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Generalized Arithmetic Mean, Harmonic Mean and Geometric Mean Divergence Measures Having One Parameter

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***Abstract*:- Whenever the reliability of the system is considered then there is a birth of divergence, according to one's need. We mainly use the divergence measure to achieve the reliability of the system. In some situations like population dynamics, exponential distributions, growth models etc. where Shannon's measure gives insufficient results we use directed-divergence measures and their generalizations. Entropy based measures have been frequently used as divergence measures. Therefore, information theoretic divergence measures either probabilistic or non-probabilistic are of great importance and have a major effect to get the target. There are a lot of information and divergence measures used in coding theory, information theory and also in mathematics and statistics. A symmetrized and smoothed form of K-L divergence, the Jensen-Shannon divergence (JSD), is of particular interest because of its sharing properties with families of other divergence measures. The uniqueness and versatility of this measure arise because of a number of attributes including generalization to any number of probability distributions. Furthermore, its entropic formulation allows its generalization in different statistical frameworks. We revisit these generalizations and propose a new generalized AM, HM and GM divergence measure in the integrated mathematical framework. We show that this generalization can be interpreted in terms of mutual information divergence measures.**

Keywords:- Divergence, Csiszar's f-divergence, Arithmetic Mean, Harmonic Mean, Geometric Mean, Generalized Mean Divergence Measure, Generalized Square Root Mean Divergence Measure

INTRODUCTION

Probabilistic mean divergence is a necessary process of collecting useful information from different and relevant sources. It extensively exists in many areas including medical diagnosis, business (share market), engineering, insurance, decision making and so on. In the literature of information theory so many techniques have been developed and discussed to gather relevant information. We know that the first measure of information theory was given by C E Shannon in 1948 while working in Bell laboratories. The most popular divergence is K-L relative information measure or cross entropy. Jenson-Shannon (JS), J-divergence and Arithmetic Geometric (AG) mean and some other divergences are also famous classical measures. All these measures have some interesting inequalities. In some situations like population dynamics, exponential distributions, growth models etc. where Shannon's measure and its generalizations are not applicable we use directed-divergence measures and their generalizations. Aczel [1948] defined some mean values. Bhatia [1990] studied some quantitative-qualitative entropy measures and did their applications in coding theory. Jain [1987] studied some characteristics of a relative useful information measure. Tuteja and Bhaker [1993], defined mean values of some useful information measures. Eve [2003] studied seven means from a mathematical and geometrical point of view, which are Harmonic, Geometric, Arithmetic, Heronian, Contra-Harmonic, Root-mean Square and Centroidal. For any type of divergence, the main relation is for the strategies which one has taken or used to achieve the goal or target. Basically the strategies are related with probabilities of that event.

Let $P = (p_1, p_2, \ldots\ldots, p_n)$ be a discrete probability distribution with n outcomes and $Q = (q_1, q_2, \ldots\ldots, q_n)$ be another probability distribution then the well known Kullback-Leibler [1951] measure of directed-divergence was defined as:

$$D(P,Q) = \sum_{i=1}^{n} log\, \frac{p_i}{q_i}$$

The KL divergence is non-symmetric in nature. If q_i is zero then the corresponding is also zero. The KL divergence measure satisfies the following conditions (which are commonly same for divergence measure).

(a) $D(P:Q)$ □ 0
(b) $D(P:Q) = 0$ iff $P = Q$
(c) $D(P:Q)$ is a convex function of P i.e. of $p_1, p_2, \ldots\ldots, p_n$

Now let $$U_n = \left[P = (p_1, p_2, \ldots, p_3) \middle| \; p_i > 0, \; \sum_{i=1}^{n} p_i = 1 \right], \; n \geq 2$$

is the set of all finite discrete probability distributions for all $P, Q \in U_n$. Then the measures are defined as:

$$K(P:Q) = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right) \quad ; \qquad H(P:Q) = \sum_{i=1}^{n} p_i \log\left(\frac{2p_i}{p_i + q_i}\right) \qquad \text{and}$$

