α-Homotopy and α-Homotopy Type Spaces

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Abstract- **In this paper, we introduce the term of α-homotopy of α-continuous maps, α-homotopy equivalence classes and their consequences, same α-homotopy type, α-contractible spaces, and few properties. Also we present induced map and their properties, and consequences respect to some notions.**

Key Words: α-homotopy, same α-homotopy type, α-contractible space.
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I. INTRODUCTION

Algebraic methods were introduced in topology by Poincare around 1895. Chronologically, fundamental group of topological spaces and homology groups were some of important notions that enable us to compute for the topological spaces (Analysis Situs, 1895) [7, 9, 12]. As a result a separate branch took birth known as algebraic topology. Later, we see in the work of Poul Heegard, Barratt etc, in the extensive reading of homotopy and their equivalence among the spaces M and N. Well, there algebraic methods enhanced the scope of the classification themes among topological spaces.

Next to these algebraic notions, we have certain aspect of associations not necessarily a algebraic structure with topological space, namely the notion homotopy equivalence classes over space of all continuous function from a space to space. Deformations being standard idea in topology which in a way converts one space into anther space without losing its qualitative property. Homotopy equivalence classis set $[M, N]$ was first systemically studied by M.G. Barratt in 1955 [9]. The issue of characterization of homeomorphic topological spaces has partially solved by help of homotopy. The problems of classification of topological spaces according to their homotopy properties are also take important role in classification theory. But homotopy equivalence of spaces not necessarily topological equivalence. Homotopy equivalence is a weaker relation than topological equivalence, hence its class is bigger. Therefore homotopy equivalence has prominent role than homeomorphism. Deformation is a standard idea in topology, as morphisms can do so. The deformation is a notion that converts one space into other by the tool of homotopy. Hence homotopy theory takes powerful tools for this purpose [1,4,8,9]. In geometry, not only homotopy as well homology and cohomology are frequently used algebraic association.

Njastad [3,11] introduced α -open sets in a topological space and studied some properties. Further α-continuous map, and respective open map and closed map in topological space is studied by A.S.Mashhour, I.A.Hasanein [2]. Semi-homotopy and semi-fundamental groups studied by Ayhan Erciyes, Ali Aytek in and Tuncar Sahan [4]. We introduce such class of α-homotopy of α-continuous maps, same α-homotopy type, α-contractible space, and post and pre induced map of a α-continuous map. These have many rich consequences concern to contractible space, induced map, usual Homotopy and fundamental group. Here we place example for few respective results.

II. PRELIMINARIES

Throughout this paper L, M, N, R and S represent the topological spaces on which no separation axioms are assumed unless otherwise mentioned. Map here mean function and for a subset F of topological space M, the $M\backslash F$ denotes the complement of F in M. We recall the following definitions.

Definition 2.1 [3, 11] A subset F of a space M is said to be α-open set, if $F \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F)))$. Respectively called α-closed set, if $M \setminus F$ is α-open set in M. We denote the set of all α-open sets in M by $\alpha O(M)$.

Definition 2.2 [2] A map $h: M \to N$ is said to be α-continuous map, if $h^{-1}(F)$ is α-closed set of M, for every closed set F of N.

Definition 2.3 A map $h: M \to N$ is said to be α-irresolute map, if $h^{-1}(F)$ is α-closed set of M, for every α-closed set Fof N.

Definition 2.4 A bijection $h: M \to N$ is called α-homeomorphism, if both h and h^{-1} are α-continuous.

Definition 2.5 (Consider) A topological space M is called η_α-space, if every α-closed set is closed set.

Theorem 2.6 Every α-irresolute maps are α-continuous.

III. α-HOMOTOPY

In this section we introduce the notions of α-homotopy of α-continuous maps, α-relative homotopy, α-contractible space, α-homotopy type and post and pre induced map of a α-continuous map also discuss few properties respectively.

Definition 3.1 Let M and N be two topological spaces and $g, h: M \to N$ be two α-continuous maps. Then a α-homotopy is a map $H: M \times I \to N$ (Here, $I = [0, 1]$) such that, for all $t \in I$ the restrictions of H, $H_t: M \to N$ by $x \to H_t(x) = H(x, t)$ is α -continuous, satisfying $H_0(x) = H(x, 0) = g(x)$ and $H_1(x) = H(x, 1) = h(x)$ for all $x \in M$. If such H is exist then H is called α-homotopy between them, and g is called α-homotopic to h. We denote it by $g \simeq_\alpha h$.

