International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 6 Issue 06, June - 2017

Image Restoration by Total Variation Image in Painting Case

Raminoson Tsiriniaina
Telecommunication- Automatic – Signal – Image- Research Laboratory/Doctoral School in Science and Technology of Engineering and Innovation/University of Antananarivo
Antananarivo, Madagascar

Ralaivao Harinaivo Hajasoa
Telecommunication- Automatic – Signal – Image- Research Laboratory/Doctoral School in Science and Technology of Engineering and Innovation/University of Antananarivo
Antananarivo, Madagascar

Randriamitantsoa Paul Auguste
Telecommunication- Automatic – Signal – Image- Research Laboratory/Doctoral School in Science and Technology of Engineering and Innovation/University of Antananarivo
Antananarivo, Madagascar

***Abstract*—This paper describes algorithms for minimizing the total variation of an image. Many regularization models are presented : Tychonov model, Rudin-Osher-Fatemi (ROF) model and Osher-Sole-Vese (OSV) model. We show applications to image inpainting.**

Keywords*— *Image restoration; regularization; total variation; image inpainting

I. INTRODUCTION

Image restoration, including image denoising, deblurring, inpainting, computed tomography, etc., plays an important role in numerous areas of applied sciences, such as medical and astronomical imaging, film restoration, and image/video coding. Its major purpose is to enhance the quality of giving image that is corrupted in various ways during the process of imaging, acquisition and communication.

Considering an original image u_0, supposed that it was degraded by an additive noise v, and evently by a fuzzy operator R. The operator R is modelling by a convolution product. From the observed image $u_d = Ru_0 + v$ a degraded image, we have to rebuild u_0. If we suppose that the additive noise is Gaussian, the method of "Maximum likelihood" conducts us to find u_0 as a solution of minimization problem.

$$\inf_u \|u_d - Ru_0\|_2^2, \tag{1}$$

$\|.\|_2$ design the norm in L^2. This is an inverse ill-posed problem : the operator is not necessary inversible (and if it is inversible, the inverse is difficult to calculate). In other terms, the existence and/or the unicity of the solutions is not ensured or the solution is not stable. To solve numerically, we have to introduce a regularization term $L(u)$, and have to consider the following problem:

$$\inf_u \|u_d - Ru_0\|_2^2 + L(u) \tag{2}$$

II. DISCRETIZATION

The size of the processed image is $M \times N$. We denote $X = \mathbb{R}^{M\times N}$ and $Y = X \times X$ with the usual scalar product in X

$$\langle u, v\rangle_X = \sum_{1\le i\le N}\sum_{1\le j\le M} u_{ij}v_{ij} \tag{3}$$

and the associate Euclidian norm : $\|u\|_X = \sqrt{\langle u,u\rangle_X}$

A. Definition 1

Let $u \in X$; then the discrete gradient of u, written $\nabla u \in Y = X \times X$, is defined by

$$(\nabla u)_{i,j} = \left((\nabla u)^1_{i,j}, (\nabla u)^2_{i,j}\right) \tag{4}$$

with $\forall i,j \in [\![0,\dots,M-1]\!] \times [\![0,\dots,N-1]\!]$

$$(\nabla u)^1_{i,j} = \begin{cases} u_{i+1,j} - u_{i,j} & if\ i < M-1 \\ 0 & if\ i = M-1 \end{cases}$$

$$(\nabla u)^2_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j} & if\ j < N-1 \\ 0 & if\ j = N-1 \end{cases}$$

