Graph Theoritical Approach:
The Complete Set of ODCH Coxeter Polyhedra P10-1

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Abstract: We classified the combinatorial structures of 3-dimensional bounded ODCH (orderable and deformable compact hyperbolic) Coxeter polyhedra up to symmetry. Using Plantri graphs and graph theory, we proved that the number of such combinatorial polyhedrons is five. One of such combinatorial polyhedra is P10-1 which has 10 vertices. Using graph theory, we find that there are five orderable and deformable compact hyperbolic polyhedra P10-1 up to symmetry.

Keywords: Planner graph, Orbifold, Coxeter polyhedron, Hyperbolic space.
MSC 2010 Codes: 15A45, 05C50, 05C69,

1. Introduction

A *n-dimensional orbifold* is a topological space with a structure based on the quotient space of $\square^n$ by a finite group action. An orbifold is called *good* if its universal cover is a manifold. We will concentrate only on good orbifolds.

To give a *hyperbolic structure* on an orbifold, we model it locally by the orbit spaces of finite subgroups of $PO(1,n)$ acting on open subsets of H^n . Similarly, to put a real projective structure on an orbifold, we model it locally by the orbit spaces of finite subgroups of $PGL(n+1,\square)$ acting on open subsets of $\square P^n$.

A real projective structure on an orbifold *M* implies that *M* has a universal cover $\tilde{M}$ and the deck transformation group $\pi_1(M)$ acting on $\tilde{M}$ so that $\dfrac{\tilde{M}}{\pi_1(M)}$ is homeomorphic to *M*.

A *convex* set in $\square P^n$ is a convex set in an affine patch. If we use Klein's model of a *n*-dimensional hyperbolic space, then is an open ball in $\square P^n$ and $PO(1,n)$ is a subgroup of $PGL(n+1,\square)$ preserving H^n . Therefore H^n can be imbedded in an $(n+1)$-dimensional real vector space *V* as an upper part of hyperboloid

$$-x_1^2 + x_2^2 + ... + x_{n+1}^2 = -1$$

Hence hyperbolic orbifolds naturally have real projective structures. But a real projective structure of an orbifold may not have hyperbolic structure.

We will concentrate on 3-dimensional compact hyperbolic orbifolds whose base spaces are homeomorphic to a convex polyhedron and whose sides are silvered and each edge is given

an order. If the dihedral angle of an edge of a compact hyperbolic polyhedron is $\dfrac{\pi}{n}$ then we say that the order of the edge is n where n is a positive number.

Definition 1.0.1. Let X be S^3, E^3, or H^3. Let *Isom(X)* denotes the group of isometries of *X*. A *Coxeter polyhedron* is a convex polytope in *X* whose dihedral angles are all integer sub-multiples of π. Let *P* be a 3-dimensional Coxeter polyhedron and Γ be the group generated by the reflections in the faces of *P*. Then Γ is a discrete group of Isom(*X*) and *P* is its fundamental polyhedron. Conversely, every discrete group Γ of Isom(*X*) can be obtained from a Coxeter polyhedron *P* such that *P* is its fundamental polyhedron. The number of faces intersect at vertex is called the *degree* of that vertex. Also the edge order of edges of a Coxeter polyhedron are positive integers.

Definition 1.0.2. Let *P* be a fixed convex polyhedron. Let us assign orders at each edge. Let *e* be the number of edges and e_2 be the numbers of order-two. Let *f* be number of sides. We remove any vertex of *P* which has more than three edges ending or with orders of the edges ending there is not of the form

$$(2,2,n), n \geq 2, (2,3,3), (2,3,4), (2,3,5),$$

i.e., orders of spherical triangular groups. This make *P* into an open 3-dimensional orbifold. Let $\hat{P}$ denote the differential orbifold with sides silvered and the edge orders realized as assigned from *P* with vertices removed. We say that $\hat{P}$ has a *Coxeter orbifold structure*.

