Global behavior of third order system of rational difference equations *

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Abstract

In this paper, our aim is to study the dynamical behavior of third-order system of rational difference equations

$$x_{n+1}=\frac{\alpha x_{n-2}}{\beta+\gamma x_n x_{n-1} x_{n-2}},\ y_{n+1}=\frac{\alpha_1 y_{n-2}}{\beta_1+\gamma_1 y_n y_{n-1} y_{n-2}},\ n=0,1,\cdots.$$

where the parameters $\alpha,\ \beta,\ \gamma,\ \alpha_1,\ \beta_1,\ \gamma_1$ and initial conditions $x_0,\ x_{-1},\ x_{-2},\ y_0,\ y_{-1},\ y_{-2}$ are positive real numbers. Some numerical examples are given to verify our theoretical results.

Keywords and phrases: System of rational difference equations, stability, global character, rate of convergence

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1 Introduction and preliminaries

The theory of difference equations occupies a central position in applicable Analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. In applications Nonlinear difference equations of order greater than one are of great importance. Such equations also appear naturally as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology,physiology, physics, engineering and economics. It is very interesting to investigate the dynamical behavior of positive solutions for system of higher-order rational difference equations.

C. Cinar [1] investigated the periodicity of the positive solutions of the system of rational difference equations:

$$x_{n+1}=\frac{1}{y_n},\ y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}}.$$

S. Stević [2] studied the system of two nonlinear difference equation:

$$x_{n+1}=\frac{u_n}{1+v_n},\ y_{n+1}=\frac{w_n}{1+s_n},$$

where $u_n,\ v_n,\ w_n,\ s_n$ are some sequences x_n or y_n.

S. Stević [3] studied the system of three nonlinear difference equations:

$$x_{n+1}=\frac{a_1 x_{n-2}}{b_1 y_n z_{n-1} x_{n-2}+c_1},\ y_{n+1}=\frac{a_2 y_{n-2}}{b_2 z_n x_{n-1} y_{n-2}+c_2},\ z_{n+1}=\frac{a_3 z_{n-2}}{b_3 x_n y_{n-1} z_{n-2}+c_3},$$

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where the parameters $a_i,\ b_i,\ c_i,\ i \in \{1,2,3\}$ are real numbers.

Ignacio Bajo and Eduardo Liz [4] investigated the global behavior of difference equation:

$$x_{n+1} = \frac{x_{n-1}}{a + bx_{n-1}x_n},$$

for all values of real parameters $a,\ b$.

S. Kalabušić, M. R. S. Kulenović and E. Pilav [5] investigated the global dynamics of the following systems of difference equations:

$$x_{n+1} = \frac{\alpha_1 + \beta_1 x_n}{A_1 + y_n},\ y_{n+1} = \frac{\gamma_2 y_n}{A_2 + B_2 x_n + y_n}.$$

A. S. Kurbanli, C. Çinar, I. Yalçinkaya [7] studied the behavior of positive solutions of the system of rational difference equation:

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1},\ y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1}.$$

N. Touafek and E.M. Elsayed [9] studied the periodic nature and got the form of the solutions of the following systems of rational difference equations:

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3}y_{n-1}},\ y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3}x_{n-1}}.$$

Similarly, N. Touafek and E.M. Elsayed [10] studied the periodicity nature of the following systems of rational difference equations:

$$x_{n+1} = \frac{y_n}{x_{n-1}(\pm 1 \pm y_n)},\ y_{n+1} = \frac{x_n}{y_{n-1}(\pm 1 \pm x_n)}.$$

Recently, Q. Zhang, L. Yang, J. Liu [11] studied the dynamics of a system of rational third-order difference equation:

$$x_{n+1} = \frac{x_{n-2}}{B + y_n y_{n-1} y_{n-2}},\ y_{n+1} = \frac{y_{n-2}}{A + x_n x_{n-1} x_{n-2}},\ n = 0, 1, \cdots.$$