And If g is α-homotophic to a constant map (i.e. $g \simeq_\alpha c_y$, where $c_y: M \to N, c_y(x) = y$ for some $y \in N$) then g is called α-nullhomotopic.

Example 3.2 Let $g, h: (\mathbb{R}, \eta_u) \to (\mathbb{R}, \eta_u)$ be $g(x) = 5x$, $h(x) = x + 2$ and $\eta_u$be usual topology on $\mathbb{R}$, then obviously map g is become α-homotopic to h, quite simple that one can check $H(x,t) = (1-t)\cdot 5x + t\cdot (x+2)$ is a α-homotopy, because we know every continuous maps are α-continuous.

Theorem 3.3 Every homotopy is α-homotopy (Every homotopy between continuous maps is α-homotopy between same α-continuous maps.).

Proof: It fallowed from the fact, every continuous map is α-continuous map.

Theorem 3.4 [1,6,8] The relation α-homotopic is an equivalence relation on the set $\alpha\text{-}C(M,N)$ of all α-continuous maps from topological space M to N.

Proof: Let M and N be two topological spaces then,

Reflexivity: If $g \in \alpha\text{-}C(M,N)$ i.e. $g: M \to N$ is α-continuous map. Define $H: M\times I \to N$ by $H(x,t) = g(x)$ for all $x \in M$ and all $t \in I$, then H become α-homotopy between g and g. Hence $g \simeq_\alpha g$.

Symmetry: Suppose $g,h \in \alpha\text{-}C(M,N)$ and $g \simeq_\alpha h$ implies there is a α-homotopy map $H: M\times I \to N$ such that, for all $t \in I$ the restrictions of H, $H_t: M \to N$ by $x \to H_t(x) = H(x,t)$ is α-continuous, satisfying $H(x,0) = g(x)$ and $H(x,1) = h(x)$ for all $x \in M$. Define $F: M\times I \to N$, by $F(x,t) = H\big(x,(1-t)\big)$, obviously for all $t \in I$ the restrictions of F, $F_t: M \to N$ by $x \to F_t(x) = H\big(x,(1-t)\big) = H(x,t_0) = H_{t_0}$ for some $t_0 \in I$, hence it is α-continuous, satisfying $F(x,0) = h(x)$ and $F(x,1) = g(x)$ and become α-homotopy between h and g, implies $h \simeq_\alpha g$.

Transitivity: Suppose $g,h,k \in \alpha\text{-}C(M,N)$ with $g \simeq_\alpha h$ and $h \simeq_\alpha k$ implies there are α- homotopies, $H: M\times I \to N$ such that, for all $t \in I$ the restrictions of H, $H_t: M \to N$ by $x \to H_t(x) = H(x,t)$ is α-continuous, satisfying $H(x,0) = g(x)$ and $H(x,1) = h(x)$ for all $x \in M$. And $F: M\times I \to N$ such that, for all $t \in I$ the restrictions of F, $F_t: M \to N$ by $x \to F_t(x) = F(x,t)$ is α-continuous, satisfying with $F(x,0) = h(x)$ and $F(x,1) = k(x)$ for all $x \in M$. Define map $G: M\times I \to N$ by,

$G(x,t) = \begin{cases} H(x,2t) & \text{if } t \in [0,1/2] \\ F(x,2t-1) & \text{if } t \in [1/2,1] \end{cases}$. Obvious G become α-homotopy between g and k, because $G_t(x) = G(x,t) = H_{t_0}$ or F_{t_0} for some $t_0 \in I$, hence it is α-continuous, satisfying $G(x,0) = g(x)$ and $F(x,1) = k(x)$ and become α-homotopy, implies $g \simeq_\alpha k$. Therefore $\simeq_\alpha$ is an equivalence relation.

Equivalence of α-homotopy on $\alpha\text{-}C(M,N)$ gives following notions.