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$$G(P:Q)=\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2p_i}\right)$$

are called Relative-Information, Relative Jenson Shannon Divergence and Relative Arithmetic Geometric Divergence. A generalization of above measures have studied by Taneja [2005] and he called these type 's' divergence. It is important to define a generalization of a divergence measure by introducing some real parameter that yields a number of new divergences. A lot of work in this area have done by Jain and Tuteja [1986], Sharma [1978], Taneja [1985], Taneja and Hooda [1983], Dragomir [2001, 2001a], Taneja [1989, 1995, 2001, 2013], Burbea and Rao [1982], etc.

A few years back Eve [2003] studied the geometrical interpretation of the seven means as:
Let a > 0, b > 0 then the means are as follows:

Arithmetic Mean : A (a : b) = $\frac{a+b}{2}$; Geometric Mean : G (a : b) = $\sqrt{ab}$ Harmonic Mean : H (a : b) = $\frac{2ab}{a+b}$; Heronian Mean : H′ (a : b) = $\frac{(a+\sqrt{ab}+b)}{3}$ Contra-Harmonic Mean : (a : b) = $\frac{a^2+b^2}{a+b}$; R-M Square : R (a : b) = $\sqrt{\frac{a^2+b^2}{2}}$ and Centroidal Mean : C (a : b) = $\frac{2(a^2+ab+b^2)}{3(a+b)}$

Now we can verify the following result for the above means.

$$HM \le \Gamma M \le H'M \le AM \le XM \le PM \le XHM$$

Now if we take M (a : b) = b f_m (a/b), where M denotes any of the above seven means, then we have

$$\phi_H(\xi) \le \phi_\Gamma(\xi) \le \phi_{H'}(\xi) \le \phi_A(\xi) \le \phi_X(\xi) \le \phi_P(\xi) \le \phi_{XH}(\xi)$$

Where $f_H(x)=\frac{2x}{x+1}$, $f_G(x)=\sqrt{x}$, $f_{H'}(x)=\frac{(x+\sqrt{x}+1)}{3}$

$$f_A(x)=\frac{(x+1)}{2},\quad f_C(x)=\frac{2(x^2+x+1)}{3(x+1)},\quad f_R(x)=\sqrt{\frac{(x^2+1)}{2}}$$

$$f_{CH}(x)=\frac{x^2+1}{x+1}\qquad \forall\ x>0,\ x\ne 1$$

If we take a = x and b = 1 in any of the above mentioned means then we get the above results. We can also write harmonic, heronian, contra-harmonic, centroidal means and root-mean-square in terms or arithmetic and geometric means.

Csiszar's f-Divergence

Given a function f : [0, ∞) → R, Csiszar [1967] introduced the f-divergence measure as:

$$c_f(P:Q)=\sum_{i=1}^{n}q_i f\left(\frac{p_i}{q_i}\right)\qquad \forall\ P, Q \in U_n$$

Corollary 1: If the function f is convex normalized, i.e. if f (1) = 0, then the f-divergence c_f (P : Q) is non-negative and convex in the pair of probability distribution (P : Q) ∈ $U_n \times U_n$.

Corollary 2: Let f : $R_+ \to R$ be differentiable convex and normalized i.e., f(1) = 0, then

$$0 \le c_f(P:Q) \le E_{c_f}(P:Q);\qquad \text{where}$$

$$E_{c_f}(P:Q)=\sum_{i=1}^{n}(p_i-q_i)f'\left(\frac{p_i}{q_i}\right)\qquad \forall\ P, Q \in U_n \text{ and } \quad i \in (1, 2, \ldots, n)$$