Our aim in this paper is to investigate the dynamical behavior of positive solution for third-order rational difference equations:

$$x_{n+1} = \frac{\alpha x_{n-2}}{\beta + \gamma x_n x_{n-1} x_{n-2}},\ y_{n+1} = \frac{\alpha_1 y_{n-2}}{\beta_1 + \gamma_1 y_n y_{n-1} y_{n-2}},\ n = 0, 1, \cdots. \tag{1}$$

where the parameters $\alpha,\ \beta,\ \gamma,\ \alpha_1,\ \beta_1,\ \gamma_1$ and initial conditions $x_0,\ x_{-1},\ x_{-2},\ y_0,\ y_{-1},\ y_{-2}$ are positive real numbers.

Let us consider six-dimensional discrete dynamical system of the form:

$$\begin{aligned} x_{n+1} &= f(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}), \\ y_{n+1} &= g(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}),\ n = 0, 1, \cdots, \end{aligned} \tag{2}$$

where $f : I^3 \times J^3 \to I$ and $g : I^3 \times J^3 \to J$ are continuously differentiable functions and $I,\ J$ are some intervals of real numbers. Furthermore, a solution $\{(x_n, y_n)\}_{n=-2}^{\infty}$ of system (2) is uniquely determined by initial conditions $(x_i, y_i) \in I \times J$ for $i \in \{-2, -1, 0\}$. Along with the system (2) we consider the corresponding vector map $F = (f, x_n, x_{n-1}, x_{n-2}, g, y_n, y_{n-1}, y_{n-2})$. An equilibrium point of (2) is a point $(\bar{x}, \bar{y})$ that satisfies

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}) \\ \bar{y} &= g(\bar{x}, \bar{x}, \bar{x}, \bar{y}\, \bar{y}, \bar{y}) \end{aligned}$$

The point $(\bar{x}, \bar{y})$ is also called a fixed point of the vector map F.

Definition 1. *Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system (2).*

(i) An equilibrium point $(\bar{x}, \bar{y})$ is said to be stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every initial condition (x_i, y_i), $i \in \{-2, -1, 0\}$ $\| \sum_{i=-2}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \| < \delta$ implies $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$ for all $n > 0$, where $\|.\|$ is the usual Euclidian norm in $\mathbb{R}^2$.

(ii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be unstable if it is not stable.

(iii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be asymptotically stable if there exists $\eta > 0$ such that $\| \sum_{i=-2}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \| < \eta$ and $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

(iv) An equilibrium point $(\bar{x}, \bar{y})$ is called global attractor if $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

(v) An equilibrium point $(\bar{x}, \bar{y})$ is called asymptotic global attractor if it is a global attractor and stable.

Definition 2. *Let $(\bar{x}, \bar{y})$ be an equilibrium point of the map*

$$F = (f, x_n, x_{n-1}, x_{n-2}, g, y_n, y_{n-1}, y_{n-2})$$

where f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of (2) about the equilibrium point $(\bar{x}, \bar{y})$ is

$$X_{n+1} = F(X_n) = F_J X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}$ *and F_J is Jacobian matrix of the system (2) about the equilibrium point $(\bar{x}, \bar{y})$.*

To construct corresponding linearized form of the system (1) we consider the following transformation:

$$(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}) \mapsto (f, f_1, f_2, g, g_1, g_2), \tag{3}$$

where $f = \frac{\alpha x_{n-2}}{\beta + \gamma x_n x_{n-1} x_{n-2}}$, $g = \frac{\alpha_1 y_{n-2}}{\beta_1 + \gamma_1 y_n y_{n-1} y_{n-2}}$, $f_1 = x_n$, $f_2 = x_{n-1}$, $g_1 = y_n$, $g_2 = y_{n-1}$. The Jacobian matrix about the fixed point $(\bar{x}, \bar{y})$ under the transformation (3) is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} A & A & B & 0 & 0 & o \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & o & C & C & D \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

where $A = -\frac{\alpha \gamma \bar{x}^3}{(\beta + \gamma \bar{x}^3)^2}$, $B = \frac{\alpha \beta}{(\beta + \gamma \bar{x}^3)^2}$, $C = -\frac{\alpha_1 \gamma_1 \bar{y}^3}{(\beta_1 + \gamma_1 \bar{y}^3)^2}$ and $D = \frac{\alpha_1 \beta_1}{(\beta_1 + \gamma_1 \bar{y}^3)^2}$.