Definition 3.5 Let M and N be two topological spaces and $g: M \to N$ be α-continuous map. Then the set of all α-continuous maps from M to N which are α-homotopic to g is called homotopy equivalence class of g. It is denoted by $[g]_\alpha$ or $\alpha\text{-}E_g$.

i.e. $[g]_\alpha = \{h \in \alpha\text{-}C(M,N) : g \simeq_\alpha h\}$.

Definition 3.6 Let M and N be two topological spaces, then the set of all α-homotopy equivalence classes over $\alpha\text{-}C(M,N)$ is called α-homotopy equivalence classes over $\alpha\text{-}C(M,N)$. This is denoted by $\alpha\text{-}C(M,N)/_{\simeq_\alpha}$ or $[M,N]_\alpha$. That is $\alpha\text{-}C(M,N)/_{\simeq_\alpha} = \{[g]_\alpha : g \in \alpha\text{-}C(M,N)\}$.

Theorem 3.7 If topological space M is η_α-space then $\alpha\text{-}C(M,N) = C(M,N)$ (Here C(M,N denotes the set of all continuous maps from M to N).

Proof: Open sets and α-open sets are same in η_α-space.

Theorem 3.8 If $g \in C(M,N)$ then $[g] \subset [g]_\alpha$ where $[g]$ is homotopy equivalence class of g. But $C(M,N)/_{\simeq}$ and $\alpha\text{-}C(M,N)/_{\simeq_\alpha}$ are not nessasarly comparable.

Proof: Since every continuous map is α-continuous and converse need not hold.

Lemma 3.9 [9] If $g,h: M \to N$ are α-continuous maps, $g \simeq_\alpha h$ and $k: N \to R$ is continuous then $kog \simeq_\alpha koh$.

Proof: The $g \simeq_\alpha h$, of $g,h: M \to N$ α-continuous maps guarantee that there is a α-homotopy $H: M\times I \to N$ such that, for all $t \in I$ the restrictions of H, $H_t: M \to N$ by $x \to H_t(x) = H(x,t)$ is α-continuous, satisfying $H(x,0) = g(x)$ and $H(x,1) = h(x)$ for all $x \in M$. Now define the map $F = koH: M\times I \to R$ clearly, for all $t \in I$ the restrictions of F, $F_t: M \to N$ by $x \to F_t(x) = koH(x,t) = koH_t(x)$ is α-continuous because H is α-continuous and k continuous. Also satisfying $F(x,0) = kog(x)$ and $F(x,1) = koh(x)$ for all $x \in M$. Hence the result.

Theorem 3.10 If N is η_α-space, $g,h: M \to N$ are α-continuous maps, $g \simeq_\alpha h$ and $k: N \to R$ is α-continuous then $kog \simeq_\alpha koh$.

Proof: Similar to lemma 3.9.

Theorem 3.11 If N is η_α-space, $g,h: M \to N$ are α-irresolute maps, $g \simeq_\alpha h$ and $k: N \to R$ is α-continuous then $kog \simeq_\alpha koh$.

Proof: Refer lemma 3.9.

Lemma 3.12 [9] If N is η_α-space $g,h: N \to R$ are α-continuous maps, and $g \simeq_\alpha h$ and $f: M \to N$ is α-continuous then $gof \simeq_\alpha hof$.

Proof: Hypothesis is $g,h: N \to R$ are α-continuous maps, and $g \simeq_\alpha h$. This implies there is α-homotopy $H: N\times I \to R$ such that, for all $t \in I$ the restrictions of H, $H_t: N \to R$ by $y \to H_t(y) = H(y,t)$ is α-continuous satisfying $H_0(y) = H(y,0) = g(y)$ and $H_1(y) = H(y,1) = h(y)$ for all $y \in N$. Define $F: M\times I \to R$ by $F(x,t) = H(f(x),t)$, this is well define map and $F_0(x) = H(f(x),0) = gof(x)$ and $F_1(x) = H(f(x),1) = hof(x)$ for all $x \in M$. Also for all $t \in I$ the restrictions of F is, $F_t: M \to R$ by $x \to F_t(x) = H(f(x),t) = H_t\big(f(x)\big) = H_t of(x)$ which is α-continuous because of N is η_α-space and $f: M \to N$ is α-continuous.

Theorem 3.13 If N is η_α-space, $g,h: N \to R$ are α-irresolute maps, $g \simeq_\alpha h$ and $f: M \to N$ is α-continuous then $gof \simeq_\alpha hof$.

