Generalized Fibonacci Polynomials and Some Identities

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Abstract:- **The Fibonacci and Lucas polynomials are famous for possessing wonderful and amazing properties and identities. In this paper, Generalized Fibonacci polynomials are introduced and defined by $u_n(x) = xu_{n-1}(x) + u_{n-2}(x), n \geq 2$ with $u_0(x) = a$ and $u_1(x) = 2a + 1$, where a is any integer. Further, some basic identities are generated and derived by standard methods.**

Keywords: Generalized Fibonacci polynomials, Generating function, Binet's Formula

1. INTRODUCTION

Fibonacci numbers are a popular topic for mathematical enrichment and popularization. They are famous for a host of interesting and surprising properties and show up in text books, magazine articles, and web sites. Various sequences of polynomials by the name of Fibonacci and Lucas polynomials occur in the literature over a century. The Fibonacci and Lucas polynomials are closely related and widely investigated. Fibonacci polynomials appear in different frameworks. These polynomials are of great importance in the study of many subjects such as algebra, geometry, combinatorics, approximation theory, statistics and number theory itself. Moreover these polynomials have been applied in every branch of mathematics.
The Fibonacci polynomials satisfy the following recurrence formula:

$$f_{n+1}(x) = xf_n(x) + f_{n-1}(x),\ n \geq 2 \ \text{ with } \ f_0(x) = 0,\ f_1(x) = 1. \tag{1.1}$$

The Lucas polynomials [1] are defined by the recurrence formula

$$l_{n+1}(x) = xl_n(x) + L_{n-1}(x),\ n \geq 2 \ \text{ with } \ l_0(x) = 2,\ l_1(x) = x \tag{1.2}$$

Generating function of Fibonacci polynomials is given by

$$\sum_{n=0}^{\infty} f_n(x) t^n = t\left(1 - xt - t^2\right)^{-1}. \tag{1.3}$$

Generating function of Lucas polynomials is given by

$$\sum_{n=0}^{\infty} l_n(x) t^n = (2 - xt)\left(1 - xt - t^2\right)^{-1}. \tag{1.4}$$

Explicit sum formula for Fibonacci polynomials is given by

$$f_n(x) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k} x^{n-1-2k}, \tag{1.5}$$

Explicit sum formula for Lucas polynomials is given by

$$l_n(x)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n}{n-k}\binom{n-k}{k}x^{n-2k}, \tag{1.6}$$

where $\binom{n}{k}$ a binomial coefficient and $[x]$ is define as the greatest integer less than or equal to x.

Fibonacci-Like polynomials [11] is defined by the recurrence relation:

$$s_n(x)=xs_{n-1}(x)+s_{n-2}(x),\ n\geq 2.\ \text{with}\ \ s_0(x)=2\ \text{and}\ s_1(x)=2x. \tag{1.7}$$

Generalized Fibonacci-Like polynomial [12] is defined by the recurrence relation:

$$b_n(x)=xb_{n-1}(x)+b_{n-2}(x),\ n\geq 2.\ \text{with}\ \ b_0(x)=2b\ \text{and}\ b_1(x)=s, \tag{1.8}$$

where b and s are integers.

The Fibonacci and Lucas polynomials possess many fascinating properties which have been studied in [2] to [12]. In this paper, generalized Fibonacci-Like polynomials are introduced with some basic identities.

2. GENERALIZED FIBONACCI POLYNOMIALS

Generalized Fibonacci polynomials $u_n(x)$ are defined by the recurrence relation

$$u_n(x)=xu_{n-1}(x)+u_{n-2}(x),\ n\geq 2.\ \text{with}\ \ u_0(x)=a\ \text{and}\ u_1(x)=2a+1, \tag{2.1}$$

where a is integer.

The first few terms of generalized Fibonacci polynomials are as follows:

$u_0(x)=a,$

$u_1(x)=2a+1,$

$u_2(x)=(2a+1)x+a,$

$u_3(x)=(2a+1)x^2+ax+(2a+1),$

$u_4(x)=(2a+1)x^3+ax^2+2(2a+1)x+a,$

$u_5(x)=(2a+1)x^4+ax^3+3(2a+1)x^2+2ax+(2a+1),$ and so on.

