\(\alpha\)-Homotopy and \(\alpha\)-Homotopy Type Spaces

Chidanand Badiger  
Department of Mathematics,  
Rani Channamma University, Belagavi-591 156,  
Karnataka, India.

T Venkatesh  
Department of Mathematics,  
Rani Channamma University, Belagavi-591 156,  
Karnataka, India.

Abstract- In this paper, we introduce the term of \(\alpha\)-homotopy of \(\alpha\)-continuous maps, \(\alpha\)-homotopy equivalence classes and their consequences, same \(\alpha\)-homotopy type, \(\alpha\)-contractible spaces, and few properties. Also we present induced map and their properties, and consequences respect to some notions.

Key Words: \(\alpha\)-homotopy, same \(\alpha\)-homotopy type, \(\alpha\)-contractible space.  
AMS Classification: 54C08, 14F35, 55Q05, 55P15, 55U35.

I. INTRODUCTION

Algebraic methods were introduced in topology by Poincare around 1895. Chronologically, fundamental group of topological spaces and homotopy groups were some of important notions that enable us to compute for the topological spaces (Analysis Situs, 1895) [7, 9, 12]. As a result a separate branch took birth known as algebraic topology. Later, we see in the work of Poul Heegard, Barratt etc., in the extensive reading of homotopy and their equivalence among the spaces \(M\) and \(N\). Well, there algebraic methods enhanced the scope of the classification themes among topological spaces.

Next to these algebraic notions, we have certain aspect of associations not necessarily a algebraic structure with topological space, namely the notion homotopy equivalence classes over space of all continuous function from a space to space. Deformations being standard idea in topology which in a way converts one space into another space without losing its qualitative property. Homotopy equivalence classis set \([M,N]\) was first systematically studied by M.G. Barratt in 1955 [9]. The issue of characterization of homeomorphic topological spaces has partially solved by help of homotopy. The problems of classification of topological spaces according to their homotopy properties are also take important role in classification theory. But homotopy equivalence of spaces not necessarily topological equivalence. Homotopy equivalence is a weaker relation than topological equivalence, hence its class is bigger. Therefore homotopy equivalence has prominent role than homeomorphism. Deformation is a standard idea in topology, as morphisms can do so. The deformation is a notion that converts one space into other by the tool of homotopy. Hence homotopy theory takes powerful tools for this purpose [1,4,8,9]. In geometry, not only homotopy as well homology and cohomology are frequently used algebraic association.

Njastad [3,11] introduced \(\alpha\)-open sets in a topological space and studied some properties. Further \(\alpha\)-continuous map, and respective open map and closed map in topological space is studied by A.S.Meshhour, I.A.Hasanein [2]. Semi-homotopy and semi-fundamental groups studied by Ayhan Erciyes, Ali Aytek in and Tuncar Sahan [4]. We introduce such class of \(\alpha\)-homotopy of \(\alpha\)-continuous maps, same \(\alpha\)-homotopy type, \(\alpha\)-contractible space, and post and pre induced map of a \(\alpha\)-continuous map. These have many rich consequences concern to contractible space, induced map, usual Homotopy and fundamental group. Here we place example for few respective results.

II. PRELIMINARIES

Throughout this paper \(L,M,N,R\) and \(S\) represent the topological spaces on which no separation axioms are assumed unless otherwise mentioned. Map here mean function and for a subset \(F\) of topological space \(M\), the \(M\setminus F\) denotes the complement of \(F\) in \(M\). We recall the following definitions.

Definition 2.1 [3, 11] A subset \(F\) of a space \(M\) is said to be \(\alpha\)-open set, if \(F \subseteq \text{Int} \left(\text{Cl}(\text{Int}(F))\right)\). Respectively called \(\alpha\)-closed set, if \(M \setminus F\) is \(\alpha\)-open set in \(M\). We denote the set of all \(\alpha\)-open sets in \(M\) by \(\alpha\text{O}(M)\).

Definition 2.2 [2] A map \(h:M \to N\) is said to be \(\alpha\)-continuous map, if \(h^{-1}(F)\) is \(\alpha\)-closed set of \(M\), for every closed set \(F\) of \(N\).