Definition 1.0.3. The deformation space $\hat{P}$ of projective structures on an orbifold $\hat{P}$ is the space of all projective structures on $\hat{P}$ quotient by isotopy group actions of $\hat{P}$.

Definition 1.0.4. We say *P* is *orderable* if we can order the sides of *P* so that each sides meets sides meets sides of higher index in less than or equal to 3 edges.
Example 1.0.5. Cube and dodecahedron are not satisfying orderability condition.

Definition 1.0.6. Let $\hat{P}$ be the orbifold structure of a 3-dimensional polyhedron *P*. We say that the orbifold structure $\hat{P}$ is *orderable* if the sides of *P* can be ordered so that each side has no more than three edges which are either of order 2 or included in a side of higher index. $\hat{P}$ is trivalent if each side *F* has three or less number of edges of order two or edges belonging to sides of higher class than *F*.

Definition 1.0.7. A combinatorial polyhedron is a 3-ball whose boundary sphere S^2 is equipped with a cell complex whose 0-cells, 1-cells and 2-cells will also be called vertices, edges and faces respectively, and which can be realized as a convex polyhedron. Topologically, a compact polyhedron *P* is a combinatorial polyhedron. A polyhedron is called *trivalent* if degree of each vertex is 3.
Remark 1.0.8. For our convenient, we will use notation in short as:
  (1) A compact hyperbolic Coxeter polyhedron is as CH-Coxeter polyhedron.
  (2) An orderable and projectively deformable compact hyperbolic Coxeter orbifold is as ODCH-Coxeter orbifold.

In my previous article [APPLICATION OF PLANTRI GRAPH: All COMBINATORIAL STRUCTURE OF ORDERABLE AND DEFORMABLE COMPACT COXETER HYPERBOLIC POLYHEDRA], we proved the following proposition.

Proposition 1.0.9. *Let P be a 3-dimensional CH-Coxeter polyhedron and $\hat{P}$ be its Coxeter orbifold structure. Suppose that $\hat{P}$ is orderable and projectively deformable. Then the total number of combinatorial polyhedral of such P is 5 and P is one of the combinatorial polyhedrons T, P6, P8-1, P10-1 and P10-2. The figure of P10-1 is as follows:*
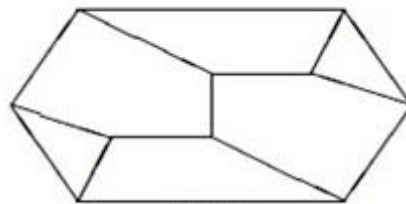
P10-1
Figure 1

In this article, we find the complete set of orderable and deformable compact Coxeter hyperbolic polyhedron P10-1.