Theorem 1. *For the system $X_{n+1} = F(X_n)$, $n = 0, 1, \cdots$, of difference equations such that $\bar{X}$ be a fixed point of F. If all eigenvalues of the Jacobian matrix J_F about $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\bar{X}$ is locally asymptotically stable. If one of them has a norm greater than one, then $\bar{X}$ is unstable.*

2 Main results

Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (1), then for $\alpha > \beta$ and $\alpha_1 > \beta_1$ system (1) has following two equilibrium points: $P_0 = (0, 0)$, $P_1 = (A, B)$, where $A = (\frac{\alpha - \beta}{\gamma})^{\frac{1}{3}}$ and $B = (\frac{\alpha_1 - \beta_1}{\gamma_1})^{\frac{1}{3}}$.

Theorem 2. *Let (x_n, y_n) be positive solution of system (1), then for every $m \geq 0$ the following results hold:*

$$0 \leq x_n \leq (\frac{\alpha}{\beta})^{m+1} x_{-2}, \ \textit{if} \ \ n = 3m+1,$$

$$0 \leq x_n \leq (\frac{\alpha}{\beta})^{m+1} x_{-1}, \ \textit{if} \ \ n = 3m+2,$$

$$0 \leq x_n \leq (\frac{\alpha}{\beta})^{m+1} x_{0}, \ \textit{if} \ \ n = 3m+3,$$

$$0 \leq y_n \leq (\frac{\alpha_1}{\beta_1})^{m+1} y_{-2}, \ \textit{if} \ \ n = 3m+1,$$

$$0 \leq y_n \leq (\frac{\alpha_1}{\beta_1})^{m+1} y_{-1}, \ \textit{if} \ \ n = 3m+2,$$

$$0 \leq y_n \leq (\frac{\alpha_1}{\beta_1})^{m+1} y_{0}, \ \textit{if} \ \ n = 3m+3,$$

Proof. The results are obviously true for $m = 0$. Suppose that results are true for $m = k \geq 1$, i.e.,

$$0 \leq x_n \leq (\frac{\alpha}{\beta})^{k+1} x_{-2}, \ \text{if} \ \ n = 3k+1,$$

$$0 \leq x_n \leq (\frac{\alpha}{\beta})^{k+1} x_{-1}, \ \text{if} \ \ n = 3k+2,$$

$$0 \leq x_n \leq (\frac{\alpha}{\beta})^{k+1} x_{0}, \ \text{if} \ \ n = 3k+3,$$

$$0 \leq y_n \leq (\frac{\alpha_1}{\beta_1})^{k+1} y_{-2}, \ \text{if} \ \ n = 3k+1,$$

$$0 \leq y_n \leq (\frac{\alpha_1}{\beta_1})^{k+1} y_{-1}, \ \text{if} \ \ n = 3k+2,$$

$$0 \leq y_n \leq (\frac{\alpha_1}{\beta_1})^{k+1} y_{0}, \ \text{if} \ \ n = 3k+3,$$

Now, for $m = k+1$ using (1) one has

$$0 \leq x_{3k+4} = \frac{\alpha x_{3k+1}}{\beta + \gamma x_{3k+3} x_{3k+2} x_{3k+1}} \leq \frac{\alpha x_{3k+1}}{\beta} \leq (\frac{\alpha}{\beta})^{k+2} x_{-2},$$

$$0 \leq x_{3k+5} = \frac{\alpha x_{3k+2}}{\beta + \gamma x_{3k+4} x_{3k+3} x_{3k+2}} \leq \frac{\alpha x_{3k+2}}{\beta} \leq (\frac{\alpha}{\beta})^{k+2} x_{-1},$$

