\right)}{\rho} + \frac{\phi}{\rho} \qquad (39)$$

Equation (6) can also be written as,

$$\frac{d\rho}{dt} = -\rho\left(\nabla.\vec{U}\right) \qquad (40)$$

Substituting (39) & (40) in equation (38) we get,

$$T\frac{dS}{dt} = -\frac{\nabla.q}{\rho} + \frac{\phi}{\rho}$$

By rearranging we get,

$$\rho\frac{dS}{dt} = -\frac{\nabla.q}{T} + \frac{\phi}{T} \qquad (41)$$

Using vector calculus the term $-\frac{\nabla.q}{T}$ is split as $-\frac{\nabla.q}{T} = -\nabla.\left(\frac{q}{T}\right) - \left(\frac{q}{T^2}\right).\nabla T$ to separate reversible $\left(-\nabla.\frac{q}{T}\right)$ and irreversible $\left(-\frac{1}{T^2}q.\nabla T\right)$ effects of heat transfer.

Hence equation (41) becomes,

$$\rho\frac{dS}{dt} = -\nabla.\frac{q}{T} - \frac{1}{T^2}q.\nabla T + \frac{\phi}{T} \qquad (42)$$

This is called Equation of Entropy

The above equation describes that rate of change of entropy during the injection moulding cycle is proportional to energy added to the thermoplastic melt during that cycle.

2.4 Volumetric Heat Absorption

Thermoplastic melt state change for a particular cycle through the runner conduit of an injection mould is considered. Accounting to runner conduit configuration heat transfer is dominant in radial direction compared to that of heat transfer in the direction of injection which is very less and hence neglected. This provides us with an advantage of considering heat conduction in radial direction alone. To begin with derivation let us consider a differential runner volume element with heat transfer taking place in r-direction alone. Since the runner geometry is best explained by cylindrical co-ordinate system, the same is taken to derive heat conduction equation.

Lemma

"Rate of volumetric thermal energy absorbed across thermoplastic melt core and conduit interface wall is proportional to difference between net rate of heat conducted radially across runner conduit cross-section and rate of internal energy change throughout an injection cycle"

Proof

The rate of volumetric heat absorbed across thermoplastic melt core and conduit wall interface is given by

$$= \dot{q}_{vh}\left(rdr\right) \qquad (43)$$

Let the rate of heat conducted into differential runner element in r-direction

$$\dot{Q}_r = -\left(k\frac{\partial T}{\partial r}\right)r$$

Let the rate of heat conducted away from differential runner element in r-direction

$$\dot{Q}_{r+dr} = \left(\dot{Q}_r + \frac{\partial \dot{Q}_r}{\partial r}\right) dr$$

Thus net rate of heat conducted into differential runner element in r-direction

$$\left(\dot{Q}_r - \dot{Q}_{r+dr}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)(rdr) \quad (44)$$

The rate of internal energy change of thermoplastic melt is given by

$$= \left(\rho C_v \frac{\partial T}{\partial t}\right)(rdr) \quad (45)$$

Hence from equations (43), (44) and (45) we have

$$\dot{q}_{vh}(rdr) = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)(rdr) - \left(\rho C_v \frac{\partial T}{\partial t}\right)(rdr)$$

Neglecting the volume terms and expanding the above terms, the equation becomes

$$\dot{q}_{vh} = k\frac{\partial^2 T}{\partial r^2} + \frac{k}{r}\frac{\partial T}{\partial r} - \left(\rho C_v \frac{\partial T}{\partial t}\right) \quad (46)$$

The above equation represents the melt state change during a particular injection moulding cycle through the runner conduit.

3. Equation of State (Tait Equation)

Thermoplastic melt specific volume streaming through runner conduit at any instant is represented as a function of its pressure P (t) and temperature T (t). This equation is popularly known as Tait equation [12].

$$\nu(T,P) = \nu_0(T)\left[1 - C\ln\left(1 + \frac{P}{B(T)}\right)\right] + \nu_t(T,P) \quad (47)$$

Where C=0.0894 is a universal constant.

$\nu_0(T)$ is given by,

$$\nu_0(T) = b_1^{(m)} + b_2^{(m)}(T - b_5), \text{ if } T > T_{trans,} \quad (48)$$

Where, subscripts (m) represent molten state of the polymer.

B (T) is given by

$$B(T) = b_3^{(m)} \exp\left[-b_4^{(m)}(T - b_5)\right], \text{ if } T > T_{trans} \quad (49)$$

For thermoplastic melt streaming through runner conduit under general injection moulding processing conditions,

$$\nu_t(T,P) = 0 \quad (50)$$

Where, T_{trans} is the glass transition temperature which is almost a linear function of pressure [13].

$$T_{trans} = b_5 + b_6 P \quad (51)$$

Hence governing equations are:-

Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla.\left(\rho\vec{U}\right) = 0$$

Linear momentum equation

$$-\nabla P + \nabla.\overleftrightarrow{\tau} = \rho\frac{d\vec{U}}{dt}$$

Angular momentum equation

$$-\nabla.\left(\vec{r}\times\overleftrightarrow{\tau}\right) = \rho\frac{d}{dt}\left(\vec{r}\times\vec{U}\right)$$

First law of thermodynamics

$$\rho\frac{d\hat{u}}{dt} = \nabla.(k\nabla T) - P\left(\nabla.\vec{U}\right) + \overleftrightarrow{\tau} : \nabla\vec{U}$$

Second law of thermodynamics

$$\rho\frac{dS}{dt} = -\nabla.\frac{q}{T} - \frac{1}{T^2}q.\nabla T + \frac{\phi}{T}$$

Volumetric heat absorption

$$\dot{q}_{vh} = k\frac{\partial^2 T}{\partial r^2} + \frac{k}{r}\frac{\partial T}{\partial r} - \left(\rho C_v \frac{\partial T}{\partial t}\right)$$

4. Conclusion

The authors have earlier proposed spatiotemporal conservation principle to design runner system for a plastic injection mould [14]. Accordingly governing equations for thermoplastic melt inside runner conduit throughout an injection moulding cycle has been obtained after implementing conservation principles. Further set of equations tabled above are then used to derive a computational model to design the ideal runner conduit size for feeding thermoplastic melt relative to specific combination of injection moulding machine available, characteristics of thermoplastic material chosen and desired features of component being moulded.

5. References

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