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Geometrically Nonlinear Free Vibration of a Beam Carrying Concentrated Masses

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***Abstract*: The objective of this paper is to establish the formulation of the problem of nonlinear transverse vibration of a clamped-clamped (CC) Bernoulli-Euler beam carrying two concentrated masses and taking into account the associated rotatory inertia. The method used is based on the principle of Hamilton and spectral analysis for non-linear free vibration occurring at large displacement amplitudes. The problem is reduced to solution of a nonlinear algebraic system using the so-called second formulation applied previously to nonlinear transverse vibration of continuous structures, such as beams and plates, nonlinear longitudinal vibration of 2-dof and multi- dof systems and to nonlinear transverse vibration of 2-dof systems. The backbone curves, corresponding to the nonlinear fundamental mode of a CC beam carrying one concentrated mass, is presented to determine the error in the measurement of the nonlinear frequency.**

Keywords— Nonlinear transverse vibration, concentrated masses, Hamilton's principle, second formulation, Newton-Raphson, spectral analysis, resonant frequencies, mode shapes, large displacement amplitude, backbone curve, nonlinear algebraic system.

I. INTRODUCTION

Free vibration problems are of considerable interest to engineers and modelers have been much studied. When the deflections of the structure under examination are small enough, a wide range of linear analysis tools, such as modal analysis, can be used, and analytical results are often available. As the deflections become larger, geometrical nonlinearities enter into play and induce many effects that are not observed in linear systems. In such situations, numerical approximate methods must be used. In this paper, a method, based on the principle of Hamilton and spectral analysis, is used to investigate nonlinear free vibrations occurring at large displacement amplitudes of CC beams carrying concentrated masses. The problem is reduced to solution of a nonlinear algebraic system using the so-called second formulation developed in [6] and applied to nonlinear transverse vibration of continuous structures, such as beams and plates, nonlinear longitudinal vibration of 2- dof and multi-dof systems, and to nonlinear transverse vibration of 2-dof systems.

A big amount of research [7-16] has been conducted to analyze the vibration of beams carrying concentrated masses, because of their practical interest in representing various physical systems, like, for example, the wings of aircrafts carrying engines. Most of these studies have treated only the linear case. In this paper, a method is presented for determining the nonlinear frequency of vibration of a CC Bernoulli-Euler beam carrying two concentrated masses. A general solution of the nonlinear problem is made, and the nonlinear frequency amplitude curve is presented, corresponding to the nonlinear fundamental mode of the CC beam carrying one concentrated mass, with a view to determine the corrections to a nonlinear frequency measurements using accelerometers

II. NONLINEAR FORMULATION

In a series of previous works, the non-linear mode shapes and resonant frequencies of beams with various boundary conditions have been examined both theoretically and experimentally [1-6]. The theory was based on Hamilton's principle and spectral analysis and had led to a series of amplitude dependent mode shapes and resonant frequencies. Similar methods are used here to formulate the geometrically nonlinear vibration problem of a CC Bernoulli-Euler beam carrying concentrated masses and taking into account the corresponding rotatory inertia.

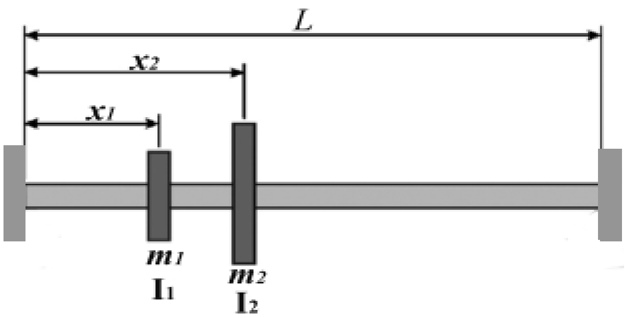

Fig. 1. CC Bernoulli-Euler beam carrying two masses.