Theorem 1.0.10. *Let P10-1 be a compact hyperbolic Coxeter polyhedrom figure-1 and $\hat{P}$10-1 be its Coxeter orbifold structure. Suppose that $\hat{P}$10-1 is orderable and projectively deformable. Then the total number of such P10-1 is 5 and P10-1 is one of the polyhedrons in figure 2.*
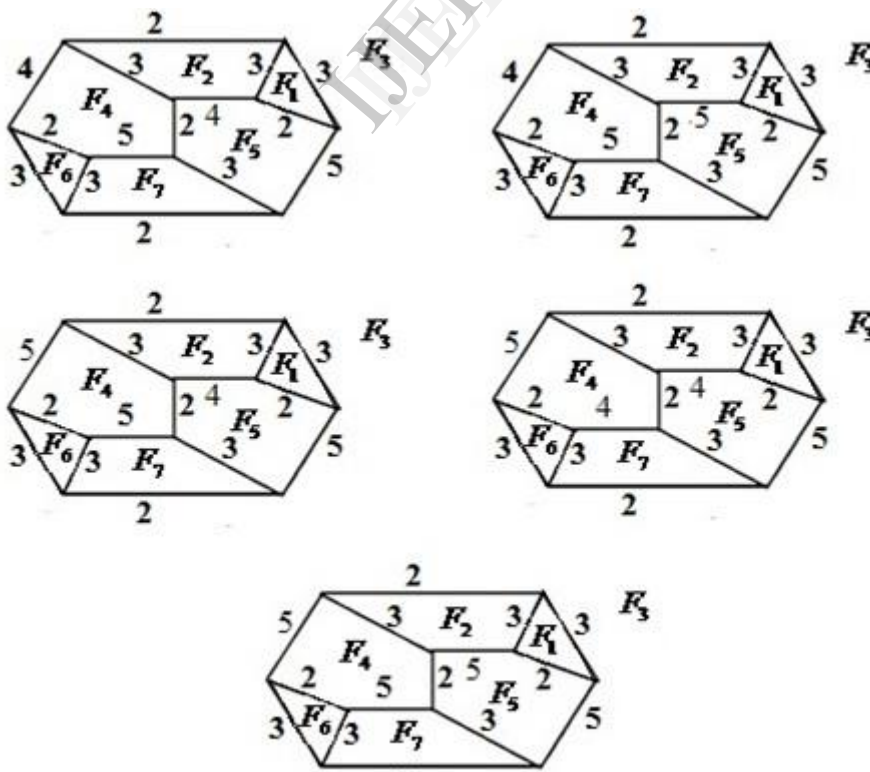

Figure 2

Proof. Using Choi's theorem, we prove that the number of edges of order 2 is exactly 5. By Andreev's condition for compact hyperbolic polyhedrons, we prove that the order of the

edges at each vertex is one of the form $(2,3,3),(2,3,4),(2,3,5)$. Using graphical properties, we can assign the orders to each edge in five different ways up to symmetry. Then we assign the order of the faces in each of these five polyhedrons to form ODCH Coxeter polyhedron.

2. PRELIMINARY

2.1. **Andreev's Theorem.** In 1970, E.M. Andreev provides a complete characterization of 3-dimensional compact hyperbolic polyhedral having non-obtuse dihedral angles on his article [2]. Therefore Andreev's theorem is a fundamental tool for classification of $3-$dimensional compact hyperbolic Coxeter polyhedron. Some elementary faces about polyhedral are essential before we state Andreev's theorem.

Definition 2.1.1. A cell complex on S^2 is called *trivalent* if each vertex is the intersection of three faces. A 3-dimensional *combinatorial polyhedron* is a cell complex C on S^2 that satisfied the following condition:
> (1) Every edges of C is the intersection of exactly two faces.
> (2) Anon-empty intersection of two faces is either an edge or a vertex.
> (3) Every faces contains not fewer than 3 edges. If a face contains n edges then n is called the *length* of the face.

Suppose C^* be the dual complex of C in S^2. Then C^* is a simplicial complex which embed in the same S^2 so that the vertex correspond to face of C, etc. A simple closed curve Γ in C^* is called *k-circuit* if it is formed by k edges of C^*. A k-circuit Γ is called *prismatic k-circuit* if the intersection of any two edges of C intersected by Γ is empty. If a prismatic k-circuit meets the edges $e_1, e_2, ..., e_k$ of C successively then we say that the edges $F_1, F_2, ..., F_k$ are an *k-prismatic element* of C.