For x = 1 and a = 0, we obtain Fibonacci Sequence.

The characteristic equation of recurrence relation (2.1) is $\lambda^2-x\lambda-1=0$. Which has two real roots $\alpha=\dfrac{x+\sqrt{x^2+4}}{2}$ and $\beta=\dfrac{x-\sqrt{x^2+4}}{2}$

Also, $\alpha\beta=-1,\ \alpha+\beta=x,\ \alpha-\beta=\sqrt{x^2+4},\ \alpha^2+\beta^2=x^2+2.$ (2.2)

Binet's formula of generalized Fibonacci polynomials is given by

$$u_n(x) = A\alpha^n + B\beta^n = A\left(\frac{x+\sqrt{x^2+4}}{2}\right)^n + B\left(\frac{x-\sqrt{x^2+4}}{2}\right)^n \quad (2.3)$$

Here, $A = \frac{(2a+1)-a\beta}{\alpha-\beta}$ and $B = \frac{a\alpha-(2a+1)}{\alpha-\beta}$

Also, $AB = \frac{-(a^2+3a+1)}{(\alpha-\beta)^2}$, $\quad A+B = u_0(x) = a$. (2.4)

Generating function of generalized Fibonacci polynomials is given by

$$\sum_{n=0}^{\infty} u_n(x)t^n = \frac{a+(2a+1-ax)t}{1-xt-t^2} \quad (2.5)$$

Now we obtain hypergeometric representation of generating function.

By generating function (2.5), we have

$$\sum_{n=0}^{\infty} u_n(x)t^n = \frac{a+(2a+1-ax)t}{1-xt-t^2}$$

$$= \left[a+(2a+1-ax)t\right]\left[1-(x+t)t\right]^{-1}$$

$$= \left[a+(2a+1-ax)t\right]\sum_{n=0}^{\infty}(x+t)^n t^n$$

$$= \left[a+(2a+1-ax)t\right]\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n}\binom{n}{k}x^{n-k}t^k$$

$$= \left[a+(2a+1-ax)t\right]\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{k!n-k!}x^{n-k}t^{n+k}$$

$$= \left[a+(2a+1-ax)t\right]\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{n+k!}{k!n!}x^n t^{n+2k}$$

$$= \left[a+(2a+1-ax)t\right]\sum_{n=0}^{\infty}\frac{(xt)^n}{n!}\sum_{k=0}^{\infty}\frac{n+k!}{k!}t^{2k}$$

$$= \left[a+(2a+1-ax)t\right]e^{xt}\sum_{k=0}^{\infty}\frac{n+k!}{k!}(t^2)^k$$

$$\sum_{n=0}^{\infty}\frac{u_n(x)}{n!}t^n = \left[a+(2a+1-ax)t\right]e^{xt}\sum_{k=0}^{\infty}\frac{n+k!}{n!}\frac{(t^2)^k}{k!}$$

$$\sum_{n=0}^{\infty}\frac{u_n(x)}{n!}t^n = \left[a+(2a+1-ax)t\right]e^{xt}\sum_{k=0}^{\infty}\frac{\overline{|n+k+1}}{\overline{|n+1}}\frac{(t^2)^k}{k!}$$

$$\sum_{n=0}^{\infty}\frac{u_n(x)}{n!}t^n = \left[a+(2a+1-ax)t\right]e^{xt}\sum_{k=0}^{\infty}(n+1)_k\frac{(1)_k}{(1)_k}\frac{(t^2)^k}{k!}$$

Hence, $\sum_{n=0}^{\infty} u_n(x)\frac{t^n}{n!} = [a+(2a+1-ax)t]e^{xt}\,{}_2F_1\left(n+1,\ 1;\ 1;\ t^2\right).$ (2.6)

3. SOME IDENTITIES OF GENERALIZED FIBONACCI POLYNOMIALS

In this section, we present some recurrence relations and identities by generating function, and explicit sum formula.

