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Generalized Prediction Intervals for the Life Time Distribution of K-Unit Series System Based on Generalized Variable Approach

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ABSTRACT
We consider a k-unit series system with life time of each unit following inverted exponential distribution with an unknown scale parameter. We derive MLE and MMLE for this scale parameter. In this article we consider the problem of setting prediction interval for future sample, when lifetime distribution of a unit in a k-unit series system has inverted exponential distribution based on Generalized Variable approach. The performance of proposed generalized prediction interval is evaluated using extensive simulation work. The proposed prediction interval found to perform well for small to moderate sample sizes.

KEYWORDS: **Series system, maximum likelihood estimator, modified maximum likelihood estimator, generalized prediction interval.**

1.INTRODUCTION:

There is a large amount of literature about the estimation of scale parameter of inverted exponential distribution using different approaches. Inverted exponential distribution is life time distribution which is used in the reliability discipline. The inverted exponential distribution (IED) has been discussed as a life time model by Lin et al (1989) in detail. They have obtained maximum likelihood estimators, confidence limits and uniformly minimum variance unbiased estimators for the parameter and reliability function with complete samples. Godase S.S., et.al (2017) explained tolerance intervals and confidence intervals for the scale parameter of Pareto-Rayleigh Distribution. Godase S.S., et.al (2017) derived interval estimation for lifetime distribution of k-unit parallel system. Godase S.S., et.al(2017) obtained prediction and tolerance intervals for the lifetime distribution of k-unit Parallel System based on Generalized Variable Approach. Godase S.S., et.al (2015) developed generalized confidence intervals for the Scale Parameter of the Inverted Exponential Distribution.
We see from the literature review that there is more work on estimation of parameter of inverted exponential distribution as compared to interval estimation. The main purpose of this article is to develop a generalized pivot variable that is simple to use for interval estimation of the parameter in life time distribution of a series system. The concept of generalized p-value was introduced by Tsui and Weerahandi (1989) for hypothesis testing. Weerahandi (1993) extended the idea for constructing confidence interval. Weerahandi (1995) gives a detailed discussion along with numerous examples. The concept of generalized confidence intervals have turned out to be very satisfactory for obtaining confidence interval for many complex problems; see Weerahandi(1993,1995), Krishnamoorthy and Mathew (2003), Guo and Krishnamoorthy (2005), Ng (2007), Ye and Wang (2008)..

In this paper, we consider the problem of setting generalized prediction interval (GPI) for the future sample, when lifetime distribution of a unit in a k-unit series system has inverted exponential distribution. Potdar and Shirke (2014) explained reliability estimation of k-unit series system based on progressively censored data.

A prediction interval for a single future observation is an interval that will, with a specified degree of confidence, contain the next (or some other prespecified) randomly selected observation from a population. Such an interval would interest the purchaser of a single unit of a particular product and is generally more relevant to such an individual than, say a confidence interval to contain average performance. Some applications involve the prediction of future observations in a population or process, based on existing data. For example, one may wish to predict the number of parts that will need to be replaced in a system over the next two months. Prediction is different than estimation of a distributional characteristic because we are interested in a finite number(perhaps only one) of individuals rather than the entire conceptual population that the distribution represents.

Extensive literature is available for constructing prediction intervals for various continuous probability distributions and other models such as one-way random model and linear regression. Awad and Raqab(2000) explained prediction intervals for the future record values from exponential distribution. Ellah(2009) obtained parametric prediction limits for generalized exponential distribution using record observations. In this article, we are concerned with prediction interval for future observation from lifetime distribution of k-unit series system based on maximum likelihood estimator and modified maximum likelihood estimator.

In section 2, we provide maximum likelihood estimator (MLE) and modified maximum likelihood estimator (MMLE) for the scale parameter, when lifetime distribution of a unit in a k-unit series system has inverted exponential distribution. Tiku and Suresh (1992) obtained a new method of estimation for location and scale parameters by using MMLE. R.P.Suresh (2004) provides estimation of location and scale parameters in the two parameter exponential distribution using MMLE. In section 3, generalized prediction interval has been developed. In section 4, we study performance of generalized prediction intervals using MLE as well as MMLE for k=2, 3 and for small sample sizes using simulation technique. The proposed GPIs are simple to compute and perform better in small sample sizes.