Some New Generalized Probabilistic Mean Divergence Measures

Let us consider the following mean of order m

$$S_m(a:b)=\left[\left(\frac{a^m+b^m}{2}\right)^{\frac{1}{M}};\quad \begin{matrix} m\ne 0 \\ \forall\ a,b\in R\end{matrix}\right] \qquad (1)$$

Now we see that the values of the above equation varies as

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$$S_m(a:b) = \begin{bmatrix} \sqrt{ab} & \text{if} & m=0 \\ \max(a:b) & \text{if} & m=\infty \\ \min(a:b) & \text{if} & m=-\infty \\ & \forall a,b \in R & \end{bmatrix}$$

It can be also easily verified that $S_m(a : b)$ is a monotonic – non – decreasing function in relation to m. Thus the following inequality holds as

$$\Sigma_{-\infty}(\alpha : \beta) \le \Sigma_{-1}(\alpha : \beta) \le \Sigma_0(\alpha : \beta) \le \Sigma_1(\alpha : \beta) \le \Sigma_2(\alpha : \beta) \le \Sigma_\infty(\alpha : \beta)$$

where $S_{-1}(a:b) = \frac{2ab}{a+b} = H(a:b)$, $S_0(a:b) = \sqrt{ab} = G(a:b)$ and $S_1(a:b) = \frac{a+b}{2} = A(a:b)$, $S_2(a:b) = \sqrt{\frac{a^2+b^2}{2}} = S(a:b)$

are called Harmonic mean, Geometric mean, Arithmetic mean and Square root mean. So it can be said that

H (a : b) □ G (a : b) □ A (a : b) □ S (a : b)

Taneja [2005, 2005_a, 2005_b] has also proposed some mean divergence measures as:

(a) Square root – AM Divergence

$$M_{SA}(P\|Q) = \sum_{i=1}^{n} \sqrt{\frac{p_i^2+q_i^2}{2}} - 1 \qquad \forall \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} q_i = 1 \text{ and } P,Q \in U_n$$

(b) Square root – GM Divergence

$$M_{SG}(P\|Q) = \sum_{i=1}^{n} \left(\sqrt{\frac{p_i^2+q_i^2}{2}} - \sqrt{p_i q_i} \right) \qquad \forall \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} q_i = 1 \text{ and } P,Q \in U_n$$

(c) Square root – HM Divergence

$$M_{SH}(P\|Q) = \sum_{i=1}^{n} \left(\sqrt{\frac{p_i^2+q_i^2}{2}} - \frac{2p_i q_i}{p_i+q_i} \right) \qquad \forall \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} q_i = 1 \text{ and } P,Q \in U_n$$

(d) Arithmetic – Geometric Mean Divergence

$$M_{AG}(P\|Q) = 1 - \sum_{i=1}^{n} \sqrt{p_i q_i} \qquad \forall \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} q_i = 1 \text{ and } P,Q \in U_n$$

(e) Arithmetic – Harmonic Mean Divergence

$$M_{AH}(P\|Q) = 1 - \sum_{i=1}^{n} \frac{2p_i q_i}{p_i+q_i} \qquad \forall \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} q_i = 1 \text{ and } P,Q \in U_n$$

(f) Geometric – Harmonic Mean Divergence

$$M_{GH}(P\|Q) = \sum_{i=1}^{n} \left(\sqrt{p_i q_i} - \frac{2p_i q_i}{p_i+q_i} \right) = \sum_{i=1}^{n} \frac{\sqrt{p_i q_i}\left(\sqrt{p_i} - \sqrt{q_i}\right)^2}{p_i+q_i} \qquad \forall \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} q_i = 1 \text{ and } P,Q \in U_n$$

from the above results we can say that the following inequalities hold

$$0 \le M_{\Sigma A}(\Pi \| \Theta) \le M_{\Sigma\Gamma}(\Pi \| \Theta) \le M_{\Sigma H}(\Pi \| \Theta) \quad \alpha\nu\delta \quad 0 \le M_{A\Gamma}(\Pi \| \Theta) \le M_{AH}(\Pi \| \Theta)$$