B. Definition 2

Let $p = (p^1, p^2) \in Y$, we define the numerical divergence operator $div: Y \to X$ such that $div = -\nabla^*$ the adjoint operator of ∇ by the following:

$$(div\ p)_{i,j} = \begin{cases} p^1_{i,j} - p^1_{i-1,j} & if\ 0 < i < M-1 \\ p^1_{i,j} & if\ i = 0 \\ -p^1_{i-1,j} & if\ i = M-1 \end{cases} + \begin{cases} p^2_{i,j} - p^2_{i,j-1} & if\ 0 < j < N-1 \\ p^2_{i,j} & if\ j = 0 \\ p^2_{i,j-1} & j = N-1 \end{cases} \tag{5}$$

$$\forall p \in Y, \forall u \in X\ \langle -div\ p, u\rangle_X = \langle p, \nabla u\rangle_Y = \langle p^1, \nabla^1 u\rangle_X + \langle p^2, \nabla^2 u\rangle_X$$

Laplacian $\Delta u = div(\nabla u)$ (6)

C. *Total variation*

In the discrete case, the total variation can be written by :

$$J(u) = \sum_{\substack{0<i<M-1\\0<j<N-1}} \left|(\nabla u)_{i,j}\right| \tag{7}$$
$$= \sum_{\substack{0<i<M-1\\0<j<N-1}} \sqrt{\left((\nabla u)^1_{i,j}\right)^2 + \left((\nabla u)^2_{i,j}\right)^2}$$

Let us observe here that this functional J is a discretization of the standard total variation, defined in the continuous setting for a function $u \in L^1(\Omega)$ (Ω open subset of $\mathbb{R}^2$) by

$$J(u) = sup\left\{\int_\Omega u(x)\, div\xi(x)\, dx;\ \xi \in C_c^1(\Omega;\mathbb{R}^2), |\xi(x)| \le 1\ \forall x \in \Omega\right\} \tag{8}$$

III. Variationnal Methods

A. *Tychonov regularization*

It is a process of regularization most classical and too short for image processing.

Let $V = H^1(\Omega)$ and $H = L^2(\Omega)$, for all $\alpha > 0$ the problem (P_α) is :

$$\min_{u\in V} \|u - u_d\|_H^2 + \alpha\|\nabla u\|_H^2 \tag{9}$$

The Euler equation to find the solution u_α is shown below:

$$u_\alpha - u_d - \alpha\Delta u_\alpha = O, u_\alpha \in H^1(\Omega)$$

And to approach u_α, we have:

$$\frac{\partial u}{\partial t} - \alpha\Delta u + u = u_d$$

Algorithm 1 Tychonov model

Input

Number of iteration N
Image to be denoised u_d

Output

$u^{(N)}$ *estimed solution*

Initialization

$u^{(0)} = u_d$

$for\ k \coloneqq 0\ to\ N-1\ do$

$$u^{(k+1)} = u^{(k)} + dt * \left(\left(u_d - u^{(k)}\right) + \alpha\Delta u^{(k)}\right)$$

The restored image u is very smoothed, because the Laplacian is an operator of “isotropic diffusion”.

B. *Rudin-Osher-Fatemi model*

They [6] introduced in regularization the total variation, the problem (P_{ROF}) is obtained after discretization of :

$$\min_{u\in X} \frac{1}{2}\|u - u_d\|^2 + \varepsilon J(u) \tag{10}$$

The discrete version of total variation is given, similarly in continuous case, by

$$J(u) = \sup_{\xi\in K} \langle u, \xi\rangle_X$$

with

$$K \coloneqq \{\xi = div(g) | g \in Y, |g_{i,j}| \le 1,\ \forall (i,j) \in \{1,\ldots,N\} \times \{1,\ldots,M\}\}$$

Hence the solution u of problem (10) is simply given by

$$u = u_d - \lambda P_{\lambda K}(u_d) \tag{11}$$

with $P_{\lambda K}$ is the nonlinear projection on λK

Algorithm 2 Projection’s algorithm of Chambolle [3]

Input

Number of iteration N
Image to be denoised u_d

Output

$u^{(N)}$ *estimed solution*

Initialization

$p^{(0)} = 0$

$for\ k \coloneqq 0\ to\ N-1\ do$

$$\begin{cases} u^{(k)} = u_d - \lambda div\left(p^{(k)}\right) \\ p^{(k+1)} = \dfrac{p^{(k)} - \frac{\tau}{\lambda}\nabla u^{(k)}}{1 + \frac{\tau}{\lambda}\|\nabla u^{(k)}\|_{\mathbb{R}^2}} \end{cases}$$

Theorem 1 [3]

Let $\tau \le {}^1/_8$. *Then* $\lambda div\, p^n$ *converges to* $P_{\lambda K}$ *as* $n \to \infty$.