Theorem 2.1.2 (Andreev, 1970), *Let C be an combinatorial polyhedron such that C is not a simplex and suppose that non-obtuse angles $0 < \alpha_{ij} \leq \dfrac{\pi}{2}$ are given corresponding to each edge $F_{ij} = F_i \cap F_j$ of C where F_i and F_j are the faces of C. Then there exist a compact hyperbolic polyhedron P in 3-dimensional hyperbolic space which realize C with dihedral angles α_{ij} at the edge F_{ij} if and only if the following five conditions hold:*
> (1) *C is trivalent.*
> (2) *If $F_{ijk} = F_i \cap F_j \cap F_k$ is a vertex of C then*
> $$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} > \pi .$$
> (3) *If Γ is a prismatic 3-circuit which intersects edges F_{ij}, F_{jk}, F_{ki} of C then*
> $$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} < \pi$$
> (4) *If Γ is a prismatic 4-circuit which intersects edges $F_{ij}, F_{jk}, F_{ki}, F_{li}$ of C then*
> $$\alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 2\pi .$$
> (5) *If F_s is a four sides faces of C with edges $F_{is}, F_{js}, F_{ks}, F_{ls}$ enumerated successively, then*
> $$\alpha_{is} + \alpha_{ks} + \alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 3\pi$$
> $$\alpha_{js} + \alpha_{ls} + \alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 3\pi$$

Furthermore, this polyhedron is unique up to hyperbolic isometries. Also Roeder, Hubbard and Dunbar proved that if C is not a triangular prism, then condition (5) is a consequence of (3) and (4) (Sec [14]). Andreev's restriction to non-obtuse dihedral angles is necessary to ensure that P be convex. Without this restriction of dihedral angles, compact hyperbolic polyhedral realizing a given abstract polyhedron may not be convex. Since dihedral angles of Coxeter polyhedron is non-obtuse, Andreev's theorem provide a complete characterization of 3-dimensional hyperbolic Coxeter polyhedron having more than four faces.

2.2. Choi's Theorem. Prof. Choi concentrated a class of Coxeter orbifolds which is called orderable Coxeter orbifolds and a certain type of orbifolds known as normal type orbifolds. In this class of orbifolds, we understand the restricted deformation space of orbifolds in real projective space from his article [6].

Definitation 2.2.1. We denote $k(P)$ the dimension of the projective group acting on a convex polyhedron P.

$$k(P) = \begin{cases} 3 \text{ if } P \text{ is a tetrahedron,} \\ 1 \text{ if } P \text{ is a cone with base} \\ \quad \text{a convex polygon which is no} \\ 0 \text{ otherwise} \end{cases}$$

Definition 2.2.2. A Coxeter group Γ is an abstract group define by a group presentation of form

$$\left(R_i, ; \left(R_i R_j \right)^{n_{ij}} \right), i, j \in I .$$

Where I is a countable index set, $n_{ij} \in N$ is symmetric for i, j and $n_{ij} = 1$.

The fundamental group of the orbifold will be a Coxeter group with a presentation

$$R_i, i = 1, 2, \dots f, \left(R_i R_j \right)^{n_{ij}} = 1$$

where R_i is associated with silvered sides and $R_{i,j}$ has order $n_{i,j}$ associated with the edge formed by the *i-th* and *j-th* side meeting.

A Coxeter orbifold whose polytope has a side F and a vertex v where all other sides are adjacent triangles to F and contains v and all edge orders of F are 2 is called a *cone-type* Coxeter orbifold. A Coxeter orbifold whose polyhedron is topologically a polygon times an interval and edges orders of top and bottom sides are 2 is called a *product-type* Coxeter orbifold. If $\hat{P}$ is not above type and has not finite fundamental group, then $\hat{P}$ is said to be a *normal-type* Coxeter orbifold.

Theorem 2.2.3 (Choi, 2006). *Let P be a convex polyhedron and $\hat{P}$ be given a normal type Coxeter orbifold structure. Let k(P) be the dimension of the group of projective automorphisms acting on P. Suppose that $\hat{P}$ is orderable. Then the restricted deformation space of projective structures on the orbifold $\hat{P}$ is a smooth manifold of dimension $3f - e - e_2 - k(P)$ if it is not empty.*

Corollary 2.2.4. *Let P be a convex polyhedron and $\hat{P}$ be given a normal type Coxeter orbifold structure. If $3f - e - e_2 < 0$ and $\hat{P}$ is orderable, then the restricted deformation space is empty.*