Definition 2.3 A map \(h:M \to N\) is said to be \(\alpha\)-irresolute map, if \(h^{-1}(F)\) is \(\alpha\)-closed set of \(M\), for every \(\alpha\)-closed set \(F\) of \(N\).

Definition 2.4 A bijection \(h:M \to N\) is called \(\alpha\)-homeomorphism, if both \(h\) and \(h^{-1}\) are \(\alpha\)-continuous.

Definition 2.5 (Consider) A topological space \(M\) is called \(\eta\alpha\)-space, if every \(\alpha\)-closed set is closed set.

Theorem 2.6 Every \(\alpha\)-irresolute maps are \(\alpha\)-continuous.

III. \(\alpha\)-HOMOTOPY

In this section we introduce the notions of \(\alpha\)-homotopy of \(\alpha\)-continuous maps, \(\alpha\)-relative homotopy, \(\alpha\)-contractible space, \(\alpha\)-homotopy type and post and pre induced map of a \(\alpha\)-continuous map also discuss few properties respectively.

Definition 3.1 Let \(M\) and \(N\) be two topological spaces and \(g,h:M \to N\) be two \(\alpha\)-continuous maps. Then \(\alpha\)-homotopy is a map \(H:M \times I \to N\) (Here, \(I = [0,1]\)) such that, for all \(t \in I\) the restrictions of \(H_t: M \to N\) by \(x \to H_t(x) = H(x,t)\) is \(\alpha\)-continuous, satisfying \(H_0(x) = H(x,0) = g(x)\) and \(H_1(x) = H(x,1) = h(x)\) for all \(x \in M\). If such \(H\) is exist then \(H\) is called \(\alpha\)-homotopy between them, and \(g\) is called \(\alpha\)-homotopic to \(h\). We denote it by \(g \simeq_{\alpha} h\).

And if \(g\) is \(\alpha\)-homotopic to a constant map \((i.e., g \simeq_{\alpha} c_y\), where \(c_y: M \to N, c_y(x) = y\) for some \(y \in N\)) then \(g\) is called \(\alpha\)-nullhomotopic.

Example 3.2 Let \(g,h:(\mathbb{R},\eta_{\alpha}) \to (\mathbb{R},\eta_{\alpha})\) be \(g(x) = 5x\), \(h(x) = x + 2\) and \(\eta_{\alpha}\) be usual topology on \(\mathbb{R}\), then obviously map \(g\) is become \(\alpha\)-homotopic to \(h\), quite simple that one can
Theorem 3.3 Every homotopy is $\alpha$-homotopy (Every homotopy between continuous maps is $\alpha$-homotopy between same $\alpha$-continuous maps.).

Proof: It followed from the fact, every continuous map is $\alpha$-continuous map.

Theorem 3.4 [1,6,8] The relation $\alpha$-homotopic is an equivalence relation on the set $\alpha$-C(M,N) of all $\alpha$-continuous maps from topological space M to N.

Proof: Let M and N be two topological spaces then,

Reflexivity: If $g \in \alpha$-C(M,N) i.e. $g: M \rightarrow N$ is $\alpha$-continuous map. Define $H: M \times I \rightarrow N$ by $H(x,t) = (1-t) \cdot g(x) + t \cdot h(x)$ for all $x \in M$ and all $t \in I$, then H become $\alpha$-homotopy between g and g. Hence $g \cong_h g$.

Symmetry: Suppose $g, h \in \alpha$-C(M,N) and $g \cong_h h$ implies there is a $\alpha$-homotopy map $H: M \times I \rightarrow N$ such that, for all $t \in I$ the restrictions of $H, H_t: M \rightarrow N$ by $x \rightarrow H_t(x) = H(x, t)$ is $\alpha$-continuous, satisfying $H(x, 0) = g(x)$ and $H(x, 1) = h(x)$ for all $x \in M$. Define $F: M \times I \rightarrow N$, by $F(x,t) = H(x, (1-t))$, obviously for all $t \in I$ the restrictions of F, $F_t: M \rightarrow N$ by $x \rightarrow F_t(x) = H(x, (1-t))$ is $\alpha$-continuous for some $t_0 \in I$. hence it is $\alpha$-continuous, satisfying $F(x, 0) = h(x)$ and $F(x, 1) = g(x)$ and become $\alpha$-homotopy between h and g, implies $h \cong_g g$.