2. MODEL AND ESTIMATION OF SCALE PARAMETER

Consider a k-unit series system with independent and identically distributed lifetimes of components. Let Y_1, $Y_2 \ldots Y_k$ be the lifetimes, where Y_i is the lifetime of i^{th} component namely inverted exponential distribution. Lifetime of the system is $X=\min(Y_1, Y_2 \ldots Y_k)$. The cdf of X is

$$F_X(x;\theta) = 1 - \left(1 - e^{-\frac{1}{\theta x}}\right)^k \quad ; \quad x \geq 0, \theta > 0 \tag{2.1}$$

The pdf of X is given by,

$$f_X(x,\theta) = \left(\frac{k}{\theta x^2}\right) e^{\frac{-1}{\theta x}} \left(1 - e^{-\frac{1}{\theta x}}\right)^{k-1} \quad ; \quad x \geq 0, \theta > 0 \tag{2.2}$$

$$= \quad 0 \qquad ; \qquad \text{herwise}$$

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2.1 MAXIMUM LIKELIHOOD ESTIMATION

Here log likelihood of the sample is given by

$$L = nlog(k) - nlog(\theta) - 2\sum_{i=1}^{n} logx_i - \frac{1}{\theta}\sum_{i=1}^{n}(\frac{1}{x_i}) + (k-1)\sum_{i=1}^{n} log\ (1 - e^{\frac{-1}{\theta x_i}}\ )$$

The MLE of θ can be obtained by solving $\frac{dL}{d\theta}$ = 0, where

$$\frac{dL}{d\theta} = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n}\frac{1}{x_i} - \frac{(k-1)}{\theta^2}\sum_{i=1}^{n}\frac{e^{\frac{-1}{\theta x_i}}}{x_i(1-e^{\frac{-1}{\theta x_i}})} \qquad (2.3)$$

The solution can be obtained by Newton-Raphson Method by taking initial solution $\widehat{\theta}_0 = \underline{X}$.

2.2 MODIFIED MAXIMUM LIKELIHOOD ESTIMATION

In the following we discuss Modified Maximum Likelihood Estimation on the lines of Tiku and Suresh (1992). The likelihood equation is given by

$$\frac{dL}{d\theta} = -\frac{n}{\theta} + \sum_{i=1}^{n} z_i - \frac{(k-1)}{\theta}\sum_{i=1}^{n}\frac{z_i e^{-z_i}}{(1-e^{-z_i})}$$

(2.4)

where $z_i = \frac{1}{\theta x_i}$

The maximum likelihood equation (2.4) does not have explicit solution for θ. This is due to the fact that the term $g(z_i) = \frac{z_i e^{-z_i}}{(1-e^{-z_i})}$ is intractable. In this paper, we use the MML approach to derive approximate MLE for θ by

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linearizing the term $g(z_i) = \frac{z_i e^{-z_i}}{(1-e^{-z_i})}$ using Taylor series expansion around the quantile point of F with reference to Tiku et. Al. (1986), Tiku and Suresh (1992), R.P.Suresh (2004). The linearization is done in such a way that the derived MML estimators retain all the desirable asymptotic properties of the maximum likelihood estimators.
Here Modified Maximum Likelihood Estimator is

$$\hat{\theta} = \frac{\sum_{i=1}^{n} \frac{1}{x_i}(1-b(k-1))}{n(a(k-1)+1)} \tag{2.5}$$

where a = $\frac{\lambda_q e^{-\lambda q}}{1-e^{-\lambda q}} - \frac{\lambda_q e^{-\lambda q}(1-\lambda_q - e^{-\lambda q})}{(1-e^{-\lambda q})^2}$, b = $\frac{e^{-\lambda q}(1-\lambda_q - e^{-\lambda q})}{(1-e^{-\lambda q})^2}$.