The solution of the problem (10) is given by :

$$u = u_d - \lambda div(p^\infty) \tag{12}$$

with $p^\infty = \lim_{n\to\infty} p^n$

Algorithm 3 Projection’s algorithm of Chambolle with $\tau \le {}^1/_8$

Input

Number of iteration N
Image to be denoised u_d

Output

$u^{(N)}$ *estimed solution*

Initialization

$p^{(0)} = 0$

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 6 Issue 06, June - 2017

for $k := 0$ *to* $N-1$ *do*

$$\begin{cases} u^{(k)} = u_d - \lambda div(p^{(k)}) \\ p^{(k+1)} = \dfrac{p^{(k)} - \frac{\tau}{\lambda}\nabla u^{(k)}}{\max\left\{1, \left\|p^{(k)} - \frac{\tau}{\lambda}\nabla u^{(k)}\right\|\right\}} \end{cases}$$

C. *Osher-Sole-Vese model*

We will present here a proposed model by Osher-Solé-Vese (P_{OSV}) stated as below :

$$\inf_{u\in X}\left(\frac{1}{2\lambda}\|u-u_d\|^2_{-1,2} + J(u)\right) \quad (13)$$

$\|.\|_{-1,2}$ is a discrete norm in $W^{-1,2}$ which is the dual space of Sobolev space $W_0^{1,2}$

Algorithm 4 Osher-Sole-Vese model

Input

Number of iteration N

Image to be denoised u_d

Output

$u^{(N)}$ *estimed solution*

Initialization

$p^{(0)} = 0$

for $k := 0$ *to* $N-1$ *do*

$$\begin{cases} u^{(k)} = u_d + \lambda(\Delta div(p^{(k)})) \\ p^{(k+1)} = \dfrac{p^{(k)} - \tau\left(\nabla(\Delta div(p^{(k)}) + f/\lambda)\right)}{1+\tau\left\|\left(\nabla(\Delta div(p^{(k)}) + f/\lambda)\right)\right\|} \end{cases}$$

IV. INPAINTING REGULARIZATION

The object of inpainting is to reconstitute the missing or damaged regions in images, in order to make it more legible and to restore its unity. Mathematically speaking, inpainting is essentially an interpolation problem.

Considering an image f defined on a domain $\Omega \subset \mathbb{R}^2$ but missing or damaged on a subset $D \subset \Omega$. Then it has to find methods or models to resolve

$$u = \Phi(f)$$

with Φ is a masking operator. It is in fact a projection operator on $\Omega - D$:

$$\Phi(u)(x) = \begin{cases} u(x) & si\ x \notin D \\ 0 & si\ x \in D \end{cases}$$

The idea of variational methods is to minimize the quantity $\|f - \Phi(u)\|$ while adding a regularization term $J(u)$

In general, we consider the following problem :

$$\begin{cases} \min \mathcal{F}(u) = \dfrac{1}{2}\|f - \Phi(u)\|^2 + \alpha J(u) \\ u \in \mathbb{R}^N \end{cases} \quad (14)$$

In the Total Variation regularization, the norm L^1 is used :

$$J(u) = \sum_{i,j}\|(\nabla u)_{i,j}\|$$

with

$$\|(\nabla u)_{i,j}\| = \left\|(\partial_{x_1}u)_{i,j}, (\partial_{x_1}u)_{i,j}\right\| = \sqrt{(\partial_{x_1}u)^2_{i,j} + (\partial_{x_2}u)^2_{i,j}}$$

