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Generalized Half Linear Canonical Transform And Its Properties

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Abstract: As generalization of the Fractional Fourier transform (FRFT), the linear canonical transform (LCT) has been used in several areas, including optical analysis and signal processing. For practical purpose half linear canonical transform (HLCT) is more useful. Hence in this paper we have proved some important results e.g. inversion theorem, linearity, differentiation, derivative property, Parseval's Identity for half linear canonical transform, also relation between HLCT and Laplace transform (LT) is discussed.

Keywords: Linear canonical transform, Fractional Fourier Transform.

Introduction: The idea of the fractional powers of Fourier operator appeared in mathematical literature as early in 1930. It has been rediscovered in quantum mechanics by Namias [5] in 1980. He had given a systematic method for the development of fractional integral transforms by means of Eigen values. In the past decade, FRFT has attracted much attention of the signal processing community as the generalization of FT. The relevant theory has been developed including uncertainty principle, sampling theory, convolution theorem. Numbers of other integral transforms also have been extended in its fractional domain for example, Akay [1] had studied fractional Mellin transform, Tayawade, Gudadhe [9] discussed Fractional Hankel transform, Fractional Hilbert transform has been developed by Zayed [10], Sontakke, Gudadhe [8] studied number of property of fractional Hartley transform, Joshi, Gudadhe [2] worked on generalized canonical sine transform etc. Bhosale and Choudhary [3] had developed fractional Fourier transform as a generalized function. These fractional transforms found number of applications in signal processing, image processing, quantum mechanics etc.

Further generalization of fractional Fourier transform known as linear canonical transform was introduced by Moshinsky [4] in 1971. Pei, Ding [6,7] had studied its eigen value aspect. Linear canonical transform is a three parameter linear integral transform which has several special cases as fractional Fourier transform, Fresnel transform, Chirp transform etc. Linear canonical transform is defined as,

$$[LCTf(t)](s) = \sqrt{\frac{1}{2\pi ib}} \int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot e^{\frac{i}{2}(a/b)t^2} \cdot e^{-i(s/b)t} f(t)\, dt, \quad \text{for } b \neq 0$$

$$= \sqrt{d}\ \ e^{\frac{i}{2}(cd)s^2} \cdot f(d\cdot s), \text{ for } b = 0, \text{ with } ad - bc = 1,$$

where a, b, c, and d are real parameters independent on s and t.

This paper emphasizes on defining half linear canonical transform, deriving its inversion theorem, then some properties of the half linear canonical transform are discussed, also relation between Half Linear Canonical Transform and Laplace transform and finally conclusions are given.

1. Testing Function Space $\mathcal{E}$:

An infinitely differentiable complex valued function ϕ on R^n belongs to $\mathcal{E}$ (R^n), if for each compact set, $I \subset S_\alpha$ where $S_\alpha = \{t : t \in R^n, |t| \le \alpha, \alpha > 0\}$ and for $k \in R^n$,

$$\gamma_{\mathcal{E},k}\,\varphi(t) = \sup_{t \in I} \left| D^k \varphi(t) \right| < \infty.$$

Note that space $\mathcal{E}$ is complete and a Frechet space, let $\mathcal{E}$' denotes the dual space of $\mathcal{E}$.

2 Half Linear Canonical Transform:

2.1 Definition:

The Half Linear Canonical Transform $f \in \mathcal{E}'(R^n)$ can be defined by,

{HLCT *f (t)*} (s) = < *f(t)*, K $_{HL}$ (t, s) > where,

$$K_{HL}(t,s) = \sqrt{\frac{2}{\pi . ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2 .} e^{-\frac{i}{b}st} \quad \text{............ (2.1.1)}$$

Hence the generalized half linear canonical transform of $f \in \mathcal{E}'(R^n)$ can be defined by,

$$\{\, HLCT\ f(t)\ \}(s) = \sqrt{\frac{2}{\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{-\frac{i}{b}st} f(t)\,dt$$

Since the range of integration for the half linear canonical transform is just [0, ∞] and not for (-∞, ∞) using half linear canonical transform is more convenient than using the canonical transform to deal with the even function.

2.2 Inversion for Generalized Half Linear Canonical Transform:

Any transform is used to solve differential equations, only if inverse of the transform is available. We obtain inverse of half linear canonical transform.