$$0 \leq x_{3k+6} = \frac{\alpha x_{3k+3}}{\beta + \gamma x_{3k+5} y_{3k+4} y_{3k+3}} \leq \frac{\alpha x_{3k+3}}{\beta} \leq (\frac{\alpha}{\beta})^{k+2} x_{0},$$

$$0 \leq y_{3k+4} = \frac{\alpha_1 y_{3k+1}}{\beta_1 + \gamma_1 y_{3k+3} y_{3k+2} x_{3k+1}} \leq \frac{\alpha_1 y_{3k+1}}{\beta_1} \leq (\frac{\alpha_1}{\beta_1})^{k+2} y_{-2},$$

$$0 \leq y_{3k+5} = \frac{\alpha_1 y_{3k+2}}{\beta_1 + \gamma_1 y_{3k+4} y_{3k+3} y_{3k+2}} \leq \frac{\alpha_1 y_{3k+2}}{\beta_1} \leq (\frac{\alpha_1}{\beta_1})^{k+2} y_{-1},$$

$$0 \leq y_{3k+6} = \frac{\alpha_1 y_{3k+3}}{\beta + 1 + \gamma + 1 y_{3k+5} y_{3k+4} y_{3k+3}} \leq \frac{\alpha_1 y_{4k+3}}{\beta_1} \leq (\frac{\alpha_1}{\beta_1})^{k+2} y_{0},$$

□

Theorem 3. *For the equilibrium point P_0 of Equation (1) the following results hold:*

(i) If $\alpha < \beta$ and $\alpha_1 < \beta_1$, then equilibrium point P_0 of the system (1) is locally asymptotically stable.

(ii) If $\alpha > \beta$ or $\alpha_1 > \beta_1$, then equilibrium point P_0 is unstable.

Proof. (i) The linearized system of (1) about the equilibrium point P_0 is given by:

$$X_{n+1} = F_J(P_0)X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}$ and $F_J(0,0) = \begin{pmatrix} 0 & 0 & \frac{\alpha}{\beta} & 0 & 0 & o \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\alpha_1}{\beta_1} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.

The characteristic polynomial of $F_J(P_0)$ is given by

$$P(\lambda) = \lambda^6 - (\frac{\alpha}{\beta} + \frac{\alpha_1}{\beta_1})\lambda^3 + \frac{\alpha\alpha_1}{\beta\beta_1}.. \tag{4}$$

The roots of $P(\lambda)$ are $\pm\frac{\alpha}{\beta}$ and $\pm\frac{\alpha_1}{\beta_1}$ repeated roots. Since $|\frac{\alpha}{\beta}| < 1$ and $|\frac{\alpha_1}{\beta_1}| < 1$, whenever $\alpha < \beta$ and $\alpha_1 < \beta_1$. Thus, by Theorem 1 P_0 is locally asymptotically stable.

(ii) It is easy to see that if $\alpha > \beta$ or $\alpha_1 > \beta_1$, then there exists at least one root λ of Equation (4) such that $|\lambda| > 1$. Hence, by Theorem 1 if $\alpha > \beta$ or $\alpha_1 > \beta_1$, then $(0,0)$ is unstable. □

Theorem 4. *If $\alpha < \beta$ or $\alpha_1 < \beta_1$, then positive equilibrium point P_1 of Equation (1) is unstable.*

Proof. The linearized system of (1) about the equilibrium point P_1 is given by:

$$X_{n+1} = F_J(P_1)X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{pmatrix}$ and $F_J(P_1) = \begin{pmatrix} -1+\frac{\beta}{\alpha} & -1+\frac{\beta}{\alpha} & \frac{\beta}{\alpha} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1+\frac{\beta_1}{\alpha_1} & -1+\frac{\beta_1}{\alpha_1} & \frac{\beta_1}{\alpha_1} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$,

One of the roots of characteristic polynomial of $F_J(P_1)$ is given by $\frac{\beta_1}{\alpha_1}$. Hence, by Theorem 1 if $\beta_1 > \alpha_1$ then P_1 is unstable. □