Remark 2.2.5. Let *P* be a convex polyhedron and $\hat{P}$ be given a normal type Coxeter orbifold structure. Let *k(P)* be the dimension of the group of projective automorphisms acting on *P*. Suppose that $\hat{P}$ is orderable. Then $\hat{P}$ is projectively deformable if and only if

$$3f - e - e_2 - k(P) > 0$$

2.3. Planar Graphs. The study of graphs is very important to understand the combinatorial structure of a polyhedron. We will discuss about the basis relation between graph theory and the 3-dimensional convex polyhedron.

Definition 2.3.1. A *planar graph* is a graph that can be drawn on the sphere(or the plane) without edge crossings. Two edges of a graph are *parallel* if they have the same endpoints. A *loop* is an edge whose endpoints are the same vertex. If there are neither parallel edges nor loops, a graph is called simple. A simple graph is called *k-connected* if the removal of any *k*-1 or fewer vertices (all the edges they are incident with) leaves a connected graph. The *dual graph* of a plane graph is a plane graph obtained from the original graph by exchanging the vertices and faces. The dual graph of a graph is *k*-connected if and only if the graph is *k*-connected. If all the faces of planar graph is triangles then the graph is called *triangulation*. The dual of a triangulation is a trivalent planar graph. A triangulation with *n* vertices has exactly $3n - 6$ edges and $2n - 4$ faces.

Definition 2.3.2. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs imbedded on the sphere such that V_1, V_2 be the set of vertices of G_1, G_2 and E_1, E_2 be the sets of edges of G_1, G_2. An *isomorphism* from G_1 to G_2 is a pair of bijections $\varphi: V_1 \to V_2$ and $\varphi: E_1 \to E_2$ which preserve the vertex-edge incidence relationship.

Definition 2.3.3. Let *P* be a convex polyhedron. The vertices and the edges of *P* from an abstract, finite, simple graph, called the graph of *P* and denoted by *G(P)*. Thus, *G(P)* is an abstract graph defined on the set of vertices vert *(P)* of *P*. Two vertices *u* and *v* in vert *(P)* are adjacent in *G(P)* if and only if *[u,v]* is an edge of *P*.

Definition 2.3.4. A 3-dimensional polyhedron is called *simplicial* polyhedron if every face contain exactly 3 vertices. A 3-dimensional polyhedron is called a simple polyhedron if each vertex is the intersection of exactly 3 faces.

Theorem 2.3.5 (Blind and Mani). *If P is convex polyhedron, then the graph G(P) determines the entire combinatorial structure of P.*
In other words, if two simple polyhedral have isomorphic graphs, then their combinatorial polyhedral are isomorphic as well.

Steinitz established the following basic theory for 3-dimensional polyhedron.

Theorem 2.3.6 (Steinitz). *G(P) is the graph of a 3-dimensional polyhedron P if and only if it is simple, planar and 3-connected.*

Corollary 2.3.7. *Every 3-connected planar graph has a representation in the plane such that all edges are straight, and all the bounded regions determined by it, as well as the union of all the bounded regions, are convex polygons.*

Since the compact hyperbolic polyhedron is simple, the combinatorial polyhedron of a compact hyperbolic polyhedron can be known from 3-connected planar graph of the polyhedron.

3. RESULTS

3.1. **Known Results from previous article.** In my previous articles [The graphical investigation of orderable and deformable compact Coxeter polyhedral in hyperbolic space] and [Application of Plantri graph: All Combinatorial structure of Orderable and Deformable Compact Coxeter Hyperbolic Polyhedra], we found the following theorems and propositions:

Let P be a CH-Coxeter polyhedron and $\hat{P}$ be its Coxeter orbifold structure of P.

Proposition 3.1.1. *If $\hat{P}$ is orderable, then P is also orderable.*

Remark 3.1.2. If P is not orderable, then $\hat{P}$ is also not orderable.