2.2.1 Inversion theorem for half linear canonical transform :

If $\{HLCT\ f(t)\}(s)$ is half linear canonical transform of *f(t)* is given by,

$$\{HLCT\ f(t)\}(s) = \sqrt{\frac{2}{\pi ib}}\ e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{-i\left(\frac{s}{b}\right)t} f(t)\,dt$$

then, $$f(t)=\sqrt{\frac{\pi i}{2b}}\, e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2} \int_0^\infty e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{i\frac{s}{b}t} \{HLCTf(t)\}(s)ds$$

Proof: The half linear canonical transform of *f(t)* is given by

$$\{HLCT\ f(t)\}(s)=\sqrt{\frac{2}{\pi ib}}\, e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{-i\left(\frac{s}{b}\right)t} f(t)\,dt$$

$$F_{HL}(s)=\frac{\sqrt{2}}{\sqrt{\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{-i\left(\frac{s}{b}\right)t} f(t)\,dt$$

where, $\{HLCT\ f(t)\}(s)=F_{HL}(s)$

$$\therefore F(s)\sqrt{\frac{\pi ib}{2}}\cdot e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} = \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{-i\left(\frac{s}{b}\right)t} f(t)\,dt$$

$$\therefore C_1(s)=\int_0^\infty g(t)\, e^{-i\left(\frac{s}{b}\right)t} dt$$

where, $C_1(s)=F_{HL}(s)\sqrt{\frac{\pi ib}{2}}\cdot e^{-\frac{1}{2}\left(\frac{d}{b}\right)s^2}$ and $g(t)=e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)$

$$C_1(s)=\int_0^\infty g(t)\, e^{-i\left(\frac{s}{b}\right)t} dt$$

$$\because \left(\frac{s}{b}\right)=\eta \quad => d\eta=\frac{1}{b}ds$$

$$\therefore C_1(s)=\int_0^\infty g(t)\, e^{-i\eta t} d\eta$$

$\therefore$ By using inverse formula

$$\therefore g(t)=\int_0^\infty C_1(s)\, e^{i\eta t} d\eta$$

$$\therefore e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)=\int_0^\infty F_{HL}(s)\sqrt{\frac{\pi ib}{2}}\, e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{i\eta t} d\eta$$

$$f(t)=e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_0^{\infty}F_{HL}(s)\sqrt{\frac{\pi ib}{2}}\,e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\,e^{i\eta t}d\eta$$

$$f(t)=e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_0^{\infty}e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\sqrt{\frac{\pi ib}{2}}\,F_{HL}(s)e^{i\frac{s}{b}t}\frac{1}{b}ds$$

$$f(t)=e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\sqrt{\frac{\pi ib}{2}}\frac{1}{b}\int_0^{\infty}e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\,e^{i\frac{s}{b}t}F_{HL}(s)\,ds$$

$$f(t)=e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\sqrt{\frac{\pi i}{2b}}\int_0^{\infty}e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\,e^{i\frac{s}{b}t}F_{HL}(s)\,ds$$

$$f(t)=\sqrt{\frac{\pi i}{2b}}\,e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_0^{\infty}e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\,e^{i\frac{s}{b}t}\left\{HLCTf(t)\right\}(s)ds$$

3 Properties for Half Linear canonical Transform:

3.3.1 Linear Property for half linear canonical transform:

If *{HLCT f(t)}(s)*, *{HLCT g(t)}(s)* denotes generalized half linear canonical transform of *f(t)*,*g(t)* and P_1,P_2 are constant then,

$$\{HLCT[P_1f(t)+P_2g(t)]\}(s)=P_1\{HLCT\ f(t)\}(s)+P_2\{HLCTg(t)\}(s)$$

Proof: The proof is simple and hence omitted.