The uniform beam, with two concentrated masses m_1 and m_2 shown in Fig.1, is made of a material of mass density ρ, Young's modulus E, length L, cross sectional area S, thickness e and second moment of area of cross section I. I_s is the moment of inertia of the attached mass m_s, for s equals 1 and 2, while r_s is its radius of gyration with respect to the neutral axis of the beam. Let x be the coordinate along the neutral axis of the beam measured from the right end, $w(x, t)$ be the transverse deflection of the beam, measured from its equilibrium position, and ψ be the slop defined as the partial derivative of w with respect to x. Neglecting the beam axial and rotary inertia, the kinetic energy of the system can be written as:

$$T = \frac{1}{2}\rho S\int_0^L\left(\frac{\partial w(x,t)}{\partial t}\right)^2 dx + \frac{1}{2}\sum_{S=1}^{2} m_S\left(\frac{\partial w(x_S,t)}{\partial t}\right)^2 + \frac{1}{2}\sum_{S=1}^{2} m_S\left(\frac{\partial w(x_S,t)}{\partial t}\right)^2 \quad (1)$$

The beam total strain energy can be written as the sum of the strain energy due to the bending denoted as V_{lin}, plus the axial strain energy due to the axial load induced by large deflexion denoted as V_{Nlin}

$$V_{Lin} = \frac{1}{2}EI\int_0^L\left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 dx \quad (2)$$

$$V_{Lin} = \frac{1}{8}\frac{ES}{L}\left[\int_0^L\left(\frac{\partial w(x,t)}{\partial x}\right)^2 dx\right]^2 \quad (3)$$

The transverse displacement function is expanded as a series of basic spatial functions (the linear modes) and the time function is supposed to be harmonic:

$$w(x, t) = q_i(t)w_i(x) = a_i\, w_i \sin(\omega t) \quad (4)$$

Where the usual summation convention for the repeated indices is used. One obtains after discretization of expressions (1) to (3)

$$T = \frac{1}{2}\omega^2 a_i a_j m_{ij}\cos^2(\omega t) \quad (5)$$

$$V_{Lin} = \frac{1}{2}a_i a_j k_{ij}\sin^2(\omega t) \quad (6)$$

$$V_{Lin} = \frac{1}{2}a_i a_j a_k a_l k_{ijkl}\sin^4(\omega t) \quad (7)$$

With:

$$m_{ij} = \rho S\int_0^L w_i w_j dx\sum_{S=1}^{2} m_S w_i(x_S)w_j(x_S) + \sum_{S=1}^{2} I_S\frac{\partial w_i(x_S)}{\partial x}\frac{\partial w_j(x_S)}{\partial x} \quad (8)$$

$$k_{ij} = EI\int_0^L\frac{\partial^2 w_i}{\partial x^2}\frac{\partial^2 w_i}{\partial x^2}dx \quad (9)$$

$$b_{ijkl} = \frac{1}{4}\frac{ES}{L}\int_0^L\frac{\partial w_i}{\partial x}\frac{\partial w_i}{\partial x}dx\int_0^L\frac{\partial w_k}{\partial x}\frac{\partial w_l}{\partial x}dx \quad (10)$$

The dynamic behavior of the structure is governed by Hamilton's principle, which is symbolically written as:

$$\partial\int_0^{\frac{2\pi}{\omega}}(V-T) = 0 \quad (11)$$

In which ∂ indicates the variation of the integral. Introducing the assumed series (4) into the energy condition (11) via equations (5-7), integrating the trigonometric functions $\sin^2(wt)$, $\cos^2(wt)$ and $\sin^4(wt)$ over the range [0,1] reduces the problem to that of finding the minimum of the function Φ given by:

$$\Phi = \frac{\pi}{2\omega}\left[a_i a_j k_{ij} + \frac{3}{4}a_i a_j a_k a_l b_{ijkl} - \omega^2 a_i a_j m_{ij}\right] \quad (12)$$

By writing:

$$\frac{\partial\Phi}{\partial a_r} = 0 \qquad r=1, \ldots\ldots, n \quad (13)$$

And taking into account the properties of symmetry usually satisfied, equations (13) appear to be equivalent to the following set of nonlinear algebraic equations:

$$2a_i k_{ir} + 3a_i a_j a_r b_{ijkr} - 2\omega^2 a_i m_{ir} = 0 \quad r=1, 2, \ldots, n \quad (14)$$