Theorem 5. *Let $\alpha < \beta$ and $\alpha_1 < \beta_1$, then the equilibrium point P_0 of Equation (1) is globally asymptotically stable.*

Proof. For $\alpha < \beta$ and $\alpha_1 < \beta_1$, from Theorem 3 P_0 is locally asymptotically stable. From Theorem 2, it is easy to see that every positive solution (x_n, y_n) is bounded, i.e., $0 \le x_n \le \mu$ and $0 \le y_n \le \nu$ for all $n = 0, 1, 2, \cdots$, where $\mu = \max\{x_{-2}, x_{-1}, x_0\}$ and $\nu = \max\{y_{-2}, y_{-1}, y_0\}$. Now, it is sufficient to prove that (x_n, y_n) is decreasing. From system (1) one has

$$\begin{aligned} x_{n+1} &= \frac{\alpha x_{n-2}}{\beta + \gamma x_n x_{n-1} x_{n-2}}, \\ &\le \frac{\alpha x_{n-2}}{\beta} < x_{n-2}. \end{aligned}$$

This implies that $x_{3n+1} < x_{3n-2}$ and $x_{3n+4} < y_{3n+1}$. Also

$$\begin{aligned} y_{n+1} &= \frac{\alpha y_{n-2}}{\beta + \gamma y_n y_{n-1} y_{n-2}}, \\ &\le \frac{\alpha y_{n-2}}{\beta} < y_{n-2}. \end{aligned}$$

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This implies that $y_{3n+1} < x_{3n-2}$ and $y_{3n+4} < x_{3n+1}$. Hence, the subsequences $\{x_{3n+1}\}$, $\{x_{3n+2}\}$, $\{x_{3n+3}\}$ and $\{y_{3n+1}\}$, $\{y_{3n+2}\}$, $\{y_{3n+3}\}$ are decreasing.Therefore the sequences $\{x_n\}$ and $\{y_n\}$ are decreasing. Hence, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. □

3 Rate of Convergence

In this section we will determine the rate of convergence of a solution that converges to the unique positive equilibrium point of the system (1). The following result gives the rate of convergence of solution of a system of difference equations

$$X_{n+1} = (A + B(n))\, X_n, \tag{5}$$

where X_n is an m-dimensional vector, $A \in C^{m\times m}$ is a constant matrix, and $B : \mathbb{Z}^+ \to C^{m\times m}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \tag{6}$$

as $n \to \infty$,where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm

$$\|(x, y)\| = \sqrt{x^2 + y^2}$$

Proposition 1. *(Perron's Theorem)[**?**] Suppose that condition* (6) *holds. If X_n is a solution of* (5) *, then either $X_n = 0$ for large n or*

$$\rho = \lim_{n\to\infty} (\|X_n\|)^{1/n}, \tag{7}$$

or

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{8}$$

exists and is equal to the norm of one the eigenvalues of the matrix A.

Assume that $\lim_{n\to\infty} x_n = \bar{x}$ and $\lim_{n\to\infty} y_n = \bar{y}$. First we will find a system of limiting equations for the map F. The error terms are given as

$$x_{n+1} - \bar{x} = \sum_{i=0}^{2} A_i\,(x_{n-i} - \bar{x}) + \sum_{i=0}^{2} B_i\,(y_{n-i} - \bar{y}),\ \ y_{n+1} - \bar{y} = \sum_{i=0}^{2} C_i\,(x_{n-i} - \bar{x}) + \sum_{i=0}^{2} D_i\,(y_{n-i} - \bar{y})\,.$$