Proof: Since every α-irresolute map is α-continuous map.

Lemma 3.14 [9] If N is η_α-space, $f,g: M \to N$ and $h,k: N \to R$ are α-continuous maps and $f \simeq_\alpha g$, $h \simeq_\alpha k$ then $hof \simeq_\alpha kog$.

Proof: Hypothesis is $f,g: M \to N$ and $h,k: N \to R$ are α-continuous maps and $f \simeq_\alpha g$, $h \simeq_\alpha k$. This implies there is α-homotopy $H(x,t)$ between f and g, such that, for all $t \in I$ the restrictions of H, $H_t: M \to N$ by $x \to H_t(x) = H(x,t)$ is α-continuous maps satisfying $H_0(x) = H(x,0) = f(x)$ and $H_1(x) = H(x,1) = g(x)$ for all $x \in M$. Also there is α-homotopy $G(y,t)$ between h and k, such that, for all $t \in I$ the restrictions of G, $G_t: M \to N$ by $y \to G_t(y) = G(y,t)$ is α-continuous map satisfying $G_0(y) = G(y,0) = h(y)$ and $G_1(y) = G(y,1) = k(y)$ for all $y \in N$. Define $F: M\times I \to R$ as $F(x,t) = G(H(x,t),t)$, which become map and $F(x,0) = G(H(x,0),0) = hof(x)$, $F(x,1) = G(H(x,1),1) = kog(x)$. Since H_t and G_t are α-continuous map and N is η_α-space gives $G_t oH_t$ is α-continuous map and $G_t oH_t(x) =$

$G_t(H_t(x)) = G(H(x,t),t) = F_t(x) = F(x,t)$. Hence for all $t \in I$ the restrictions of F, $F_t: M \to R$ by $x \to F_t(x) = F(x,t)$ is α-continuous map. Therefore F is α-homotopy between hof and kog.

Corollary 3.15 [9] If N is η_α-space $p: N \to R$ is α-continuous and for all $f \in \alpha\text{-}C(M,N)$ then there exist map $p_*: [M,N]_\alpha \to [M,R]_\alpha$ by $p_*([f]_\alpha) = [pof]_\alpha$, (call pre induced map).

Proof: Here N is η_α-space $p: N \to R$ is α-continuous. Defined $p_*: [M,N]_\alpha \to [M,R]_\alpha$ by $p_*([f]_\alpha) = [pof]_\alpha$, is clearly well defined because, $\forall\theta \in [M,N]_\alpha$ implies $\theta = [f]_\alpha$ for some $f: M \to N$ α-continuous. This gives $pof: M \to R$ is α-continuous, implies$[pof]_\alpha = p_*([f]_\alpha) \in [M,R]_\alpha$.

Also suppose for $\theta \in [M,N]_\alpha$ such that $\theta = [f]_\alpha = [g]_\alpha$, implies $f \simeq_\alpha g$ by theorem 3.10 $pof \simeq_\alpha pog$, so $[pof]_\alpha = [pog]_\alpha$ and $p_*([f]_\alpha) = p_*([g]_\alpha)$. Hence p_* is well defined.

Theorem 3.16 If N, R are η_α-spaces, $p: N \to R$, $q: R \to S$ are α-continuous maps and for all $\forall\theta \in [M,N]_\alpha$ then $(qop)_*(\theta) = q_*op_*(\theta)$.

Proof: N, R are η_α-space and $p: N \to R$, $q: R \to S$ are α-continuous maps implies $qop: N \to S$ is α-continuous. Therefore each map induce $p_*: [M,N]_\alpha \to [M,R]_\alpha$, $q_*: [M,R]_\alpha \to [M,S]_\alpha$ and $(qop)_*: [M,N]_\alpha \to [M,S]_\alpha$, hence q_*op_* possible also q_*op_* and $(qop)_*$ have same domain and codamain.