Theorem 3.1: Prove that

$$u_{n+1}(x) - u_{n-1}(x) = xu_n(x),\ n \geq 1. \quad (3.1)$$

Proof: By generating function of generalized Fibonacci polynomials, we have

$$\sum_{n=0}^{\infty} u_n(x)t^n = [a+(2a+1-ax)t]\left(1-xt-t^2\right)^{-1}$$

Differentiating both sides with respect to t, we get

$$\sum_{n=0}^{\infty} nu_n(x)t^{n-1} = [a+(2a+1-ax)t](x+2t)\left(1-xt-t^2\right)^{-2} + (2a+1-ax)\left(1-xt-t^2\right)^{-1}$$

$$\left(1-xt-t^2\right)\sum_{n=0}^{\infty} nu_n(x)t^{n-1} = [a+(2a+1-ax)t](x+2t)\left(1-xt-t^2\right)^{-1} + (2a+1-ax)$$

$$\left(1-xt-t^2\right)\sum_{n=0}^{\infty} nu_n(x)t^{n-1} = (x+2t)\sum_{n=0}^{\infty} u_n(x)t^n + (2a+1-ax)$$

$$\sum_{n=0}^{\infty} nu_n(x)t^{n-1} - \sum_{n=0}^{\infty} nxu_n(x)t^n - \sum_{n=0}^{\infty} nu_n(x)t^{n+1} = \sum_{n=0}^{\infty} xu_n(x)t^n - \sum_{n=0}^{\infty} 2u_n(x)t^{n+1} + (2a+1-ax)$$

Now equating the coefficient of t^n on both sides we get,

$$(n+1)u_{n+1}(x) - nxu_n(x) - (n-1)u_{n-1}(x) = xu_n(x) + 2u_{n-1}(x)$$

$$(n+1)u_{n+1}(x) - (n+1)u_{n-1}(x) = (n+1)xu_n(x)$$

$$u_{n+1}(x) - u_{n-1}(x) = xu_n(x)$$

This is required result.

Theorem 3.2: Prove that

$$u'_{n+1}(x) = xu'_n(x) + u_n(x) + u'_{n-1}(x),\ n \geq 1 \quad (3.2)$$

Proof: By (3.1), we have

$$u_{n+1}(x) - u_{n-1}(x) = xu_n(x),\ n \geq 1.$$

Differentiating both sides with respect to x, we get

$$u'_{n+1}(x) - u'_{n-1}(x) = xu'_n(x) + u_n(x),$$

$$u'_{n+1}(x) = xu'_n(x) + u_n(x) + u'_{n-1}(x).$$

Theorem 3.3: Prove that

$nu_n(x) = xu'_n(x) - 2u'_{n-1}(x),\ n \geq 1$ and $xu'_{n+1}(x) = (n+1)u_{n+1}(x) - 2u'_n(x),\ n \geq 1.$

Proof: By generating function of generalized Fibonacci polynomials, we have

$$\sum_{n=0}^{\infty} u_n(x)t^n = \left[a+(2a+1-ax)t\right]\left(1-xt-t^2\right)^{-1}$$

Differentiating both sides with respect to t, we get

$$\sum_{n=0}^{\infty} nu_n(x)t^{n-1} = (2a+1-ax)\left(1-xt-t^2\right)^{-1} + \left[a+(2a+1-ax)t\right]\left(x+2t\right)\left(1-xt-t^2\right)^{-2} \quad (3.3)$$

Differentiating both sides with respect to x, we get

$$\sum_{n=0}^{\infty} u_n'(x)t^n = \left[a+(2a+1-ax)t\right]\left(1-xt-t^2\right)^{-2} t - at\left(1-xt-t^2\right)^{-1}$$

$$\sum_{n=0}^{\infty} u_n'(x)t^{n-1} = \left[a+(2a+1-ax)t\right]\left(1-xt-t^2\right)^{-2} - a\left(1-xt-t^2\right)^{-1}$$

$$\sum_{n=0}^{\infty} u_n'(x)t^{n-1} + a\left(1-xt-t^2\right)^{-1} = \left[a+(2a+1-ax)t\right]\left(1-xt-t^2\right)^{-2} \quad (3.4)$$

Using (3.4) in (3.3), we get

$$\sum_{n=0}^{\infty} nu_n(x)t^{n-1} = (2a+1-ax)\left(1-xt-t^2\right)^{-1} + \left(x+2t\right)\left\{\sum_{n=0}^{\infty} u_n'(x)t^{n-1} + a(1-xt-t^2)^{-1}\right\}.$$

$$\sum_{n=0}^{\infty} nu_n(x)t^{n-1} = (2a+1-ax)\left(1-xt-t^2\right)^{-1} + \left(x+2t\right)\sum_{n=0}^{\infty} u_n'(x)t^{n-1} + a\left(x+2t\right)(1-xt-t^2)^{-1}.$$