Transitivity: Suppose $g, h, k \in \alpha$-C(M,N) with $g \cong_h h$ and $h \cong_k k$ implies there are $\alpha$- homotopies, $H: M \times I \rightarrow N$ such that, for all $t \in I$ the restrictions of $H, H_t: M \rightarrow N$ by $x \rightarrow H_t(x) = H(x, t)$ is $\alpha$-continuous, satisfying $H(x, 0) = g(x)$ and $H(x, 1) = h(x)$ for all $x \in M$. And $F: M \times I \rightarrow N$ such that, for all $t \in I$ the restrictions of F, $F_t: M \rightarrow N$ by $x \rightarrow F_t(x) = F(x, t)$ is $\alpha$-continuous, satisfying $F(x, 0) = h(x)$ and $F(x, 1) = k(x)$ for all $x \in M$. Define map $G: M \times I \rightarrow N$ by,

$$G(x,t) = \begin{cases} \frac{H(x,2t)}{2} & \text{if } t \in [0,1/2] \\ \frac{F(x,2t-1)}{2} & \text{if } t \in [1/2,1] \end{cases}$$

Obvious G become $\alpha$-homotopy between g and k. because $G_t(x) = \frac{H_t(x)}{2}$ or $F_t(x)$ for some $t_0 \in I$, hence it is $\alpha$-continuous, satisfying $G(x, 0) = g(x)$ and $G(x, 1) = k(x)$ and become $\alpha$-homotopy, implies $g \cong_k k$. Therefore $\cong_k$ is an equivalence relation.

Equivalence of $\alpha$-homotopy on $\alpha$-C(M,N) gives following notions.

Definition 3.5 Let M and N be two topological spaces and $g: M \rightarrow N$ be $\alpha$-continuous map. Then the set of all $\alpha$-continuous maps from M to N which are $\alpha$-homotopic to g is called homotopy equivalence class of g. It is denoted by $[g]_\alpha$ or $\alpha$-$E_g$.

i.e. $[g]_\alpha = \{ h \in \alpha$-C(M,N) : $g \cong_h h \}$.

Definition 3.6 Let M and N be two topological spaces, then the set of all $\alpha$-homotopy equivalence classes over $\alpha$-C(M,N) is called $\alpha$-homotopy equivalence classes over $\alpha$-C(M,N).

This is denoted by $\alpha$-C(M,N)/$\cong$ or $[M,N]_\alpha$. That is $\alpha$-C(M,N)/$\cong = \{ [g]_\alpha : g \in \alpha$-C(M,N) \}.

Theorem 3.7 If topological space M is $\eta_\alpha$-space then $\alpha$-C(M,N) = C(M,N) (Here C(M,N denotes the set of all continuous maps from M to N).

Proof: Open sets and $\alpha$-open sets are same in $\eta_\alpha$-space.
\(G_{\alpha}(H(x)) = G(H(x)) = F_{\alpha}(x) = F(x)\). Hence for all \(t \in I\) the restrictions of \(F, F_{\alpha}: M \to R\) by \(x \to F_{\alpha}(x) = F(x)\) is \(\alpha\)-continuous. Therefore \(F\) is \(\alpha\)-homotopy between \(h_{\alpha}\) and \(\log\).

**Corollary 3.15** [9] If \(N\) is \(\eta_{\alpha}\)-space \(p: N \to R\) is \(\alpha\)-continuous and for all \(f \in \alpha\)-\((C, M, N)\) then there exist map \(p_{\#}:[M, N]_{\alpha} \to \{[M, N]_{\alpha}\}\) by \(p_{\#}([f])_{\alpha} = [\text{po\textunderscore }f]_{\alpha}\), (call pre induced map).