Putting $b_{ij}(\{A\}) = a_k a_l b_{ijkl}$, the nonlinear geometrical rigidity matrix [B] is defined. Each term of matrix [B] is a quadratic function of the column matrix of coefficients $\{A\} = [a_1\ a_2\ \ldots\ a_n]^T$. Introducing matrix [B] in equations (14) leads to the following matrix equation:

$$2[K]\{A\} + 3[B(\{A\})]\{A\} = 2\omega^2[M]\{A\} \quad (15)$$

Where $[K]$ and $[M]$ are the classical rigidity and mass matrices respectively, which are well known in linear theory. To obtain non-dimensional parameters, we put:

$$\eta = \frac{x}{L}\ ;\ w_i(x) = e w_i^*(\eta)\ ;\ \frac{\omega^2}{\omega^{*2}} = \frac{EI}{\rho S L^4}$$
$$\frac{m_{ij}}{m_{ij}^*} = \rho S e^2 L\ ;\ \frac{k_{ij}}{k_{ij}^*} = \frac{EIe^2}{L^3}\ ;\ \frac{b_{ij}}{b_{ij}^*} = \frac{EIe^2}{L^3} \quad (16)$$

m_{ij}^*, k_{ij}^* and b_{ijkl}^* are non-dimensional tensors given by:

$$m_{ij} = \rho S\int_0^1 w_i^* w_j^* d\eta\sum_{S=1}^{2} M_S w_i^*(\eta_S)w_j^*(\eta_S) + \sum_{S=1}^{2} M_S C_S^2\frac{\partial w_i^*(\eta_S)}{\partial\eta}\frac{\partial w_j^*(\eta_S)}{\partial\eta} \quad (17)$$

$$k^*_{ij} = \int_0^1 \frac{\partial^2 w_i^*}{\partial \eta^2} \frac{\partial^2 w_j^*}{\partial \eta^2} d\eta \qquad (18)$$

$$b^*_{ijkl} = \alpha \int_0^1 \frac{\partial w_i^*}{\partial \eta} \frac{\partial w_j^*}{\partial \eta} d\eta \int_0^1 \frac{\partial w_k^*}{\partial \eta} \frac{\partial w_l^*}{\partial \eta} d\eta \qquad (19)$$

In which the non−dimensional parameters α, $M_{s\ and}$ C_s are defined by:

$$M_S = \frac{m_S}{\rho SL} \;;\; C_s = \frac{r_S}{L} \;;\; \alpha = \frac{Se^2}{4I} \qquad (20)$$

Substituting these equations into equation (15) leads to:

$$2\left[K^*\right]\{A\} + 3\left[B^*(\{A\})\right]\{A\} = 2\omega^2\left[M^*\right]\{A\} \quad (21)$$

III. DETERMINATION OF THE LINEAR MODE SHAPES OF THE SYSTEM BEAM-MASSES

Before examining the nonlinear case, we start in this section by determination of the linear mode shapes, in order to use them as basic functions in the nonlinear theory. The transverse displacement function w of the beam shown in figure1 can be defined at each span by:

$$w^*(\eta) = \begin{cases} w_1^*(\eta) &]0,\eta_1[\\ w_2^*(\eta) &]\eta_1,\eta_2[\\ w_3^*(\eta) &]\eta_2,1[\end{cases} \qquad (22)$$

The general solution for transverse vibration in each span, for the i^{th} mode can be written as:

$$w_{1i}^*(\eta) = a_1\cosh(\beta i(\eta)) + a_2\sinh(\beta i(\eta)) + a_3\cos(\beta i(\eta)) + a_4\sin(\beta i(\eta)) \qquad (23)$$

$$w_{2i}^*(\eta) = a_5\cosh(\beta i(\eta-\eta_1)) + a_6\sinh(\beta i(\eta-\eta_1)) + a_7\cos(\beta i(\eta-\eta_1)) + a_8\sin(\beta i(\eta-\eta_1)) \qquad (24)$$

$$w_{3i}^*(\eta) = a_9\cosh(\beta i(\eta-\eta_2)) + a_{10}\sinh(\beta i(\eta-\eta_2)) + a_{11}\cos(\beta i(\eta-\eta_2)) + a_{12}\sin(\beta i(\eta-\eta_2)) \qquad (25)$$

With:

$$\beta_i = \sqrt[4]{\frac{\rho S\omega_i^2}{EI}} \qquad i = 1, 2, \ldots. \qquad (26)$$