Set $e^1_n = x_n - \bar{x}$ and $e^2_n = y_n - \bar{y}$, one has

$$e^1_{n+1} = \sum_{i=0}^{2} A_i e^1_{n-i} + \sum_{i=0}^{2} B_i e^2_{n-i},\ \ e^2_{n+1} = \sum_{i=0}^{2} C_i e^1_{n-i} + \sum_{i=0}^{2} D_i e^2_{n-i}.$$

where $A_0 = -\dfrac{\alpha\gamma\bar{x}\prod_{i=1}^{2} x_{n-i}}{\left(\beta+\gamma\prod_{i=0}^{2} x_{n-i}\right)(\beta+\gamma\bar{x}^3)}$, $A_1 = -\dfrac{\alpha\gamma\bar{x}^2 x_{n-2}}{\left(\beta+\gamma\prod_{i=0}^{2} x_{n-i}\right)(\beta+\gamma\bar{x}^3)}$, $A_2 = -\dfrac{\alpha\beta}{\left(\beta+\gamma\prod_{i=0}^{2} x_{n-i}\right)(\beta+\gamma\bar{x}^3)}$,

$B_i = 0$ for $i \in \{0, 2\}$, $C_i = 0$ for $i \in \{0, 2\}$, $D_0 = -\dfrac{\alpha_1\gamma_1\bar{y}\prod_{i=1}^{2} y_{n-i}}{\left(\beta_1+\gamma_1\prod_{i=0}^{2} y_{n-i}\right)(\beta_1+\gamma_1\bar{y}^3)}$, $D_1 = -\dfrac{\alpha_1\gamma_1\bar{y}^2 y_{n-2}}{\left(\beta_1+\gamma_1\prod_{i=0}^{2} y_{n-i}\right)(\beta_1+\gamma_1\bar{y}^3)}$,

$D_2 = -\dfrac{\alpha_1\beta_1}{\left(\beta_1+\gamma_1\prod_{i=0}^{2} y_{n-i}\right)(\beta_1+\gamma_1\bar{y}^3)}$, Taking the limits, we obtain $\lim_{n\to\infty} A_i = -\frac{\alpha\gamma\bar{x}^3}{(\beta+\gamma\bar{x}^3)^2}$ for $i \in \{0,1\}$, $\lim_{n\to\infty} A_2 = \frac{\alpha\beta}{(\beta+\gamma\bar{x}^3)^2}$, $\lim_{n\to\infty} B_i = 0$ for $i \in \{0,2\}$, $\lim_{n\to\infty} C_i = 0$ for $i \in \{0,2\}$, $\lim_{n\to\infty} D_i = -\frac{\alpha_1\gamma_1\bar{y}^3}{(\beta_1+\gamma_1\bar{y}^3)^2}$ for $i \in \{0,1\}$, $\lim_{n\to\infty} D_2 = \frac{\alpha_1\beta_1}{(\beta_1+\gamma_1\bar{y}^3)^2}$, So, the limiting system of error terms can be written as $E_{n+1} = F_J(0,0)E_n$, where where $E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_{n-2}^1 \\ e_n^2 \\ e_{n-1}^2 \\ e_{n-2}^2 \end{pmatrix}$. Using proposition (1), one has following result.

Theorem 6. *Assume that $\{(x_n, y_n)\}$ be a positive solution of the system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$, and $\lim_{n\to\infty} y_n = \bar{y}$, where $(\bar{x},\bar{y}) = (0,0)$. Then, the error vector E_n of every solution of (1) satisfies both of the following asymptotic relations*

$$\lim_{n\to\infty}(\|E_n\|)^{\frac{1}{n}} = |\lambda F_J(\bar{x},\bar{y})|, \quad \lim_{n\to\infty}\frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda F_J(\bar{x},\bar{y})|,$$

where $\lambda F_J(\bar{x},\bar{y})$ are the characteristic roots of the Jacobian matrix $F_J(\bar{x},\bar{y})$ about $(0,0)$.

4 Examples

In order to verify our theoretical results we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to the system of nonlinear difference equations (1). All plots in this section are drawn with mathematica.