3.3.2 Differentiation property of half linear canonical transform:

If *{HLCT f(t)}(s)* denotes generalized half linear canonical transform of *f(t)* then,

$$\{HLCT\ f'(t)\}(s)=i\left\{\left(\frac{s}{b}\right)\{HLCT\ f(t)\ \}(s)-\left(\frac{a}{b}\right)\{HLCT[tf(t)]\}(s)\right\}$$

Proof: We have, $\{HLCT\ f'(t)]\}(s)=\sqrt{\frac{2}{\pi ib}}\,e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_0^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot e^{\frac{-ist}{b}}\cdot f'(t)dt$

$$=C_1(s)\int_0^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot e^{\frac{-i(st)}{b}}\cdot f'(t)dt$$

where, $C_1(s)=\sqrt{\frac{2}{\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}$

$$\{HLCT\ f'(t)\}(s)=C_1(s)\int_0^\infty f'(t)\cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot e^{\frac{-ist}{b}}\cdot dt$$

$$=C_1(s)\left\{\left[e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot e^{\frac{-ist}{b}}\cdot f(t)\right]_0^\infty-\int_0^\infty\left[e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot e^{\frac{-ist}{b}}\cdot\left(\frac{-is}{b}\right)+e^{\frac{-ist}{b}}\cdot i\left(\frac{a}{b}\right)te^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\right]f(t)dt\right\}$$

$$\{HLCT\ f'(t)\}(s)=C_1(s)\{0-\int_0^\infty\left[e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot e^{\frac{-i(st)}{b}}\left(\frac{-is}{b}\right)+e^{\frac{-ist}{b}}\cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot i\left(\frac{a}{b}\right)t\right]f(t)dt\ \}$$

$$\{HLCT\ (f'(t))\}(s)=C_1(s)\left(\frac{is}{b}\right)\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot e^{\frac{-ist}{b}}\cdot f(t)dt-C_1(s)i.\left(\frac{a}{b}\right)\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cdot e^{\frac{-ist}{b}}\cdot t\cdot f(t)dt$$

$$\{HLCT\ f'(t)\}(s)=i\left\{\left(\frac{s}{b}\right)\{HLCT\ f(t)\ \}(s)-\left(\frac{a}{b}\right)\{HLCT[tf(t)]\}(s)\right\}$$

3.3.3 Derivative property of half linear canonical transform:

If *{HLCT f(t)}(s)* denotes generalized half linear canonical transform, then,

$$\frac{d}{ds}[\{HLCTf(t)\}(s)]=i\left\{s.\left(\frac{d}{b}\right)\{HLCTf(t)\}(s)-\frac{1}{b}\{HLCT[t.f(t)]\}(s)\right\}$$

Proof: We have,

$$\frac{d}{ds}\{HLCT\ f(t)\}(s)=\frac{d}{ds}\left\{\sqrt{\frac{2}{\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{-i\left(\frac{st}{b}\right)}\ f(t)dt\right\}$$

$$\frac{d}{ds}\{HLCT\ f(t)\}(s)=\sqrt{\frac{2}{\pi ib}}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\frac{\partial}{\partial s}\left(e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}.e^{-i\left(\frac{st}{b}\right)}\right)f(t)dt$$

$$=\sqrt{\frac{2}{\pi ib}}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\left(e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}.e^{-i\left(\frac{st}{b}\right)}\left(\frac{-it}{b}\right)+i\left(\frac{d}{b}\right).s.e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{-i\left(\frac{st}{b}\right)}\right)f(t)dt$$

$$=\sqrt{\frac{2}{\pi ib}}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}.e^{-i\left(\frac{st}{b}\right)}\left(\frac{-it}{b}\right)f(t)dt+\sqrt{\frac{2}{\pi ib}}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}i\left(\frac{d}{b}\right).s.e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{-i\left(\frac{st}{b}\right)}\ f(t)dt$$

$$\frac{d}{ds}\{HLCT\ f(t)\}(s)=-\frac{i}{b}\{HLCT[t.f(t)]\}(s)+is.\left(\frac{d}{b}\right)\{HLCTf(t)\}(s)$$

$$\frac{d}{ds}\left[\{HLCTf(t)\}(s)\right]=i\left\{s.\left(\frac{d}{b}\right)\{HLCTf(t)\}(s)-\frac{1}{b}\{HLCT[t.f(t)]\}(s)\right\}$$

4.1 Parseval's Identity for half linear canonical transform:

If *f(t)* and *g(t)* are the inversion half linear canonical transform of $F_{HL}(s)$ and $G_{HL}(s)$ respectively, then

(1) $\int_0^\infty f(t).\overline{g(t)}\,dt=(-i)\frac{\pi}{2}\int_0^\infty F_{HL}(s).\overline{G_{HL}(s)}\,ds$ and (2) $\int_0^\infty |f(t)|^2\,dt=(-i)2\pi\int_0^\infty |F_{HL}(s)|^2\,ds$