Also $\forall\theta \in [M,N]_\alpha$ implies $\theta = [f]_\alpha$ for some $f: M \to N$ α-continuous. Consider,

$$\begin{aligned}(qop)_*(\theta) &= (qop)_*([f]_\alpha)\\ &= [(qop)of]_\alpha\\ &= [qo(pof)]_\alpha\\ &= q_*([pof]_\alpha)\\ &= q_*op_*([f]_\alpha)\\ &= q_*op_*(\theta)\end{aligned}$$

Theorem 3.17 If N is η_α-space, $Id_N: N \to N$ defined as $Id_N(x) = x$ and for all $f \in \alpha\text{-}C(M,N)$ then $(Id_N)_*([f]_\alpha) = Id_{[M,N]_\alpha}([f]_\alpha)$, $\forall[f]_\alpha \in [M,N]_\alpha$.

Proof: Here N is η_α-space, $Id_N: N \to N$ defined as $Id_N(x) = x$ is α-continuous. This implies pre induced map become, $(Id_N)_*: [M,N]_\alpha \to [M,N]_\alpha$ by $(Id_N)_*([f]_\alpha) = [Id_N of]_\alpha = [f]_\alpha = Id_{[M,N]_\alpha}([f]_\alpha), \forall[f]_\alpha \in [M,N]_\alpha$. Hence the proof.

Corollary 3.18 [9] If N is η_α-space $q: M \to N$ is α-continuous and for all $f \in \alpha\text{-}C(N,R)$ then there exist map $q_\#: [N,R]_\alpha \to [M,R]_\alpha$ by $q_\#([f]_\alpha) = [foq]_\alpha$ (call post induced map).

Proof: Here Nis η_α-space, $q: M \to N$ is α-continuous. Defined $q_\#: [N,R]_\alpha \to [M,R]_\alpha$ by $q_\#([f]_\alpha) = [foq]_\alpha$, is clearly well defined because, $\forall\theta \in [N,R]_\alpha$ implies $\theta = [f]_\alpha$ for some $f: N \to R$ α-continuous. This gives $foq: M \to R$ is α-continuous, implies $[foq]_\alpha = q_\#([f]_\alpha) \in [M,R]_\alpha$.

Also suppose for $\theta \in [N,R]_\alpha$ such that $\theta = [f]_\alpha = [g]_\alpha$, implies $f \simeq_\alpha g$ by lemma 3.12 $foq \simeq_\alpha goq$, so $[foq]_\alpha = [goq]_\alpha$ and $q_\#([f]_\alpha) = q_\#([g]_\alpha)$. Hence $q_\#$ is well defined.

Theorem 3.19 If M, N are η_α-space, $p: L \to M$, $q: M \to N$ are α-continuous maps and for all $\forall\theta \in [N,R]_\alpha$ then $(qop)_\#(\theta) = p_\# oq_\#(\theta)$.

Proof: M, N are η_α-space and $p: L \to M$, $q: M \to N$ are α-continuous maps implies $qop: L \to N$ is α-continuous. Therefore each map induce $p_\#: [M,R]_\alpha \to [L,R]_\alpha$, $q_\#: [N,R]_\alpha \to [M,R]_\alpha$ and $(qop)_\#: [N,R]_\alpha \to [L,R]_\alpha$, hence $p_\# oq_\#$ possible also $p_\# oq_\#$ and $(qop)_\#$ have same domain and codamain.

Also $\forall\theta \in [N,R]_\alpha$ implies $\theta = [f]_\alpha$ for some $f: N \to R$ α-continuous. Consider,

$$\begin{aligned}(qop)_\#(\theta) &= (qop)_\#([f]_\alpha)\\ &= [fo(qop)]_\alpha\\ &= [(foq)op]_\alpha\\ &= p_\#([foq]_\alpha)\\ &= p_\# oq_\#([f]_\alpha)\\ &= p_\# oq_\#(\theta)\end{aligned}$$

Theorem 3.20 If M is η_α-space, $Id_M: M \to M$ defined as $Id_M(x) = x$ and for all $f \in \alpha\text{-}C(M,N)$ then $(Id_M)_\#([f]_\alpha) = Id_{[M,N]_\alpha}([f]_\alpha)$, $\forall[f]_\alpha \in [M,N]_\alpha$.

Proof: Here M is η_α-space, $Id_M: M \to M$ defined as $Id_M(x) = x$ is α-continuous. This implies post induced map, $(Id_M)_\#: [M,N]_\alpha \to [M,N]_\alpha$ become $(Id_M)_\#([f]_\alpha) = [(foId_M)]_\alpha = [f]_\alpha = Id_{[M,N]_\alpha}([f]_\alpha), \forall[f]_\alpha \in [M,N]_\alpha$. Hence the proof.