*Proof:* Here \(N\) is \(\eta_{\alpha}\)-space \(p: N \to R\) is \(\alpha\)-continuous. Defined \(p_{\#}:[M, N]_{\alpha} \to \{[M, N]_{\alpha}\}\) by \(p_{\#}([f])_{\alpha} = [\text{po\textunderscore }f]_{\alpha}\), is clearly well defined because, \(\forall \theta \in \{[M, N]_{\alpha}\}\) implies \(\theta = [\text{po\textunderscore }f]_{\alpha}\) for some \(f: M \to N\) \(\alpha\)-continuous. This gives \(p_{\#}\) \(M \to R\) is \(\alpha\)-continuous, implies \([\text{po\textunderscore }f]_{\alpha} \in \{[M, N]_{\alpha}\}\).

Also suppose for \(\theta \in \{[M, N]_{\alpha}\}\) such that \(\theta = [\text{po\textunderscore }f]_{\alpha}\), implies \(f \simeq_{\alpha} \text{po\textunderscore }f\) by theorem 3.10 \(\text{po\textunderscore }f \simeq_{\alpha} \text{po}\), so \([\text{po\textunderscore }f]_{\alpha} = [\text{po\textunderscore }f]_{\alpha}\) and \([\text{po\textunderscore }f]_{\alpha} = [\text{po\textunderscore }f]_{\alpha}\). Hence \(p_{\#}\) is well defined.

**Theorem 3.16** If \(R, N\) are \(\eta_{\alpha}\)-spaces, \(p: N \to R, q: R \to S\) are \(\alpha\)-continuous maps implies \(q_{\#}p_{\#}: N \to S\) is \(\alpha\)-continuous. Therefore each map induce \(p_{\#}:[M, N]_{\alpha} \to \{[M, N]_{\alpha}\}\), \(q_{\#}:[M, N]_{\alpha} \to \{[M, S]_{\alpha}\}\) and \((q_{\#}p_{\#})_{\alpha}:[M, N]_{\alpha} \to \{[M, N]_{\alpha}\}\) hence \(q_{\#}p_{\#}\) possible also \(q_{\#}\) and \((q_{\#}p_{\#})_{\alpha}\) have some domain and codomain.

Also \(\forall \theta \in \{[M, N]_{\alpha}\}\) implies \(\theta = [\text{po\textunderscore }f]_{\alpha}\) for some \(f: M \to N\) \(\alpha\)-continuous. Consider, \(\text{po\textunderscore }f_{\alpha}([\text{po\textunderscore }f_{\alpha}]_{\alpha}) = \{[\text{po\textunderscore }f_{\alpha}]_{\alpha}\}\)

\(\forall [f]_{\alpha} \in \{[M, N]_{\alpha}\}\).

*Proof:* Here \(R, N\) are \(\eta_{\alpha}\)-spaces, \(p: N \to R, q: R \to S\) are \(\alpha\)-continuous maps implies \(q_{\#}p_{\#}: N \to S\) is \(\alpha\)-continuous. Therefore each map induce \(p_{\#}:[M, N]_{\alpha} \to \{[M, N]_{\alpha}\}\), \(q_{\#}:[M, N]_{\alpha} \to \{[M, S]_{\alpha}\}\) and \((q_{\#}p_{\#})_{\alpha}:[M, N]_{\alpha} \to \{[M, N]_{\alpha}\}\) hence \(q_{\#}p_{\#}\) possible also \(q_{\#}\) and \((q_{\#}p_{\#})_{\alpha}\) have some domain and codomain.

Also \(\forall \theta \in \{[N, R]_{\alpha}\}\) implies \(\theta = [\text{po\textunderscore }f]_{\alpha}\) for some \(f: N \to R\) \(\alpha\)-continuous. Consider, \(\text{po\textunderscore }f_{\alpha}([\text{po\textunderscore }f_{\alpha}]_{\alpha}) = \{[\text{po\textunderscore }f_{\alpha}]_{\alpha}\}\)

\(\forall [f]_{\alpha} \in \{[N, R]_{\alpha}\}\).

*Proof:* Here \(M\) is \(\eta_{\alpha}\)-space, \(Id_{M}: M \to M\) defined as \(\text{Id}_{M}(x) = x\) and for all \(f \in \alpha\)-\((C, M, N)\) then \(\text{Id}_{M}(f)_{\alpha} = \text{Id}_{M}([f]_{\alpha} f)_{\alpha} = \{f\}_{\alpha} \in \{[M, N]_{\alpha}\}\).