Proof: By definition of HLCT,

$$\{HLCT\;\; g(t)\}(s)=\sqrt{\frac{2}{\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{-i\left(\frac{1}{b}\right)st}\cdot g(t)\,dt \qquad \text{---------------- (4.1.1)}$$

Using the inversion formula of HLCT

$$g(t)=\sqrt{\frac{\pi i}{2b}}\cdot e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_0^\infty e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{i\left(\frac{1}{b}\right)st}G_{HL}(s)\,ds$$

Taking complex conjugate we get,

$$\overline{g(t)}=\sqrt{\frac{-\pi i}{2b}}\cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_0^\infty e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{-i\left(\frac{1}{b}\right)st}\,\overline{G_{HL}(s)}\,ds$$

$$\int_0^\infty f(t).\overline{g(t)}\,dt=\int_0^\infty f(t)dt\left(\sqrt{\frac{-\pi i}{2b}}\cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_0^\infty e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{-i\left(\frac{1}{b}\right)st}\,\overline{G_{HL}(s)}\,ds\right)$$

Changing the order of integration, we get,

$$\int_0^\infty f(t).\overline{g(t)}\,dt=\sqrt{\frac{-\pi i}{2b}}\int_0^\infty \overline{G_{HL}(s)}\,ds\,\frac{1}{\sqrt{\frac{2}{\pi ib}}}\left(\sqrt{\frac{2}{\pi ib}}\;e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{-i\left(\frac{1}{b}\right)st}\;f(t)dt\right)$$

$$\int_0^\infty f(t).\overline{g(t)}\,dt=(-i)\frac{\pi}{2}\int_0^\infty \overline{G_{HL}(s)}\,.F_{HL}(s)ds$$

$$\int_0^\infty f(t).\overline{g(t)}\,dt=(-i)\frac{\pi}{2}\int_0^\infty F_{HL}(s).\overline{G_{HL}(s)}\,ds \qquad \text{---------------------(4.1.2)}$$

Hence proved

(ii) Putting $f(t)=g(t)$ in equation (4.1.2), we get

$$\int_0^\infty |f(t)|^2 \, dt = (-i)\frac{\pi}{2}\int_0^\infty |F_{HL}(s)|^2 \, ds$$

4. Relation between half linear canonical transform and Laplace transform:

4.4.1 Relation between half linear canonical transform and Laplace transform:

If $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then HLCT change into Laplace transform.

Proof: By definition of HLCT,

$$\{ HLCT\ f(t) \}(s) = \sqrt{\frac{2}{\pi i b}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right).t^2.} e^{-i.\left(\frac{s}{b}t\right).} f(t)\, dt,$$

For $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we get

$$\{ HLCTf(t) \}(s) = \sqrt{\frac{2}{\pi i i}} \cdot e^{\frac{i}{2}\left(\frac{0}{b}\right)s^2} \int_0^\infty e^{\frac{i}{2}\left(\frac{0}{b}\right).t^2.} e^{-i.\left(\frac{s}{i}t\right).} f(t)\, dt$$

$$\{ HLCTf(t) \}(s) = \sqrt{-\frac{2}{\pi}} \cdot \int_0^\infty e^{-st.} f(t)\, dt$$

$$\{ HLCTf(t) \}(s) = \sqrt{-\frac{2}{\pi}} \cdot L[f(t)]$$

This is the Laplace transform.

Table for half linear canonical transform

S.N.	*f(t)*	$F_{HS}(s)$
1	$f^1(t)$	$i\left[\left(\frac{s}{b}\right)\{HCCT\ f(t)\}(s) - \left(\frac{a}{b}\right)\{HCST\ t.f(t)\}(s)\right]$
2	$[P_1 f(t) + P_2 g(t)]$	$P_1\{HLCT\ f(t)\}(s) + P_2\{HLCTg(t)\}(s)$

Conclusion: In this paper, brief introduction of the generalized half linear canonical transform is given and its Inversion theorem is proved. Linearity, differentiation, derivative property, Parseval's Identity, are obtained for half linear canonical transform which will be useful in solving differential equations occurring in signal processing and many other branches of engineering.

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