Example 1. *Consider the system (1) with initial conditions $x_{-2} = 7.9,\ x_{-1} = 0.19,\ x_0 = 1.2,\ y_{-2} = 3.6,\ y_{-1} = 2.3,\ y_0 = 9.1$. Moreover, choosing the parameters $\alpha = 970,\ \beta = 990,\ \gamma = 110,\ \alpha_1 = 770,\ \beta_1 = 790,\ \gamma_1 = 90$. Then, the system (1) can be written as:*

$$x_{n+1} = \frac{970x_{n-2}}{990+110x_nx_{n-1}x_{n-2}},\ y_{n+1} = \frac{770y_{n-2}}{790+90y_ny_{n-1}y_{n-2}},\ n = 0,1,\cdots, \tag{9}$$

$n = 0,1,\cdots$ *and with initial conditions $x_{-2} = 7.9,\ x_{-1} = 0.19,\ x_0 = 1.2,\ y_{-2} = 3.6,\ y_{-1} = 2.3,\ y_0 = 9.1$ .The plot of system (9) is shown in Figure (1) and its global attractor is shown in Figure(2).*

Example 2. *Consider the system (1) with initial conditions $x_{-2} = 2.9,\ x_{-1} = 0.19,\ x_0 = 1.2,\ y_{-2} = 3.6,\ y_{-1} = 5.3,\ y_0 = 1.1$. Moreover, choosing the parameters $\alpha = 197,\ \beta = 199,\ \gamma = 210,\ \alpha_1 = 177,\ \beta_1 = 179,\ \gamma_1 = 190$. Then, the system (1) can be written as:*

$$x_{n+1} = \frac{197x_{n-2}}{199+210x_nx_{n-1}x_{n-2}},\ y_{n+1} = \frac{177y_{n-2}}{179+190y_ny_{n-1}y_{n-2}},\ n = 0,1,\cdots, \tag{10}$$

$n = 0,1,\cdots$ *and with initial conditions $x_{-2} = 2.9,\ x_{-1} = 0.19,\ x_0 = 1.2,\ y_{-2} = 3.6,\ y_{-1} = 5.3,\ y_0 = 1.1$ .The plot of system (10) is shown in Figure (3) and its global attractor is shown in Figure (4).*

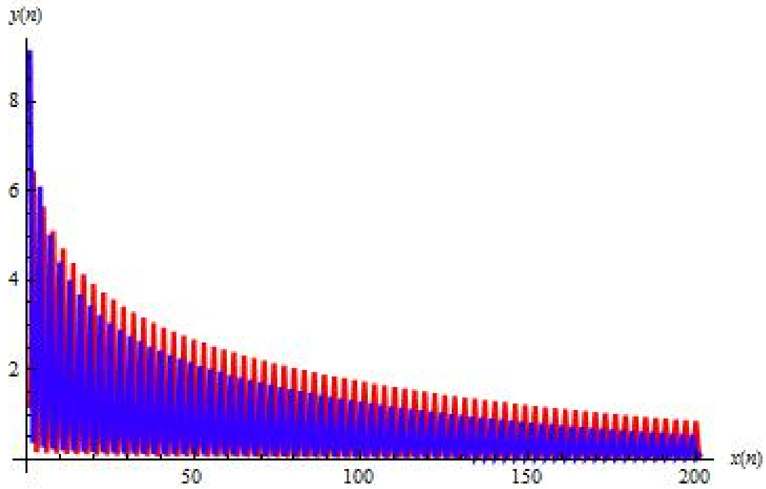

Figure 1: Plot of system (9)