Definition 3.21 Let M and N be two topological spaces $A \subset M$ and $g, h: M \to N$ be two α-continuous maps. Then a α-relative homotopy respect to A is a map $H: M \times I \to N$ such that, for all $t \in I$ the restrictions of H, $H_t: M \to N$ by $x \to H_t(x) = H(x,t)$ is α-continuous, satisfying $H_0(x) = H(x,0) = g(x)$, $H_1(x) = H(x,1) = h(x)$ for all $x \in M$ and $H(a,t) = g(a) = h(a), \forall a \in A, \forall t \in I$. If such H is exist then H is called α-relative homotopy between them, and g is called α-relative homotopic to h respect to A. We denote it by $g \simeq_{\alpha(A)} h$.

Example 3.22 Let $g, h: \mathbb{R} \to \mathbb{R}$ be $g(x) = x$ and

$h(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } x \in [0,1] \\ 1 & \text{if } x \ge 1 \end{cases}$ then $g \simeq_{\alpha(A)} h$, where $A = [0,1]$.

Theorem 3.23 Every relative homotopy respect to A is α-relative homotopy respect to A.

Proof: Refer theorem 3.3.

Theorem 3.24 α-relative homotopy respect to A is an equivalence relation on set of all α-continuous maps from M to N.

Proof: Refer theorem 3.4.

Theorem 3.25 [9] If N is η_α-space, $f, g: M \to N$ are α-relative homotopy respect to $A \subset M$, and $h, k: N \to R$ are α-relative homotopy respect to $B \subset N$ and $f(A) \subset B$ then $hof \simeq_{\alpha(A)} kog$.

Proof: Hypothesis $f, g: M \to N$ and, $h, k: N \to R$ are α-continuous maps and $f \simeq_{\alpha(A)} g$, $h \simeq_{\alpha(B)} k$. This implies there is α-relative homotopy $H(x,t)$ respect to A, between f and g, such that, for all $t \in I$ the restrictions of H, $H_t: M \to N$ by $x \to H_t(x) = H(x,t)$ is α-continuous maps satisfying $H_0(x) = H(x,0) = f(x)$, $H_1(x) = H(x,1) = g(x)$ for all $x \in M$ and $H(a,t) = f(a) = g(a), \forall a \in A, \forall t \in I$. Also there is α-relative homotopy $G(y,t)$ respect to B, between h and k, such that, for all $t \in I$ the restrictions of G, $G_t: M \to N$ by $y \to G_t(y) = G(y,t)$ is α-continuous map satisfying $G_0(y) = G(y,0) = h(y)$, $G_1(y) = G(y,1) = k(y)$ for all $y \in N$ and $G(b,t) = h(b) = k(b), \forall b \in B, \forall t \in I$.

Define $F: M \times I \to R$as $F(x,t) = G(H(x,t),t)$become map and $F(x,0) = G(H(x,0),0) = hof(x)$, $F(x,1) = G(H(x,1),1) = kog(x)$ and $F(a,t) = G(H(a,t),t) =$

$G(f(a),t) = hof(a) = kof(a) = kog(a), \forall a \in A, \forall t \in I$. Since H_t and G_t are α-continuous map and N is η_α-space gives $G_t o H_t$ is α-continuous map and $G_t o H_t(x) = G_t(H_t(x) = G(H(x,t),t) = F_t(x) = F(x,t)$. Hence for all $t \in I$ the restrictions of F, $F_t: M \to R$ by $x \to F_t(x) = F(x,t)$ is α-continuous map. Therefore F is α-relative homotopy respect to A between hof and kog .

Definition 3.26 Let M and N be two topological spaces and $g: M \to N$ be α-continuous map if there is α-continuous map $h: M \to N$ such that $hog \simeq_\alpha Id_M$ and $goh \simeq_\alpha Id_N$. Then h is called α-homotopy equivalence to g, or α-homotopy inverse of g. Here g, h are called α-homotopy equivalences between M and N.

Definition 3.27 Let M and N be two topological spaces, if there is a α-homotopy equivalences between them, then M and N are called α-homotopy equivalent to each other or same α-homotopy type.