β_i are the eigenvalue parameters for the system beam-masses. The constants a_i are determined by the boundary conditions. At η_1 and η_2 the beam is carrying a concentrated mass and the continuity conditions (displacement, slope, moment and shear discontinuity) are used as in [7] for s=1, 2:

$$w^*_{(S+1)i}(\eta_j) = w^*_{Si}(\eta_S) \qquad (27)$$

$$\frac{\partial w^*_{(S+1)i}}{\partial \eta}\Big|_{\eta=\eta_S} = \frac{\partial w^*_{Si}}{\partial \eta}\Big|_{\eta=\eta_S} \qquad (28)$$

$$\frac{d^2 w^*_{(S+1)i}}{\partial \eta^2}\Big|_{\eta=\eta_S} = \frac{d^2 w^*_{Si}}{d\eta^2}\Big|_{\eta=\eta_S} - M_S C_S^2 (\beta_i L)^4 \frac{dw^*_{Si}}{d\eta}\Big|_{\eta=\eta_S} \qquad (29)$$

$$\frac{d^3 w^*_{(S+1)i}}{\partial \eta^3}\Big|_{\eta=\eta_S} = \frac{d^3 w^*_{Si}}{d\eta^3}\Big|_{\eta=\eta_S} - M_S C_S^2 (\beta_i L)^4 w^*_{Si}\Big|_{\eta=\eta_S} \qquad (30)$$

The boundary conditions are:

$$w_{1i}^*(0) = 0 \qquad (31)$$

$$\frac{dw_{1i}^*(0)}{d\eta} = 0 \qquad (32)$$

$$w_{3i}^*(1) = 0 \qquad (33)$$

$$\frac{dw_{3i}^*(1)}{d\eta} = 0 \qquad (34)$$

Substituting Eq. (23-25) into Eqs. (27-34), one obtains, after appropriate non-dimensionalization, a linear homogeneous system of equations which can written in a matrix form as follows:

$$\begin{bmatrix} t_{1-1} & t_{1-2} & . & . & . & t_{1-12} \\ t_{2-1} & t_{2-2} & & & & t_{2-12} \\ . & & & & & . \\ . & & & & & . \\ . & & & & & . \\ t_{12-1} & t_{12-2} & . & . & . & t_{12-12} \end{bmatrix} \begin{pmatrix} a_1 \\ . \\ . \\ . \\ . \\ a_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ . \\ . \\ . \\ . \\ 0 \end{pmatrix} \qquad (35)$$

The non-triviality condition is established by solving:

$$Det[\mathrm{T}] = 0 \qquad (36)$$

Where $[T]$ is the 12×12 matrix of the coefficients t_{ij} of the system and the roots β_i are the eigenvalues of the problem.

IV. NUMERICAL DETAILS AND APPLICATION

The values of the parameters β_i were computed by solving numerically the nonlinear transcendental equation (36), using the standard Newton-Raphson iterations. The corresponding parameters β_i for i = 1, .. , 6, in a case of CC Bernoulli-Euler beam with one concentered mass in the middle, are given in table 1 and the corresponding curves are plotted in Fig. 2. The parameters m^*_{ij}, k^*_{ij} and b^*_{ijkl} have been computed numerically using Simpson's rule in the range [0, 1] and the nonlinear algebraic system (21) has been solved using the so-called second formulation, developed and used for the first time in [6]. It is based on an approximation which consists on writing the contribution vector as: $\{A\}=[a_1, \varepsilon_2, \varepsilon_3, \ldots\varepsilon_6]$ and neglecting in the expression of equations (21) the second terms with respect to ε, i.e., terms of the type $a_i a_j \varepsilon_k b_{1jkr}$ when considering the first nonlinear mode, in equation (21). Separating in the nonlinear expression a terms proportional to a_1^3, terms proportional to $a_1^3\varepsilon_i$ and neglecting terms proportional to $a_1\varepsilon_i\varepsilon_j$ leads to:

$$a_i a_j a_k b^*_{ijkr} = a_1 b^*_{111r} + a_1^2 \varepsilon_i b^*_{11ir} \qquad (37)$$

After substituting and rearranging, equation (21) can be written in matrix form as:

$$\left(\left[K^*_{RI}\right] - 2\omega^2\left[M^*_{RI}\right]\right) A_{RI} + \frac{3}{2}\left[\alpha^*_1\right] A_{RI} = -\frac{3}{2} a_1^3 b^*_{i111} \qquad (38)$$