*Proof:* Here \(M\) is \(\eta_{\alpha}\)-space, \(Id_{M}: M \to M\) defined as \(\text{Id}_{M}(x) = x\) is \(\alpha\)-continuous. This implies post induced map, \(\text{Id}_{M}(f)_{\alpha} = \text{Id}_{M}([f]_{\alpha} f)_{\alpha} = \{f\}_{\alpha} \in \{[M, N]_{\alpha}\}\).

Hence the proof.

**Definition 3.21** Let \(M\) and \(N\) be two topological spaces \(A \subseteq M\) and \(g, h: M \to N\) be two \(\alpha\)-continuous maps. Then a \(\alpha\)-relative homotopy respect to \(A\) is a map \(H: M \times I \to N\) such that, for all \(t \in I\) the restrictions of \(H, h: M \to N\) by \(x \to H_{\alpha}(x, t)\) is \(\alpha\)-continuous, satisfying \(H_{\alpha}(x, 0) = g(x)\), \(H_{\alpha}(x, 1) = h(x)\) for all \(x \in M\) and \(H_{\alpha}(x, 0) = g(x)\), \(\forall a \in A, \forall t \in I\). If such \(H\) is exist then \(H\) is called \(\alpha\)-relative homotopy between them, and \(g\) is called \(\alpha\)-relative homotopic to \(h\) respect to \(A\). We denote it by \(g \simeq_{\alpha}(A)\).

**Example 3.22** Let \(x, y: \mathbb{R} \to \mathbb{R}\) be \(g(x) = x\) and \(h(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x \in [0, 1] \\ 1 & \text{if } x \geq 1 \end{cases}\)

*Proof:* Refer theorem 3.3.

**Theorem 3.23** Every relative homotopy respect to \(A\) is \(\alpha\)-relative homotopy respect to \(A\).

*Proof:* Refer theorem 3.4.

**Theorem 3.24** \(\alpha\)-relative homotopy respect to \(A\) is an equivalence relation on set of all \(\alpha\)-continuous maps from \(M\) to \(N\).

*Proof:* Refer theorem 3.4.

**Theorem 3.25** [9] If \(N\) is \(\eta_{\alpha}\)-space, \(f, g: M \to N\) are \(\alpha\)-relative homotopy respect to \(A \subseteq M\) and \(h, k: N \to R\) are \(\alpha\)-relative homotopy respect to \(B \subseteq N\) and \(f(A) \subseteq B\) then \(h \simeq_{\alpha}(A, k)\) kog.
G(f(a), t) = hof(a) = kof(a) = kog(a), ∀a ∈ A, ∀t ∈ I. Since H₁ and G₁ are α-continuous maps and N is ηα-space gives 
\( G₀H₁ \) is α -continuous map and \( G₀H₁(x) = G( H₁(x), t) = F₁(x) = F(x, t) \). Hence for all t \( t \in I \) the
restrictions of F, F_1: M \rightarrow R by x \rightarrow F₁(x) = F(x, t) is α -continuous map. Therefore F is α-relative homotopy respect to
A between hof and kog.

**Definition 3.26** Let M and N be two topological spaces and g: M \rightarrow N be α-continuous map if there is an α-continuous map
h: M \rightarrow N such that h of = IdM and g of = IdN. Then h is
called α-homotopy equivalence to g, or α-homotopy inverse of g. Here h, g are called α-homotopy equivalences between M and N.

**Definition 3.27** Let M and N be two topological spaces, if there is an α-homotopy equivalences between them, then M and N are called α-homotopy equivalent to each other or same α-homotopy type.

**Example 3.28** The cylinder and circle are same α-homotopy type. It will follow by theorem 3.29.

**Theorem 3.29** Same homotopy type spaces are also same α-

**Note 3.30** Generally composition of two α-continuous maps

**Theorem 3.31** [9] The relation same α-homotopy type is
equivalence relation on set of ηα -topological spaces.

**Proof:** Reflexive: for every ηα-space M, obvious IdM, IdM is
a α-homotopy equivalences.

Symmetric: Let M, N and R are ηα-spaces, suppose M and N are same α-homotopy type, implies there is g, h a α-