Example 3.28 The cylinder and circle are same α-homotopy type. It will follow by theorem 3.29.

Theorem 3.29 Same homotopy type spaces are also same α-homotopy type spaces. Converse need not hold.

Proof: Fallowed by the theorem 3.3.

Note 3.30 Generally composition of two α-continuous maps are need not α-continuous. Therefore same α-homotopy type relation even thou reflexive, symmetric but not transitive, so it will not induce equivalence relation. Therefore our intuition, partition of topological space under same α-homotopy type is not possible. Since every α-irresolute map are α-continuous map and composition of two α-irresolute maps are α-irresolute, also confining set of all topological spaces are η_α-spaces on which the relation same α-homotopy type become equivalence relation as fallow.

Theorem 3.31 [9] The relation same α-homotopy type is equivalence relation on set of η_α- topological spaces.

Proof: Reflexive: for every η_α- space M, obvious Id_M, Id_M is a α-homotopy equivalences.

Symmetric: Let M, N and R are η_α-spaces, suppose M and N are same α-homotopy type, implies there is g, h a α-homotopy equivalences. Obvious h, g gives α-homotopy equivalences between N and M.

Transitive: Let M, N and R are η_α-spaces and M, N are same α-homotopy type and N, R are same α-homotopy type. This implies there exist a α-homotopy equivalences f, g between M and N, that is gof $\simeq_\alpha Id_M$and fog $\simeq_\alpha Id_N$ via a α-homotopy. Also there exist a α-homotopy equivalences h, k between N and R, such that koh $\simeq_\alpha Id_N$ and hok $\simeq_\alpha Id_R$ via a α-homotopy. It is true that $hof: M \to R$ and $gok: R \to M$ become α-homotopy equivalences between M and R. Because since the hypothesis koh $\simeq_\alpha Id_N$ and fact $f \simeq_\alpha f$, and by the lemma 3.14 we have kohof $\simeq_\alpha Id_N$of again by lemma 3.14 we can have gokohof $\simeq_\alpha$ go$Id_N$of , this implies gokohof $\simeq_\alpha$ go$Id_N$of $\simeq_\alpha Id_M$ by hypothesis. Similarly hofogok $\simeq_\alpha Id_R$. Therefore hof and gok are α-homotopy equivalences between M and R. Hence same α-homotopy type is equivalence relation.

Theorem 3.32 [9, 12] If N, R are η_α-space then,

i) If $p: N \to R$ is a α-homotopy equivalence and for any space M then $p_*: [M,N]_\alpha \to [M,R]_\alpha$ is bijective.

ii) If $p: N \to R$ is α-homeomorphism and for any space M then $p_*: [M,N]_\alpha \to [M,R]_\alpha$ is bijective.

Proof: i) Here $p: N \to R$ is α-homotopy equivalence implies existence of its α-homotopy inverse say $q: R \to N$ such that poq $\simeq_\alpha Id_R$ and qop $\simeq_\alpha Id_N$. For any space M, since $p_*: [M,N]_\alpha \to [M,R]_\alpha$ is well defined function, suppose $p_*([f]_\alpha) = p_*([g]_\alpha)$ for $[f]_\alpha, [g]_\alpha \in [M,N]_\alpha$, gives $[pof]_\alpha = [pog]_\alpha$ which implies pof $\simeq_\alpha$ pog. The hypothesis $q: R \to N$ is α-continuous, R is η_α-space and pre composition theorem 3.10 implies qopof $\simeq_\alpha$ qopog equivalently $f \simeq_\alpha g$ so $[f]_\alpha = [g]_\alpha$, therefore p_* is injective.

Also for any $[h]_\alpha \in [M,R]_\alpha$ implies $h: M \to R$ is α-continuous map, and hypothesis $q: R \to N$ is α-continuous, R is η_α-space guarantee $qoh: M \to N$ is α-continuous. Therefore $[qoh]_\alpha \in [M,N]_\alpha$ such that $p_*([qoh]_\alpha) = [poqoh]_\alpha = [h]_\alpha$, hence surjective.

ii) $p: N \to R$ is α-homeomorhism then p and p^{-1} become α-homotopy equivalences.