Now equating the coefficient of t^{n-1} on both sides, we get

$$nu_n(x) = xu_n'(x) + 2u_{n-1}'(x). \quad (3.5)$$

Again equating the coefficient of t^n on both sides, we get

$$(n+1)u_{n+1}(x) = xu_{n+1}'(x) + 2u_n'(x),$$

$$xu_{n+1}'(x) = (n+1)u_{n+1}(x) - 2u_n'(x). \quad (3.6)$$

Theorem 3.4: Prove that

$$(n+1)u_n(x) = u_{n+1}'(x) + u_{n-1}'(x),\ n \ge 1.$$

Proof: By (3.1), we have

$$u_{n+1}(x) - u_{n-1}(x) = xu_n(x),\ n \ge 1.$$

Differentiating both sides with respect to x, we get

$$u_{n+1}'(x) - u_{n-1}'(x) = xu_n'(x) + u_n(x),$$

$$xu_n'(x) + u_n(x) = u_{n+1}'(x) - u_{n-1}'(x). \quad (3.7)$$

Using (3.5) in (3.7), we get

$$nu_n(x) - 2u_{n-1}'(x) + u_n(x) = u_{n+1}'(x) - u_{n-1}'(x).$$

$$nu_n(x) + u_n(x) = u_{n+1}'(x) + 2u_{n-1}'(x) - u_{n-1}'(x),$$

$$(n+1)u_n(x) = u_{n+1}'(x) + u_{n-1}'(x). \quad (3.8)$$

ISSN: 2278-0181
ATCSMT-2015 Conference Proceedings

Theorem 3.5: Prove that
$xu_n'(x) = 2u_{n+1}'(x) - (n+2)u_n(x),\ n \geq 0.$

Proof: Using (3.5) in (3.8), we get

$$(n+1)u_n(x) = u_{n+1}'(x) + \frac{1}{2}\left[nu_n(x) - xu_n'(x)\right],$$

$$2(n+1)u_n(x) = 2u_{n+1}'(x) + \left[nu_n(x) - xu_n'(x)\right],$$

$$xu_n'(x) = 2u_{n+1}'(x) + nu_n(x) - (2n+2)u_n(x),$$

$$xu_n'(x) = 2u_{n+1}'(x) + (n-2n-2)u_n(x), \tag{3.9}$$

Theorem 3.6: Prove that
$(n+1)xu_n'(x) = nu_{n+1}'(x) - (n+2)u_{n-1}'(x),\ n \geq 1.$
Proof: Using (3.8) in (3.2), we get

$$(n+1)\left\{u_{n+1}'(x) - xu_n'(x) - u_{n-1}'(x)\right\} = u_{n+1}'(x) + u_{n-1}'(x),$$

$$(n+1)u_{n+1}'(x) - (n+1)xu_n'(x) - (n+1)u_{n-1}'(x) = u_{n+1}'(x) + u_{n-1}'(x),$$

$$(n+1)u_{n+1}'(x) - (n+1)u_{n-1}'(x) - u_{n+1}'(x) - u_{n-1}'(x) = (n+1)xu_n'(x),$$

$$nu_{n+1}'(x) - (n+2)u_{n-1}'(x) = (n+1)xu_n'(x),$$

$$(n+1)xu_n'(x) = nu_{n+1}'(x) - (n+2)u_{n-1}'(x). \tag{3.10}$$

Theorem 3.7: (Explicit Sum Formula) The explicit sum formula for generalized Fibonacci polynomials is given by

$$u_n(x) = a\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}x^{n-2k}. \tag{3.11}$$