Some studies have done by Bhattacharyya [1943], Osterreicher and Vajda [2003], etc. related to the mean divergence measures.
Now we propose some new probabilistic divergence measures having one or two parameters.
First, we study divergence measures having one parameter
The measure is defined as

$$M_\lambda(a:b) = \sum_{i=1}^{n} \left[\left(\frac{a_i^2+b_i^2}{a_i+b_i} \right) - \left(\frac{a_i^\lambda+b_i^\lambda}{2} \right)^{\frac{1}{\lambda}} \right] \qquad \text{where } \lambda \in]-\infty, 0 [U] 0, \infty [\qquad (2)$$

which is known as generalized mean divergence measure.
The equation (2) becomes AM, HM and GM divergence for different values of λ as:

$$M_A(a:b)=\sum_{i=1}^{n}\left[\left(\frac{a_i^2+b_i^2}{a_i+b_i}\right)-\left(\frac{a_i+b_i}{2}\right)\right]; \quad \text{when } \lambda=1$$

$$M_H(a:b)=\sum_{i=1}^{n}\left[\left(\frac{a_i^2+b_i^2}{a_i+b_i}\right)-\left(\frac{2a_ib_i}{a_i+b_i}\right)\right]; \quad \text{when } \lambda=-1$$

$$M_G(a:b)=\sum_{i=1}^{n}\left[\left(\frac{a_i^2+b_i^2}{a_i+b_i}\right)-\sqrt{a_ib_i}\right]; \quad \text{when } \lambda\to 0$$

The above results can be easily verified according to the values of λ (for λ = 1, -1, 0)
Bhatia and Singh [2013] have proposed some AM, HM and GM measures. The proposed measure satisfies all the conditions of a probabilistic divergence measure. Thus we can say that the proposed generalized measure is valid and reliable.
Again we discuss some other one parametric probabilistic divergence measures
The proposed measure is

$$M_\lambda^s(a:b)=\sum_{i=1}^{n}\left[\sqrt{\frac{a_i^2+b_i^2}{2}}-\left(\frac{a_i^\lambda+b_i^\lambda}{2}\right)^{\frac{1}{\lambda}}\right] \quad \text{where} \quad \lambda\in\,]-\infty,0\,[\text{U}]\,0,\infty\,[\qquad (3)$$

which is known as generalized square root mean divergence measure and satisfies the properties of a divergence measure. Again we find that the equation (3) varies for different values of λ and contains some particular cases. **Particular Cases:**

$$M_{SA}(a:b)=\sum_{i=1}^{n}\left[\sqrt{\frac{a_i^2+b_i^2}{2}}-\frac{a_i+b_i}{2}\right]; \quad \text{when } \lambda=1$$

$$M_{SG}(a:b)=\sum_{i=1}^{n}\left[\sqrt{\frac{a_i^2+b_i^2}{2}}-\left(\sqrt{a_ib_i}\right)\right]; \quad \text{when } \lambda\to 0$$

$$M_{SH}(a:b)=\sum_{i=1}^{n}\left[\sqrt{\frac{a_i^2+b_i^2}{2}}-\left(\frac{2a_ib_i}{a_i+b_i}\right)\right]; \quad \text{when } \lambda=-1$$

Similarly we can make generalized Arithmetic – Geometric mean divergence measure, Arithmetic – Harmonic mean divergence measure and Geometric – Harmonic mean divergence measure which will satisfy some properties and also have some limiting cases.
Now we prove non negativity and convexity of the proposed measure (equation (3)).

take $a=x,\ b=1$ and $\lambda=1$

Let us consider

$$D_{SA}(x:1)=\sqrt{\frac{x^2+1}{2}}-\frac{x+1}{2} \quad ; \quad x\in(0,\infty)$$