Let e be the number of edges of P and e_2 be the number of edges of edge order 2. Let v be the number of vertices of P and f be the number of faces of P.

Proposition 3.1.3. *Let P be a 3*-dimensional compact hyperbolic Coxeter polyhedron and $\hat{P}$ be Coxeter orbifold structure of P. Suppose $\hat{P}$ is orderable and projectively deformable. Then

    a. *Every vertex is incident with exactly three edges.*
    b. *Every vertex is incident with at least one edge of edge order 2.*
    c. $\dfrac{v}{2} \le e_2 \le 5 \Rightarrow v \le 10$.
    d. *v is even.*

3.2 **Main Results.** Now we are ready to establish the main results.

Proposition 3.2.1. *Let P be a 3-dimensional CH-Coxeter polyhedron with 10 vertices and $\hat{P}$ be its Coxeter orbifold structure of P. If $\hat{P}$ is orderable and projectively deformable then $e_2 = 5$.*

Proof. By Choi's theorem 2.2.3, $3f - e - e_2 - k(P) > 0$. By definition 2.2.1, for the polyhedra with 10 vertices P, $k(P) = 0$.

Since the combinatorial structure of orderable and projectively deformable compact hyperbolic coxeter polyhedra with 10 vertices are $P10\text{-}1, P10\text{-}2$, therefore $f = 7, e = 15$. Since $f = 7, e = 15$, therefore

$$3f - e - e_2 - k(T) > 0 \Rightarrow 3 \cdot 7 - 15 - e_2 - 0 > 0 \Rightarrow 6 > e_2$$

By proposition 3.1.3, every vertex is incident with at least one edge of edge order 2. Since there are 10 vertices of T and each vertex incident with exactly two vertices, therefore the number of edges of order 2 is at least $\dfrac{v}{2} = 5$. Hence $e_2 \geq 5$.

Since $5 \leq e_2 < 6$, therefore $e_2 = 5$.

Proposition 3.2.2. *Let P be a 3-dimensional CH-Coxeter polyhedra with 10 vertices and $\hat{P}$ be its Coxeter orbifold structure of P. If $\hat{P}$ is orderable and projectively deformable then the order of the edges at each vertex is one of the form* $(2,3,3), (2,3,4), (2,3,5)$.

Proof. Since there are 5 edges of P of order 2 and 10 vertices in P and each vertex is incident with at least one edge of order 2, therefore each vertex is incident with exactly one edge of order 2.

Suppose that r_1, r_2 be the order of the two edges at a vertex other than 2. Then $r_1, r_2 \geq 3$.

By Andreev's first condition, we have

$$\frac{1}{2} + \frac{1}{r_1} + \frac{1}{r_2} > 1$$

$$\frac{1}{r_1} + \frac{1}{r_2} > \frac{1}{2}$$

Since $r_1, r_2 \geq 3$, therefore $\dfrac{1}{r_1} + \dfrac{1}{r_2} < \dfrac{2}{3}$. Hence $\dfrac{1}{2} < \dfrac{1}{r_1} + \dfrac{1}{r_2} \leq \dfrac{2}{3}$.

If $r_1, r_2 \geq 4$ then $\dfrac{1}{r_1} + \dfrac{1}{r_2} \leq \dfrac{1}{2}$. This is a contradiction. Hence $r_1 = 3$ or $r_2 = 3$.

Assume that $r_2 = 3$. If $r_1 \geq 6$ then

$$\frac{1}{2} < \frac{1}{r_1} + \frac{1}{r_2} \Rightarrow \frac{1}{2} < \frac{1}{6} + \frac{1}{r_2} \Rightarrow \frac{1}{2} - \frac{1}{6} < \frac{1}{r_2} \Rightarrow \frac{2}{6} < \frac{1}{r_2} \Rightarrow r_2 < 3 \Rightarrow r_2 = 2$$

This is a contradiction. Hence $3 \leq r_1 \leq 5$.