Proof: By generating function (2.5), we have

$$\sum_{n=0}^{\infty} u_n(x)t^n = \left[a + (2a+1-ax)t\right]\left(1 - xt - t^2\right)^{-1}$$

$$= \left[a + (2a+1-ax)t\right]\left[1 - (x+t)t\right]^{-1}$$

$$= \left[a + (2a+1-ax)t\right]\sum_{n=0}^{\infty}(x+t)^n t^n$$

$$= \left[a + (2a+1-ax)t\right]\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n}\binom{n}{k}x^{n-k}t^k$$

$$= \left[a + (2a+1-ax)t\right]\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{k!n-k!}x^{n-k}t^{n+k}$$

$$= \left[a + (2a+1-ax)t\right]\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{n+k!}{k!n!}x^n t^{n+2k}$$

$$=\left[a+(2a+1-ax)t\right]\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n-k!}{k!n-2k!}x^{n-2k}t^{n}$$

$$u_n(x)=a\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}x^{n-2k}.$$

Equating coefficients of t^n on both sides, we get required explicit formula.

Theorem 3.8: For positive integer $n \geq 0$, prove that

$$u_n(x)=ax^n\,{}_2F_1\left(\frac{-n}{2},\ \frac{-n+1}{2};\ -n;\ \frac{-4}{x^2}\right). \tag{3.12}$$

Proof. By explicit sum formula (3.11), it follows that

$$u_n(x)=ax^n\sum_{k=0}^{[n/2]}\frac{n-k!}{k!n-2k!}x^{-2k}$$

$$=ax^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(1)_n(-n)_{2k}}{(-n)_k(-1)^{2k}(1)_n}\frac{x^{-2k}}{k!}$$

$$=ax^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k\,2^{2k}\left(\frac{-n}{2}\right)_k\left(\frac{-n+1}{2}\right)_k}{(-n)_k(-1)^{2k}}\frac{x^{-2k}}{k!}$$

$$=ax^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\left(\frac{-n}{2}\right)_k\left(\frac{-n+1}{2}\right)_k}{(-n)_k}\frac{\left(\frac{-4}{x^2}\right)^k}{k!}$$

Hence, $u_n(x)=ax^n\,{}_2F_1\left(\frac{-n}{2},\ \frac{-n+1}{2};\ -n;\ \frac{-4}{x^2}\right).$

Theorem 3.9: For positive integer $n \geq 0$, prove that

$$\sum_{n=0}^{\infty}(c)_n u_n(x)\frac{t^n}{n!}=a(1-xt)^{-c}\,{}_3F_2\left(\frac{c}{2},\frac{c+1}{2},n+1;\frac{n+1}{2};\frac{n+2}{2};\frac{t^2}{(1-xt)^2}\right). \tag{3.13}$$

Proof. Multiplying both sides of the explicit sum formula by $(c)_n\frac{t^n}{n!}$ and summing between the limit $n=0$ to $n=\infty$, we obtain

$$\sum_{n=0}^{\infty}(c)_n u_n(x)\frac{t^n}{n!}=a\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n-k!}{k!\,n-2k!}(c)_n\, x^{n-2k}\frac{t^n}{n!}$$

$$=a\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{n+k!}{k!\,n!\,n+2k!}(c)_{n+2k}\, x^n t^{n+2k}$$

$$=a\left\{\sum_{n=0}^{\infty}(c+2k)_n\frac{(xt)^n}{n!}\right\}\sum_{k=0}^{\infty}\frac{n+k!}{k!\,n+2k!}(c)_{2k}\,t^{2k}$$

$$=a\sum_{k=0}^{\infty}(1-xt)^{-(c+2k)}\frac{n+k!}{k!\,n+2k!}(c)_{2k}\,t^{2k} \quad ,$$

$$\sum_{n=0}^{\infty}(c)_n u_n(x)\frac{t^n}{n!}=a(1-xt)^{-c}\sum_{k=0}^{\infty}\frac{n+k!}{k!\,n+2k!}(c)_{2k}\left[\frac{t^2}{(1-xt)^2}\right]^k$$

$$=a(1-xt)^{-c}\sum_{k=0}^{\infty}\frac{n+k!}{n+2k!}2^{2k}\left(\frac{c}{2}\right)_k\left(\frac{c+1}{2}\right)_k\left[\frac{t^2}{(1-xt)^2}\right]^k\Big/k!$$

$$=a(1-xt)^{-c}\sum_{k=0}^{\infty}\frac{(n+1)_k}{(n+1)_{2k}}2^{2k}\left(\frac{c}{2}\right)_k\left(\frac{c+1}{2}\right)_k\left[\frac{t^2}{(1-xt)^2}\right]^k\Big/k!,$$

$$=a(1-xt)^{-c}\sum_{k=0}^{\infty}\frac{\left(\frac{c}{2}\right)_k\left(\frac{c+1}{2}\right)_k(n+1)_k}{\left(\frac{n+1}{2}\right)_k\left(\frac{n+2}{2}\right)_k}\left[\frac{t^2}{(1-xt)^2}\right]^k\Big/k!$$