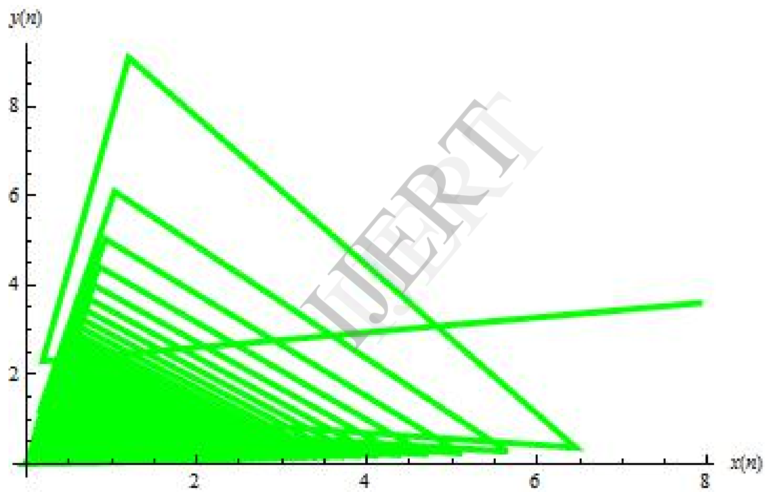

Figure 2: An attractor of system (9)

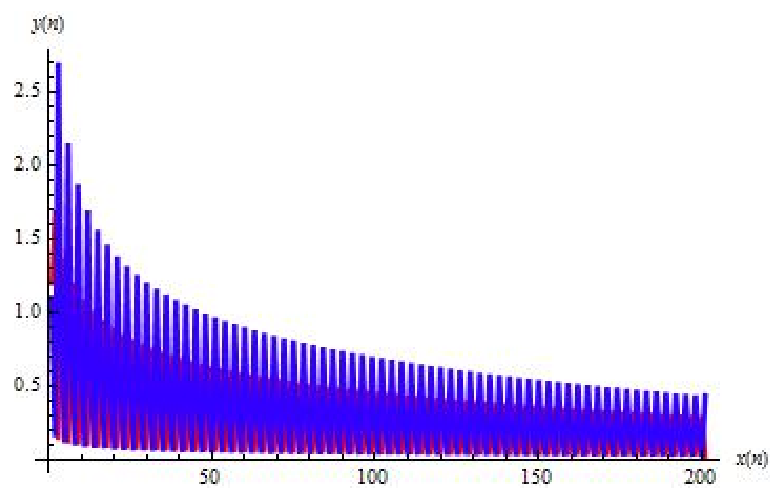

Figure 3: Plot of system (10)

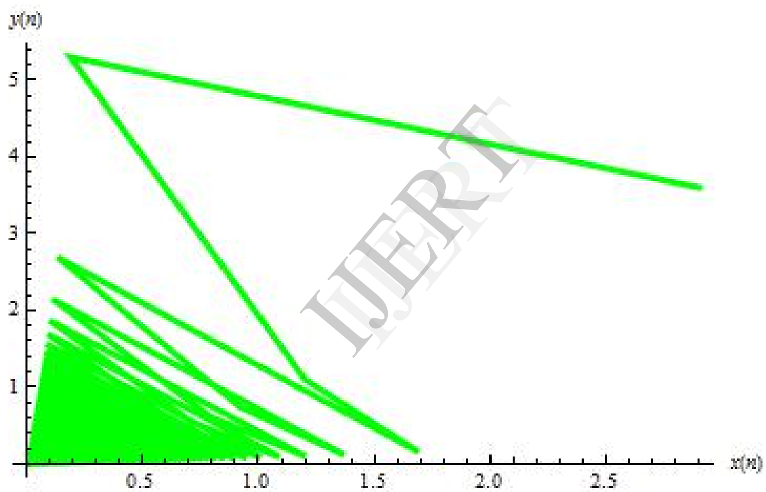

Figure 4: An attractor of system (10)

Conclusion

In the paper, we investigate some dynamics of a six-dimensional discrete system. The system has two positive equilibrium points. The linearization method is used to show that equilibrium point $(0,0)$ is locally asymptotically stable. The main objective of dynamical systems theory is to predict the global behavior of a system based on the knowledge of its present state. An approach to this problem consists of determining the possible global behaviors of the system and determining which initial conditions lead to these long-term behaviors. In case of higher-order dynamical systems, it is very difficult to discuss global behavior of the system. Some powerful tools such as semiconjugacy and weak contraction cannot be used to analyze global behavior of system (1). In the paper,using simple techniques to prove the global asymptotic stability of equilibrium point $(0,0)$. Some numerical examples are provided to support our theoretical results.

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