Therefore the edge order at each vertex is one of the form $(2,3,3), (2,3,4), (2,3,5)$.

Theorem 3.2.3. *Let P10-1 be a compact hyperbolic Coxeter polyhedron as figure-1 and $\hat{P}$10-1 be its Coxeter orbifold structure. Suppose that $\hat{P}$10-1 is orderable and projectively deformable. Then the total number of such P10-1 is 5 and P10-1 is one of the tetrahedrons in figure 2.*

Proof. Suppose that the order of the edges be $r_1, r_2, ..., r_{15}$ and level the faces of *P10-1* as follows:
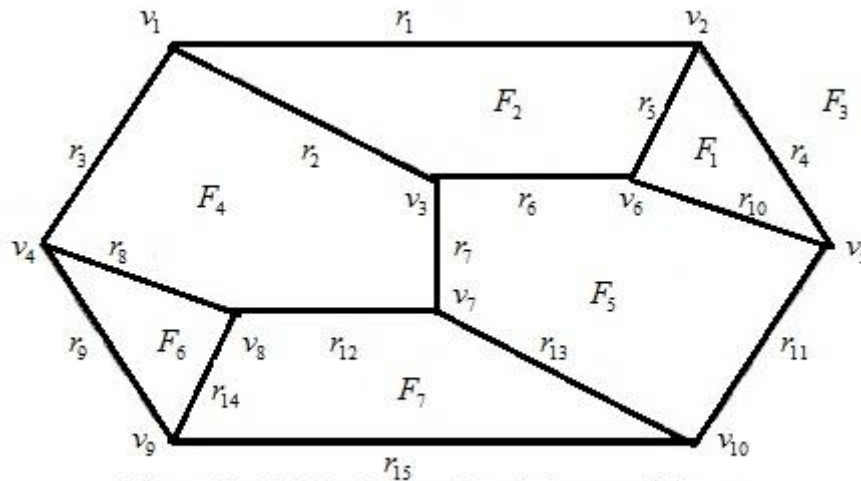
Figure 3 : P10-1 with levels of edges and faces

From the above figure-3, the 3-prismetric circuits are $(F_2, F_5, F_3), (F_4, F_7, F_3), (F_4, F_5, F_3)$. Therefore by Andreev's second condition, we have

$$\frac{1}{r_1} + \frac{1}{r_6} + \frac{1}{r_{11}} < 1, \frac{1}{r_3} + \frac{1}{r_7} + \frac{1}{r_{11}} < 1, \frac{1}{r_3} + \frac{1}{r_{12}} + \frac{1}{r_{15}} < 1 \qquad \dots (1)$$

Therefore only one edge can be order of 2 in each set of $\{r_1, r_6, r_{11}\}, \{r_3, r_7, r_{11}\}, \{r_3, r_{12}, r_{15}\}$ and $\{2,3\}$ cannot be a subset of each set. Since there are five edges of order 2 and each vertex is incident with an edge of order 2, therefore all the edges of order 2 are disjoint.
By definition of orderability 1.0.4, the order of the faces depends on the edges of order 2 and faces of higher index. Therefore we first assign edge order 2.

Suppose that $r_1 = 2$. Then $r_6, r_{11} > 2$ by equation (1). Since all the edges of order 2 are disjoint, therefore $r_5, r_4, r_3, r_2 > 2$ and hence $r_{10} = r_7 = 2$. Since $r_7 = 2$, therefore $r_{13}, r_{12} > 2$. Since at the vertex v_{10}, one edge is of order 2, therefore $r_{15} = 2$ and hence $r_8 = 2$. Since five edges are order 2, therefore remaining edges must have order other than 2. In this case the order of the faces are $F_1, F_6, F_2, F_7, F_4, F_5, F_3$.