A necessary and sufficient condition for u_α to be a solution of (14) is :

$$0 \in \partial\left(\alpha J(u_\alpha) + \frac{1}{2}\|\Phi u_\alpha - f\|^2_X\right) = \alpha\partial J(u_\alpha) + \Phi u_\alpha - f$$

Then, we obtain the following concept :

$$f - \Phi u_\alpha = \alpha P_K\left(\frac{f - \Phi u_\alpha + u_\alpha}{\alpha}\right) = P(f - \Phi u_\alpha + u_\alpha)$$

Algorithm 5 TV algorithm for inpainting

Input

Number of iteration N

Image to be denoised u_d

Output

$u^{(N)}$ *estimed solution*

Initialization

$u^{(0)} = u_d$

for $k := 0$ *to* $N-1$ *do*

$$u^{(k+1)} = P_f\left[u^{(k)} + \tau(u_d - P_{\alpha K}(u_d + u_n))\right]$$

V. NUMERICAL RESULTS

In this section we present some of the results obtained with the regularization models sush as Tychonov, ROF and OSV. For the numerical examples, we use “mandril image”, Fig. 1.

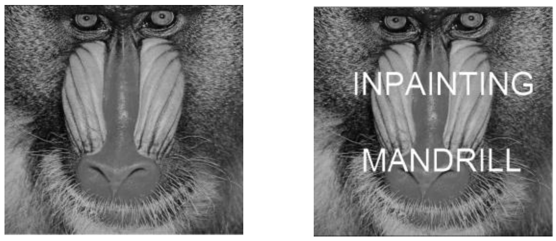

Fig. 1. Mandril image : original and mandril with mask

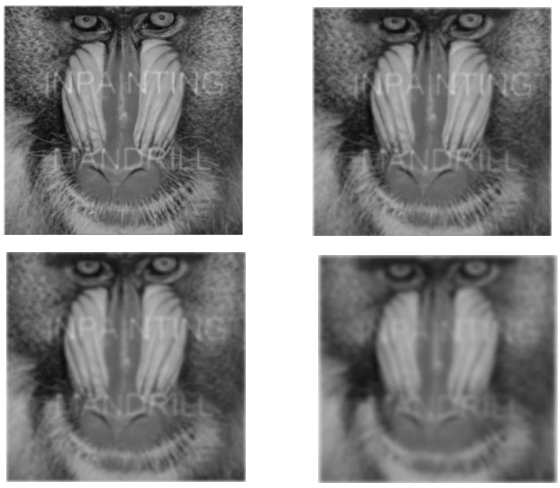

Fig. 2. Tychonov model : Top left to right bottom

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 6 Issue 06, June - 2017

Fig. 2 shows the smoothed images, implementation of the algorithm number 1 : Tychonov regularization.