Theorem 3.33 [9,12] If N, M are η_α-space then,

i) $q: M \to N$ is a α-homotopy equivalence and for any space R then $q_\#: [N,R]_\alpha \to [M,R]_\alpha$ is bijective.

ii) $q: M \to N$ is α-homeomorhism and for any space R then $q_\#: [N,R]_\alpha \to [M,R]_\alpha$ is bijective.

Proof: i) Here $q: M \to N$ is α-homotopy equivalence implies existence of its α-homotopy inverse say $p: N \to M$ such that poq $\simeq_\alpha Id_M$ and qop $\simeq_\alpha Id_N$. For any space M, since $q_\#: [N,R]_\alpha \to [M,R]_\alpha$ is well defined function, suppose $q_\#([f]_\alpha) = q_\#([g]_\alpha)$ for $[f]_\alpha, [g]_\alpha \in [N,R]_\alpha$, gives $[foq]_\alpha = [goq]_\alpha$ which implies foq $\simeq_\alpha$ goq. The hypothesis $p: N \to M$ is α-continuous, Mis η_α-space and post composition lemma 3.12 implies foqop $\simeq_\alpha$ goqop equivalently $f \simeq_\alpha g$ so $[f]_\alpha = [g]_\alpha$, therefore $q_\#$ is injective.

Also for any $[h]_\alpha \in [M,R]_\alpha$ implies $h: M \to R$ is α-continuous map, and hypothesis $p: N \to M$ is α-continuous, M is η_α-space guarantee $hop: N \to R$ is α-continuous. Therefore $[hop]_\alpha \in [N,R]_\alpha$ such that $q_\#([hop]_\alpha) = [hopoq]_\alpha = [h]_\alpha$, hence surjective.

ii) $q: N \to R$ is α-homeomorhism then q and q^{-1} become α-homotopy equivalences.

Definition 3.34 A topological space M is called α-contractible if Id_M is α-null homotophic.

Theorem 3.35 Every contractible space is α-contractible.

Proof: Since every homotopy is α-homotopy.

Example 3.36 Every star convex space is α-contractible.

Theorem 3.37 If M is α-contractible then it is same α-homotopy type to a point space in M.

Proof: Let M be a α-contractible space, implies identity map Id_M on M is α-null homotopic. That is there exist $c_{x_0}: M \to M$ by $c_{x_0}(x) = x_0$, such that $Id_M \simeq_\alpha c_{x_0}$. Define $h: \{x_0\} \to M$ by $h(x) = x_0$, and $g: M \to \{x_0\}$ by $g(x) = x_0$. Then $hog: M \to M$ become $hog(x) = x_0 = c_{x_0}$, so $hog = c_{x_0} \simeq_\alpha Id_M$, also $goh: \{x_0\} \to \{x_0\}$ become $goh(x) = x_0 = Id_{\{x_0\}}$, so $goh(x) \simeq_\alpha Id_{\{x_0\}}(x)$. Therefore $\{x_0\}$ and M are same α-homotopy type.

Theorem 3.38 Every α-contractible space is same α-homotopy type of a point space.

Proof: Let M be a α-contractible space, and $\{*\}$ be any point space. By theorem 3.37 there exist a point space $\{x_0\}$ in M such that $\{x_0\}$ and M are same α-homotopy type. Since $\{x_0\}$ and $\{*\}$ are homeomorphic space, hence same α-homotopy type. Here transivity works by same α-homotopy type relation gives M is same α-homotopy type of point space $\{*\}$.

Theorem 3.39 If N is α-contractible η_α-space and for any space M then $[M, N]_\alpha$ is singleton.

Proof: Consider any $g, h \in \alpha\text{-}C(M, N)$, implies $g, h : M \to N$ are α-continuous maps. Hypothesis is N is α-contractible space implies $Id_M \simeq_\alpha c_{x_0}$, where $Id_M, c_{x_0} : N \to N$ by lemma 3.12 $Id_M og \simeq_\alpha c_{x_0} og$ implies $g \simeq_\alpha k_{x_0}$, where $k_{x_0} = c_{x_0} og : M \to N$ a constant map. Similarly $Id_M oh \simeq_\alpha c_{x_0} oh$ implies $h \simeq_\alpha k_{x_0}$. Transitivity of α-homotopy gives $g \simeq_\alpha h$, hence $\simeq_\alpha$ is universal relation. There fore $[M, N]_\alpha$ is singleton.

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