Suppose that $r_5 = 2$. Since all the edges of order 2 are disjoint, therefore $r_1, r_4, r_6, r_{10} > 2$.
Since at the vertex v_5, one edge is of order 2, therefore $r_{11} = 2$. From the equation (1), we have $r_3, r_7, r_1, r_6 > 2$. Therefore $r_2 = 2$. Since $r_{11} = 2$ at vertex v_{10}, therefore $r_{13}, r_{15} > 2$. Then $r_{12} = 2$ and $r_9 = 2$. Since five edges are order 2, therefore remaining edges must have order other than 2. Since $\{2,3\}$ cannot be a subset of each set $\{r_1, r_6, r_{11}\}, \{r_3, r_7, r_{11}\}, \{r_3, r_{12}, r_{15}\}$ and $2 \in \{r_1, r_6, r_{11}\} \bigcap \{r_3, r_{12}, r_{15}\}$, therefore $3 \notin \{r_1, r_6, r_{11}\} \bigcup \{r_3, r_{12}, r_{15}\}$. Thus $r_1 \neq 3, r_3 \neq 3$. But the edge order at each vertex is one of the form $(2,3,3), (2,3,4), (2,3,5)$ by proposition 3.2.2. At the vertex v_1, the edge order $r_2 = 2$, therefore one of r_1, r_3 must be 3. This is a contradiction.

Suppose that $r_4 = 2$. By symmetry $r_4 \ \square \ r_9$, we will arrive similar situation as the case $r_5 = 2$. Therefore there is only one polyhedral P10-1 with edge order 2 which is as follows:

Figure 4 : P10-1 with edge order 2

Since $3 \notin \{r_1, r_6, r_{11}\} \cup \{r_3, r_{12}, r_{15}\}$, therefore $r_3 \neq 3, r_6 \neq 3, r_{11} \neq 3$. Since at each vertex there is at least one edge of order 3, therefore $r_2 = r_4 = r_5 = r_9 = r_{13} = 3$. Since $r_{12} \neq 3$, therefore $r_{14} = 3$.

Figure 5 : P10-1 with edges orders 2, 3

From the equations (1), we have
$$\frac{1}{r_6} + \frac{1}{r_{11}} < \frac{1}{2}, \frac{1}{r_3} + \frac{1}{r_{11}} < \frac{1}{2}, \frac{1}{r_3} + \frac{1}{r_{12}} < \frac{1}{2}$$

Therefore $4 \leq r_6, r_{11}, r_3, r_{11}, r_{12} \leq 5$ and $(r_6, r_{11}), (r_3, r_{11}), (r_3, r_{12})$ are not identical with $(4, 4)$.

Suppose that $r_3 = 4$. Then $r_{11} = r_{12} = 5$. We can choose r_6 as 4 or 5.

Suppose that $r_3 = 5$. If $r_{11} = 4$ then we will get same polyhedron by symmetry $r_3 \square \ r_{11}$. Therefore we choose $r_{11} = 5$. By symmetry $r_{12} \square \ r_6$, $r_{12} = 4, r_6 = 5$ and $r_{12} = 5, r_6 = 4$ will give same polyhedron up to symmetry. Therefore we can choose (r_{12}, r_6) as $(4, 4), (4, 5), (5, 5)$.

Therefore we have exactly five orderable and deformable compact hyperbolic coxeter polyhedra *P10-1* and these are as in figure-2. The order of the faces is $F_1, F_6, F_2, F_7, F_4, F_5, F_3$ for each of the above polyhedrons. Therefore each of these above polyhedrons has ODCH Coxeter Orbifold structure.

4. CONCLUSION

In this article, we find that the number of orderable and deformable compact hyperbolic Coxeter polyhedral *P10-1* in real projective space is exactly five. It can be extended to find all the 3-dimensional compact hyperbolic Coxeter polyhedral which are orderable and deformable in real projective space.

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