Hence, $\sum_{n=0}^{\infty}(c)_n u_n(x)\frac{t^n}{n!}=a(1-xt)^{-c}\,{}_3F_2\left(\frac{c}{2},\ \frac{c+1}{2},\ n+1;\ \frac{n+1}{2};\ \frac{n+2}{2};\ \frac{t^2}{(1-xt)^2}\right).$

Theorem 3.10 (Catalan's Identity): Let $u_n(x)$ be the n^{th} term of generalized Fibonacci polynomials, then

$$u_n^2(x)-u_{n+r}(x)u_{n-r}(x)=\frac{(-1)^{n-r}}{a^2+3a+1}\left[(2a+1)u_r(x)-au_{r+1}(x)\right],\ n>r\geq 1 \qquad (3.14)$$

Proof: Using Binet's formula (2.5), we have

$$u_n^2(x)-u_{n+r}(x)u_{n-r}(x)=(A\alpha^n+B\beta^n)^2-(A\alpha^{n+r}+B\beta^{n+r})(A\alpha^{n-r}+B\beta^{n-r})$$
$$=AB(\alpha\beta)^n\left(2-\alpha^r\beta^{-r}-\alpha^{-r}\beta^r\right)$$

$$= -AB(-1)^{n-r}\left(\alpha^r - \beta^r\right)^2$$

$$= \frac{(a^2+3a+1)}{(\alpha-\beta)^2}(-1)^{n-r}\left(\alpha^r - \beta^r\right)^2$$

$$= (a^2+3a+1)(-1)^{n-r}\frac{\left(\alpha^r - \beta^r\right)^2}{(\alpha-\beta)^2}$$

Since $\dfrac{\alpha^r-\beta^r}{\alpha-\beta} = \dfrac{(2a+1)u_r(x) - au_{r+1}(x)}{(2a+1)^2 - a(2a+1) - a^2} = \dfrac{(2a+1)u_r(x) - au_{r+1}(x)}{(a^2+3a+1)}$

$$u_n^2(x) - u_{n+r}(x)u_{n-r}(x) = \frac{(-1)^{n-r}}{(a^2+3a+1)}\left[(2a+1)u_r(x) - au_{r+1}(x)\right]^2,\ n > r \geq 1.$$

Theorem 3.11(Cassini's Identity): Let $u_n(x)$ be the n[th] term of generalized Fibonacci polynomials, then

$$u_n^2(x) - u_{n+1}(x)u_{n-1}(x) = (-1)^{n-1}(a^2+3a+1),\ n \geq 1 \quad (3.15)$$

Proof. If r = 1 in the Catalan's Identity, then obtained required result.

Theorem 3.12(d'Ocagne's Identity): Let $u_n(x)$ be the n[th] term of generalized Fibonacci polynomials, then

$$u_m(x)u_{n+1}(x) - u_{m+1}(x)u_n(x) = (-1)^n\left[(2a+1)u_{m-n}(x) - au_{m-n+1}(x)\right],\ m \geq 1, n \geq 0, m > n. \quad (3.16)$$