In which $[K^*_{RI}] = [k^*_{ij}]$ and $[M^*_{RI}] = [m^*_{ij}]$ are reduced rigidity and mass matrices associated with the first nonlinear mode, obtained by varying i and j in the set (2-6), $[\alpha^*_1]$ is a 5x5 square matrix, depending on a_1, whose general term α^*_{ij} is equal to $a^2_1\varepsilon_i b^*_{11ir}$, and $\{(-3/2)a^3_1 b^*_{i111}\}$ is a column vector representing the right side of the linear system (38) in which the reduced unknown vector is $\{A_{RI}\}^T = [\varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6]$. The modal contributions can be obtained easily by solving the linear system (38) of five equations and five unknowns. Higher nonlinear mode shapes may be obtained in a similar manner, using appropriate reduced matrix in each case.

Table I Eigenvalues parameters β_i for M_1=5.2/62.64, M_2=0 C_1=C_2=0

i	β_i
1	4.3432856603
2	7.8532046240
3	10.6425729185
4	14.1371654913
5	16.7730466476
6	20;4203522872

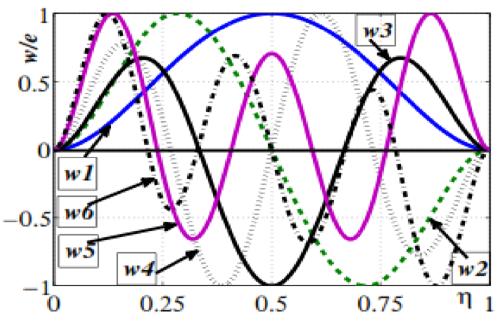

Fig. 2. Beam functions for i= 1,..,6

As has been mentioned in the introduction, various applications may be made of the present model, which would exceed the scope of the present work. In this paper, an application has been chosen of the present model in order to try and answer a question which has been often raised concerning the experimental measurements of the dynamic structural response in the nonlinear range. The current practice at the Institute of Sound and Vibration Research (ISVR) in a series of significant research projects was the use of non-contacting optical vibration transducers (OVT) in order to measure the nonlinear mode shapes and frequencies of beams and plates. The question examined here was: How would the results of measurements have been affected if a very light accelerometer has been used instead of the OVT? To answer this question, an application has been made of the theory developed above to the beam tested by Bennouna whose characteristics, reported in [17], are: 2×20×580mm, made of aluminium alloy DTD5070. To which a 5.2 g piezoelectric accelerometer of type (A/32/S) has been attached at the beam middle point and calculations have been made of the beam backbone curves of the CC beam with and without the attached mass of the accelerometer. The results are shown in Fig. 3. It can be seen that for amplitude of vibration of the order of two times the beam thickness, the error induced by the presence of the added mass on the nonlinear frequency is about 4.44%. This confirms the sensitivity of the nonlinear testing to small perturbations and the necessity of being very careful in their interpretation.

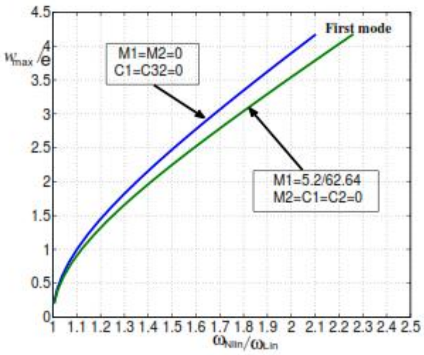

Fig. 3. The first non-dimensional backbone curve of the Bernoulli-Euler beam with a concentred mass at the middle

V. CONCLUSION

A theoretical model for large vibration amplitudes of bernoulli-euler beam carrying masses and taking into account the rotary inertia has been developed, based on Hamilton's principle and spectral analysis, to obtain numerical results. The theory effectively reduces a nonlinear free vibration problem to a set of nonlinear algebraic equations.

An application has been made of the theory to estimate the effect of an attached accelerometer on measurements carried out on beams at large vibration amplitudes. The results have shown that the nonlinearity may induce a significant effect of the mass added on the measurement results, confirming that it may be much more accurate to use non contacting transducers, as it was commonly practiced at the ISVR.

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