For more details one may refer to Tiku and Suresh (1992) and Suresh (2004).

Lemma 2.1: Distribution of $\left(\frac{\widehat{\theta_n}}{\theta}\right)$ and $\left(\frac{\hat{\theta}}{\theta}\right)$, both are free from θ.

Proof: The proof is similar to the one given by Gulati and Mi (2006).
While constructing generalized pivot this lemma can be used.
In the following, we shall see method of finding prediction interval for future observation x using generalized variable approach based on maximum likelihood estimator and modified maximum likelihood estimator of the scale parameter θ.

3. PREDICTION INTERVALS

3.1 GENERALIZED VARIABLE APPROACH

Suppose that X=(X_1, X_2 ,..., X_n) form a random sample from a distribution which depends on the parameters $\theta = (\psi, \upsilon)$ where ψ is the parameter of interest and ν^T is a vector of nuisance parameters. A generalized pivot $Q(X; x, \psi, \upsilon)$ where x is a observed value of X, for interval estimation defined by Weerahandi (1993), has the following properties:
i) $Q(X; x, \psi, \upsilon)$ has a distribution free of unknown parameters.
ii) The value of $Q(X; x, \psi, \upsilon)$ is ψ.
The percentiles of $Q(X; x, \psi, \upsilon)$can then be used to obtain confidence intervals for θ. Such confidence intervals are referred to as generalized confidence intervals. For example, if $Q_{1-\propto}$ denotes the $100_{1-\alpha}$ th percentile of $Q(X; x, \psi, \upsilon)$, then $Q_{1-\propto}$ is a generalized upper confidence limit for θ. A lower confidence limit or two-sided confidence limits can be similarly defined. Thus GCI is obtained by using a generalized pivot. The generalized pivotal quantity based on $\hat{\theta}_n$ is $Q_i = \frac{\theta}{\hat{\theta}_\iota}\hat{\theta}_0 = \frac{\hat{\theta}_0}{\left(\frac{\hat{\theta}_\iota}{\theta}\right)}$ $i = 1,2, \ldots N$. Obviously, the observed value of Q_i is θ. Moreover, the distribution of Q_i does not depend on unknown parameter. Therefore, Q_i is a generalized pivot for θ.Let x_1, x_2, ..., x_n is a sample from F(.). The MLE of θ is given by $\hat{\theta}_n$. Furthermore, $\frac{xi}{\widehat{\theta_n}}$,i=1,2,...,n are ancillary statistics and their distribution does not depend on θ.(Lawless(1982)). Using this result, a GPQ for x can be obtained as $\frac{x}{\widehat{\theta_n}}\hat{\theta}_0$ where $\hat{\theta}_n$ is the MLE based on a random sample of size n from (2.1) and $\hat{\theta}_0$ is an observed value of $\hat{\theta}_n$. The variables $\hat{\theta}_n$, x are mutually independent. Let q=$\frac{x}{\widehat{\theta_n}}$ and q_p denotes the pth quantile of q. Then $q_{1-\alpha}\hat{\theta}_0$ is a 1-α upper prediction limit for a future observation x based on GV approach. As the distribution of q does not depend on any unknown parameters, its percentiles can be estimated using Monte Carlo simulation. Specifically, q is distributed as $\frac{x*}{\widehat{\theta_n}}$, where x^* follows inverted exponential distribution with θ =1 in our model and $\hat{\theta}_n$* is the MLE based on this sample, and so the percentiles of q can be obtained using Monte Carlo simulation. A 1-α prediction interval for a future observation x based on GV approach that is generalized prediction interval is given by $(q_{\alpha/2}\hat{\theta}\ ,\ q_{1-\alpha/2}\,\hat{\theta}\ )$ where q_p denotes the pth quantile of $\frac{x*}{\widehat{\theta_n}}$ and x* and $\hat{\theta}_n$*are as defined in the preceding paragraph.
We can replace MLE of scale parameter Î¸ by MMLE and obtain generalized prediction interval based on MMLE.