**Theorem 3.32** [9, 12] If N, R are ηα-space then,

**ii) If p: N \rightarrow R is α-homeomorphism and for any space M then p_\alpha: [M, N]_\alpha \rightarrow [M, R]_\alpha is bijective.**

**Proof:** i) Here p: N \rightarrow R is α-homotopy equivalence implies existence of its α-homotopy inverse say q: R \rightarrow N such that
p of \approx qh and q of \approx IdN. For any space M, since p_\alpha: [M, N]_\alpha \rightarrow [M, R]_\alpha is well defined function, suppose
p_\alpha ([f]_\alpha) = p_\alpha ([g]_\alpha) for [f]_\alpha, [g]_\alpha \in [M, N]_\alpha, gives [poq]_\alpha = [poq]_\alpha which implies pof \approx qg. The hypothesis q: R \rightarrow N
is α-continuous, R is ηα-space and pre composition theorem 3.10 implies qof \approx qoq equivalently f \approx g so
[f]_\alpha = [g]_\alpha, therefore p_\alpha is injective. Also for any [h]_\alpha \in [M, R]_\alpha implies h: M \rightarrow R is α -

**Theorem 3.33** [9, 12] If N, M are ηα-space then,

**ii) q: M \rightarrow N is α-homotopy equivalence and for any space R then q_\alpha: [N, R]_\alpha \rightarrow [M, R]_\alpha is bijective.**

**Proof:** i) Here q: M \rightarrow N is α-homotopy equivalence implies existence of its α-homotopy inverse say p: N \rightarrow M such that
p of \approx IdM and q of \approx IdN. For any space M, since q_\alpha: [N, R]_\alpha \rightarrow [M, R]_\alpha is well defined function, suppose
q_\alpha ([f]_\alpha) = q_\alpha ([g]_\alpha) for [f]_\alpha, [g]_\alpha \in [N, R]_\alpha, gives [foq]_\alpha = [foq]_\alpha which implies foq \approx gq. The hypothesis p: N \rightarrow M
is α-continuous, Mis ηα-space and post composition lemma 3.12 implies qof \approx qoq equivalently f \approx g so
[f]_\alpha = [g]_\alpha, therefore q_\alpha is injective. Also for any [h]_\alpha \in [M, R]_\alpha implies h: M \rightarrow R is α -

**Theorem 3.34** A topological space M is called α-

**Example 3.36** Every star convex space is α-contracible.

**Theorem 3.37** If M is an α-contracible then it is same α-

**Theorem 3.38** Every α-contracible space is same α-

**Proof:** Since every homotopy is α-homotopy.
such that \( \{x_0\} \) and \( M \) are same \( \alpha \)-homotopy type. Since \( \{x_0\} \) and \( \ast \) are homeomorphic space, hence same \( \alpha \)-homotopy type. Here transivity works by same \( \alpha \)-homotopy type relation gives \( M \) is same \( \alpha \)-homotopy type of point space \( \ast \).

**Theorem 3.39** If \( N \) is \( \alpha \)-contractible \( \eta_\alpha \)-space and for any space \( M \) then \( [M,N]_\alpha \) is singleton.

Proof: Consider any \( g, h \in \alpha-C(M,N) \), implies \( g, h : M \to N \) are \( \alpha \)-continuous maps. Hypothesis is \( N \) is \( \alpha \)-contractible space implies \( \text{Id}_M \approx_\alpha c_{x_0} \), where \( \text{Id}_M, c_{x_0} : N \to N \) by lemma 3.12 \( \text{Id}_M \circ g \approx_\alpha c_{x_0} \circ g \) implies \( g \approx_\alpha k_{x_0} \), where \( k_{x_0} = c_{x_0} \circ g : M \to N \) a constant map. Similarly \( \text{Id}_M \circ h \approx_\alpha c_{x_0} \circ h \) implies \( h \approx_\alpha k_{x_0} \). Transitivity of \( \alpha \)-homotopy gives \( g \approx_\alpha h \), hence \( \approx_\alpha \) is universal relation. Therefore \( [M,N]_\alpha \) is singleton.

ACKNOWLEDGMENT

The research work of the first author is supported by the Council of Scientific and Industrial Research, India under grant award no: 09/1284(0001)2019-EMR-I with JRF exam roll no 400528.

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