Proof: Using Binet's formula (2.5), we have

$$u_m(x)u_{n+1}(x) - u_{m+1}(x)u_n(x) = (A\alpha^n + B\beta^m)(A\alpha^{n+1} + B\beta^{n+1}) - (A\alpha^{m+1} + B\beta^{m+1})(A\alpha^n + B\beta^n)$$

$$= AB\left(\alpha^m\beta^{n+1} + \alpha^{n+1}\beta^m - \alpha^n\beta^{m+1} - \alpha^{m+1}\beta^n\right)$$

$$= AB(\alpha\beta)^n\left[\beta\left(\alpha^{m-n} - \beta^{m-n}\right)\alpha\left(\alpha^{m-n} - \beta^{m-n}\right)\right]$$

$$= AB(-1)^n(\beta-\alpha)\left(\alpha^{m-n} - \beta^{m-n}\right)$$

$$= \frac{(a^2+3a+1)}{(\alpha-\beta)^2}(-1)^n(\alpha-\beta)\left(\alpha^{m-n} - \beta^{m-n}\right)$$

$$= (a^2+3a+1)(-1)^n\frac{\left(\alpha^{m-n} - \beta^{m-n}\right)}{(\alpha-\beta)}$$

Since, $\dfrac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta}=\dfrac{(2a+1)u_{m-n}(x)-au_{m-n+1}(x)}{(2a+1)^2-a(2a+1)-a^2}=\dfrac{(2a+1)u_{m-n}(x)-au_{m-n+1}(x)}{(a^2+3a+1)}$, we obtain

$$u_m(x)u_{n+1}(x)-u_{m+1}(x)u_n(x)=(-1)^n\left[(2a+1)u_{m-n}(x)-au_{m-n+1}(x)\right], m\geq 1, n\geq 0, m>n.$$

Theorem 3.13 (Generalized Identity): Let $u_n(x)$ be the n[th] term of generalized Fibonacci polynomials, then

$$u_m(x)u_n(x)-u_{m-r}(x)u_{n+r}(x)=(a^2+3a+1)(-1)^{m-r}\left[(2a+1)u_r(x)-au_{r+1}(x)\right]\left[(2a+1)u_{n-m+r}(x)-au_{n-m+r+1}(x)\right],\ n>m\geq r\geq 1 \quad (3.17)$$

Proof: Using Binet's formula (2.5), we have

$$u_m(x)u_n(x)-u_{m-r}(x)u_{n+r}(x)=(A\alpha^m+B\beta^m)(A\alpha^n+B\beta^n)-(A\alpha^{m-r}+B\beta^{m-r})(A\alpha^{n+r}+B\beta^{n+r}),$$

$$=AB(\alpha^r-\beta^r)\left[\frac{\alpha^m\beta^n}{\alpha^r}-\frac{\alpha^n\beta^m}{\beta^r}\right]$$

$$=AB(-1)^{-r}(\alpha^r-\beta^r)(\alpha^m\beta^{n+r}-\alpha^{n+r}\beta^m)$$

$$=AB(-1)^{-r}(\alpha^m\beta^m)(\alpha^r-\beta^r)(\beta^{n-p+r}-\alpha^{n-p+r})$$

$$=-AB(-1)^{-r}(\alpha^m\beta^m)(\alpha^r-\beta^r)(\alpha^{n-p+r}-\beta^{n-p+r})$$

$$=\frac{(a^2+3a+1)}{(\alpha-\beta)^2}(-1)^{-r}(\alpha^m\beta^m)(\alpha^r-\beta^r)(\alpha^{n-p+r}-\beta^{n-p+r})$$

Using subsequent results of Binet's formula, we get

Since, $\dfrac{\alpha^r-\beta^r}{\alpha-\beta}=\dfrac{(2a+1)u_r(x)-au_{r+1}(x)}{(a^2+3a+1)}$, and $\dfrac{\alpha^{n-m+r}-\beta^{n-m+r}}{\alpha-\beta}=\dfrac{(2a+1)u_{n-m+r}(x)-au_{n-m+r+1}(x)}{(a^2+3a+1)}$.

$$u_m(x)u_n(x)-u_{m-r}(x)u_{n+r}(x)=(a^2+3a+1)(-1)^{m-r}\left[(2a+1)u_r(x)-au_{r+1}(x)\right]\left[(2a+1)u_{n-m+r}(x)-au_{n-m+r+1}(x)\right],\ n>m\geq r\geq 1$$

The identity (3.13) provides Catalan's identity, Cassini's and d'Ocagne and other identities.

4. CONCLUSION

In this paper, generalized Fibonacci polynomials is introduced and presented some basic results. Further some recurrence relations and identities are described with derivation by standard methods. The concept of generalized Fibonacci-Like polynomials can be extended in two and three variables with basic results and identities.

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