We take 1st and 2nd derivative of above function and get

$$D'_{SA}(x:1)=\frac{1}{2\left(\frac{x^2+1}{2}\right)^{\frac{1}{2}}}\cdot\frac{2x}{2}-\frac{1}{2}=\frac{x}{\sqrt{2}\sqrt{x^2+1}}-\frac{1}{2}$$

And the 2nd derivative is

$$D''_{SA}(x:1)=\frac{\left\{2(x^2+1)\right\}^{\frac{1}{2}}-\frac{x}{2}\left\{2(x^2+1)\right\}^{-\frac{1}{2}}.4x}{2(x^2+1)} \; ; \; =\frac{\left\{2(x^2+1)\right\}^{\frac{1}{2}}-\frac{2x^2}{\left\{2(x^2+1)\right\}^{\frac{1}{2}}}}{2(x^2+1)}$$

$$=\frac{2(x^2+1)-2x^2}{2\sqrt{2}\sqrt{x^2+1}(x^2+1)}=\frac{1}{\sqrt{2}\sqrt{x^2+1}(x^2+1)}$$

Thus we have $D''_{SA}(x:1)>0 \quad \forall\ x\in(0,\infty)$. Also we have $D_{SA}(x:1)=0$ for $x=1$. Thus we can say that the function is non negative and convex.

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Again we have

$$D_{SG}(x:1)=\sqrt{\frac{x^2+1}{2}}-\sqrt{x}; \qquad x\in(0,\infty)$$

where $a=x,\ b=1$ and $\lambda\to 0$

Again we find the 1st and 2nd derivative of the above function and get

$$D'_{SG}(x:1)=\frac{1}{2\left(\frac{x^2+1}{2}\right)^{\frac{1}{2}}}\cdot\frac{2x}{2}-\frac{1}{2x^{\frac{1}{2}}}$$

$$=\frac{x}{2\sqrt{\frac{x^2+1}{2}}}-\frac{1}{2\sqrt{x}}=\frac{1}{\sqrt{2}}\left(\frac{x}{\sqrt{x^2+1}}-\frac{1}{\sqrt{2x}}\right)$$

Again differentiate w. r. t. x and we get

$$D''_{SG}(x:1)=\frac{1}{\sqrt{2}}\left[\left\{\frac{\sqrt{x^2+1}-\frac{x}{2\sqrt{x^2+1}}.2x}{x^2+1}\right\}-\left\{\frac{\frac{-1}{2}x^{-\frac{3}{2}}}{\sqrt{2}}\right\}\right]$$

$$=\frac{1}{\sqrt{2}}\left[\frac{\sqrt{x^2+1}-\frac{x^2}{\sqrt{x^2+1}}}{x^2+1}+\frac{1}{2\sqrt{2}}x^{\frac{3}{2}}\right]$$

$$=\frac{1}{\sqrt{2}}\left[\frac{\frac{x^2+1-x^2}{\sqrt{x^2+1}}}{x^2+1}+\frac{1}{2\sqrt{2}}x^{\frac{3}{2}}\right]=\frac{1}{\sqrt{2}(x^2+1)\sqrt{x^2+1}}+\frac{1}{4x\sqrt{x}}$$

Again, we find that $D''_{SG}(x:1)>0\ \ \forall\ x\in(0,\infty)$. We have also $D_{SG}(x:1)=0$ for $x=1$. Thus it can be said that the above equation is non negative and convex in the pair of probability distributions.