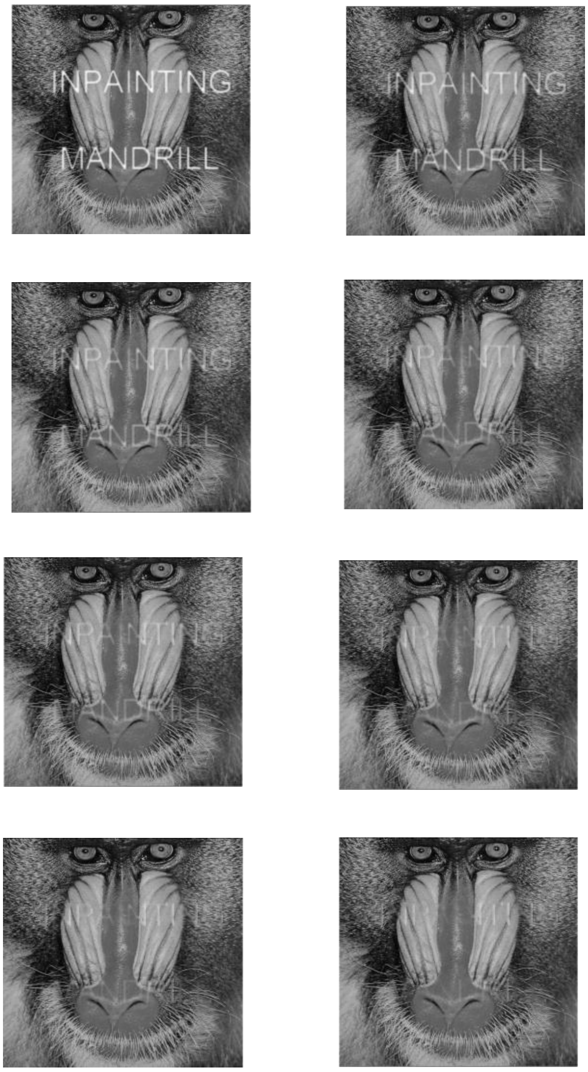

Fig. 3. On the left ROF model, on the right OSV model :

1st line : 10 iterations

2nd line : 20 iterations

3rd line : 50 iterations

4th line : 100 iterations

We use Peak signal-to-noise ratio **(**PSNR) to evaluate the performance of these algorithms.

$$PSNR = 10 log_{10}\left(\frac{255^2}{MSE}\right) \quad (14)$$

with the Mean Square Error $MSE = \frac{\sum_{M,N}[I_1(m,n)-I_2(m,n)]^2}{M*N}$

TABLE I. VARIATION OF PSNR PER MODEL

No iterations	PSNR (db)		
	Tychonov model	*ROF model*	*OSV model*
0	11.73	11.73	11.73
10	15.36	13.77	14.64
20	17.09	16.56	18.15
50	19.4	18.378	20.26
75	20.89	20.7	22.74
100	22.72	23.87	25.85
150	25.1	28.59	29.94
200	28.5	37.48	36.47

VI. CONCLUSION AND PERSPECTIVES

This paper focuses on the theory and on the implementation of minimizing the total variation on images. We have implemented many algorithms and we have compared the numerical results applied on image inpainting. The algorithms ROF and OSV are the most models but Tychonov model is very smoothed.

The one perspective is to apply the total variation on wavelet decomposition to restore an image inpainting. And it is also possible to implement these algorithms on sequence images or on videos.

REFERENCES

[1] G. Aubert and P. Kornprobst, Mathematical Problems in Image Processing:Partial Differential Equations and the Calculus of Variations, Applied Mathematical Sciences Vol. 147, in Springer.
[2] J.F. Aujol, “Traitement d’images par approches variationnelles et équations aux dérivées partielles,” ENIT Tunis., 2005, pp.55.
[3] A. Chambolle, “An algorithm for total variation minimization and applications,” Journal of Mathematical Imaging and Vision 20, 2004, pp. 89–97.
[4] S. Durand and J. Froment, “Reconstruction of wavelet coefficients using total variation minimization,” SIAM, Journal on scientific computing, 24 (5), 2003, pp.1754-1767.
[5] S. N. Ghate, S. Achaliya and S. Raveendran, “An algorithm of total variation for image inpainting,”, International Journal of Computer and Electronics Research, vol. 1, issue 3, October 2012, pp. 124-130.
[6] L. I. Rudin. S. Osher and E. Fatemi, N”onlinear total variation based noise removal algorithms,” Physica D: Nonlinear Phenomena, Volume 60, Issue 1-4, 1992, pp. 259-268
[7] T. Zeng, X. Li and M. Ng, “Alternating minimization method for total variation based wavelet shrinkage model,” Commun. Comput. Phys., vol. 8, no. 5, November 2010, pp. 976-994.