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4. SIMULATION STUDY

We conduct extensive simulation experiments to evaluate performance of Prediction Interval based on MLE and MMLE. In the simulation study, we generate n observations on X=Min{ Xi, i=1,2,…,k}, where Xi, i=1,2,…,k are iid inverted exponential distribution with scale parameter θ. Using Newton Raphson method, we obtain MLE and using Suresh's approach, we obtain MMLE based on the generated n observations. Repeating the process 10,000 times, we estimate coverage of both intervals for n= 3,4,5,6,7,8,9,10,15,30,50,We choose different values of θ, k, n and α. Results are tabulated in Table (1-2). Figures in the 1st row are based on MLE, while figures in the 2nd row are based on MMLE. From tables 1-2, we observe that simulated coverage of generalized prediction interval does not differ significantly whether it can be computed from MLE as well as MMLE. Also the performance of the proposed generalized prediction interval does not depend on θ. However, as the sample size is small, the generalized prediction intervals are efficient. The results reported in this paper can be extended to other members of inverted scale family of distributions given by Potdar and Shirke (2013).

Table 1: Mean coverage of Prediction Intervals for Inverted Exponential distribution (k-unit series system)using Generalized variable approach when θ=1,k=2

n/coverage	0.90	0.95	0.97	0.99
3	0.9027	0.9489	0.9737	0.9942
	0.9027	0.9467	0.9639	0.9827
4	0.9027	0.9509	0.9737	0.9979
	0.9011	0.9527	0.9684	0.9890
5	0.9047	0.9513	0.9782	0.9931
	0.9027	0.9548	0.9738	0.9911
6	0.9053	0.9567	0.9865	0.9930
	0.9075	0.9534	0.9734	0.9924
7	0.9028	0.9554	0.9764	0.9927
	0.9071	0.9508	0.9719	0.9936
8	0.9036	0.9564	0.9765	0.9937
	0.9037	0.9536	0.9765	0.9987
9	0.9045	0.9588	0.9714	0.9977
	0.9074	0.9519	0.9758	0.9984
10	0.9024	0.9569	0.9737	0.9925
	0.9064	0.9523	0.9765	0.9908
15	0.9023	0.9542	0.9732	0.9932
	0.9034	0.9565	0.9725	0.9985
30	0.9064	0.9536	0.9735	0.9997
	0.9035	0.9564	0.9754	0.9948
50	0.9138	0.9527	0.9748	0.9984
	0.9032	0.9565	0.9782	0.9924

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Table 2: Mean coverage of Prediction Intervals for Inverted Exponential distribution (k-unit series system) using Generalized variable approach when θ=2,k=2

n/coverage	0.90	0.95	0.97	0.99
3	0.9027	0.9489	0.9737	0.9942
	0.9027	0.9467	0.9639	0.9827
4	0.9027	0.9509	0.9737	0.9979
	0.9011	0.9527	0.9684	0.9890
5	0.9047	0.9513	0.9782	0.9931
	0.9027	0.9548	0.9738	0.9911
6	0.9053	0.9567	0.9865	0.9930
	0.9075	0.9534	0.9734	0.9924
7	0.9028	0.9554	0.9764	0.9927
	0.9071	0.9508	0.9719	0.9936
8	0.9036	0.9564	0.9765	0.9937
	0.9037	0.9536	0.9765	0.9987
9	0.9045	0.9588	0.9714	0.9977
	0.9074	0.9519	0.9758	0.9984
10	0.9024	0.9569	0.9737	0.9925
	0.9064	0.9523	0.9765	0.9908
15	0.9023	0.9542	0.9732	0.9932
	0.9034	0.9565	0.9725	0.9985
30	0.9064	0.9536	0.9735	0.9997
	0.9035	0.9564	0.9754	0.9948
50	0.9138	0.9527	0.9748	0.9984
	0.9032	0.9565	0.9782	0.9924

CONCLUSION

Generalized prediction intervals are provided for the future sample when lifetime distribution of a unit in a k-unit series system has inverted exponential distribution. The proposed prediction interval performs satisfactory for small to moderate sample sizes.

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