Also, we have

$$D_{SH}(x:1)=\sqrt{\frac{x^2+1}{2}}-\frac{2x}{x+1} \quad ; \quad x\in(0,\infty)$$

When $a=x,\ b=1$ and $\lambda=-1$; we again do derivative of the equation and find

$$D'_{SH}(x:1)=\frac{1}{2\sqrt{\frac{x^2+1}{2}}}\cdot\frac{2x}{2}-2\left[\frac{x+1-x}{(x+1)^2}\right]$$

$$=\frac{x}{\sqrt{2}\sqrt{x^2+1}}-\frac{2}{(x+1)^2}$$

Again differentiate, we get

$$D''_{SH}(x:1)=\frac{1}{\sqrt{2}}\left[\frac{\sqrt{x^2+1}-\frac{x}{2\sqrt{x^2+1}}.2x}{x^2+1}\right]+\frac{2.2}{(x+1)^3}$$

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$$=\frac{1}{\sqrt{2}}\left[\frac{x^2+1-x^2}{(x^2+1)\sqrt{x^2+1}}\right]+\frac{4}{(x+1)^3}=\frac{1}{\sqrt{2}(x^2+1)\sqrt{x^2+1}}+\frac{4}{(x+1)^3}$$ Again we see that $D''_{SH}(x:1)>0 \quad \forall\ x\in(0,\infty)$, and $D_{SH}(x:1)=0$ for $x=1$. Thus it can be also said that equation fulfills the condition of non-negativity and convexity.

Now we prove some theorems.

Theorem 1:- The following inequalities hold for arithmetic and harmonic mean differences.

(a) $0\le M_{SA}(P\|Q)\le\frac{1}{3}M_{SH}(P\|Q)$

and

(b) $0\le \Delta_{SA}(P\|Q)\le\frac{1}{3}\Delta_{SH}(P\|Q)$

Proof: Let us consider

$$K_{SA-SH}(x)=\frac{D''_{SA}(x)}{D''_{SH}(x)}$$

$$=\frac{\frac{1}{\sqrt{2}(x^2+1)\sqrt{x^2+1}}}{\frac{1}{\sqrt{2}(x^2+1)\sqrt{x^2+1}}+\frac{4}{(x+1)^3}}\quad ;\quad x\in(0,\infty)$$

$$=\frac{(x+1)^3}{(x+1)^3+4\sqrt{2}\left(x^2+1\right)^{\frac{3}{2}}}$$

Again differentiate the above equation w. r. t. x, we get

$$K'_{SA-SH}(x)=\left[\frac{D''_{SA}(x)}{D''_{SH}(x)}\right]'$$

$$=\frac{\left[(x+1)^3+4\sqrt{2}(x^2+1)^{\frac{3}{2}}\right]3(x+1)^2.1-(x+1)^3\left[3(x+1)^2+4\sqrt{2}.\frac{3}{2}(x^2+1)^{\frac{1}{2}}.2x\right]}{\left[(x+1)^3+4\sqrt{2}(x^2+1)^{\frac{3}{2}}\right]^2}$$

$$=\frac{3(x+1)^2\left[(x+1)^3+4\sqrt{2}(x^2+1)^{\frac{3}{2}}\right]-(x+1)^3\left[3(x+1)^2+12\sqrt{2}x\sqrt{(x^2+1)}\right]}{\left[(x+1)^3+4\sqrt{2}(x^2+1)^{\frac{3}{2}}\right]^2}$$

$$=\frac{12\sqrt{2}(x+1)^2(x^2+1)\sqrt{(x^2+1)}-12\sqrt{2}x(x+1)^3\sqrt{(x^2+1)}}{\left[(x+1)^3+4\sqrt{2}(x^2+1)^{\frac{3}{2}}\right]^2}$$

$$=\frac{-12\sqrt{2}(x+1)^2\sqrt{(x^2+1)}(x-1)}{\left[(x+1)^3+4\sqrt{2}(x^2+1)^{\frac{3}{2}}\right]^2}$$

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$$=\frac{-24(x-1)(x^2+1)(x+1)^2}{\sqrt{2(x^2+1)}\left[(x+1)^3+4\sqrt{2}(x^2+1)^{\frac{3}{2}}\right]^2}$$

The value of equation varies as

≥ 0 φορ $\xi \leq 1$ ανδ

≤ 0 ωηεν $\xi \geq 1$

Thus we can say that the equation given above increases in $x \in (0, 1)$ and decreases in $x \in (1, \infty)$.

Theorem 2:- The inequalities related to arithmetic and geometric mean holds as

(a) $$0 \leq M_{SA}(P\|Q) \leq \frac{1}{2} M_{SG}(P\|Q)$$

and

(b) $$0 \leq \Delta_{SA}(P\|Q) \leq \frac{1}{2} \Delta_{SG}(P\|Q)$$

Proof: Let us take

$$K_{SA-SH}(x)=\frac{D''_{SA}(x)}{D''_{SG}(x)}$$

$$=\frac{\dfrac{1}{\sqrt{2}(x^2+1)\sqrt{x^2+1}}}{\dfrac{1}{\sqrt{2}(x^2+1)\sqrt{x^2+1}}+\dfrac{1}{4x\sqrt{x}}}$$

$$=\frac{\dfrac{1}{\sqrt{2}\sqrt{x^2+1}(x^2+1)}}{\dfrac{4x\sqrt{x}+\sqrt{2}(x^2+1)\sqrt{x^2+1}}{\left[\sqrt{2}(x^2+1)\sqrt{x^2+1}\right]4x\sqrt{x}}}$$

$$=\frac{4x^{\frac{3}{2}}}{4x^{\frac{3}{2}}+\sqrt{2}(x^2+1)\sqrt{(x^2+1)}}$$

Again, differentiate the above w. r. t. x, we have

$$K'_{SA-SG}(x)=\left[\frac{D''_{SA}(x)}{D''_{SG}(x)}\right]'$$

$$=\frac{\left(4x^{\frac{3}{2}}+\sqrt{2}(x^2+1)\sqrt{(x^2+1)}\right)4.\dfrac{3}{2}x^{\frac{1}{2}}-4x^{\frac{3}{2}}\left[6x^{\frac{1}{2}}+\sqrt{2}(x^2+1).\dfrac{2x}{2\sqrt{(x^2+1)}}+\sqrt{2}\sqrt{(x^2+1)}.2x\right]}{\left[4x^{\frac{3}{2}}+\sqrt{2}(x^2+1)\sqrt{(x^2+1)}\right]^2}$$

$$=\frac{6\sqrt{2}(x^2+1)\sqrt{x(x^2+1)}-12\sqrt{2}x^{\frac{5}{2}}\sqrt{(x^2+1)}}{\left[4x^{\frac{3}{2}}+\sqrt{2}(x^2+1)^{\frac{3}{2}}\right]^2}$$

Again we find that the value of equation variates as

≥ 0 ωηεν $\xi \leq 1$ ανδ

≤ 0 ωηεν $\xi \geq 1$

Again we can say that the above equation or function increases in $x \in (0, 1)$ and decreases in $x \in (1, \infty)$.

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Theorem 3:- The following inequalities related to harmonic mean and geometric mean holds

(a) $0 \le M_{SH}(P\|Q) \le \frac{3}{2} M_{SG}(P\|Q)$

(b) $0 \le \Delta_{SH}(P\|Q) \le \frac{3}{2} \Delta_{SG}(P\|Q)$

Proof: Let us again consider

$$K_{SH-SG}(x) = \frac{D''_{SH}(x)}{D''_{SG}(x)}$$

$$= \frac{\frac{1}{\sqrt{2}(x^2+1)\sqrt{x^2+1}} + \frac{4}{(x+1)^3}}{\frac{1}{\sqrt{2}(x^2+1)\sqrt{x^2+1}} + \frac{1}{4x\sqrt{x}}}$$

$$= \frac{\frac{(x+1)^3 + 4\sqrt{2}(x^2+1)\sqrt{(x^2+1)}}{\sqrt{2}(x^2+1)\sqrt{(x^2+1)}(x+1)^3}}{\frac{4x\sqrt{x} + \sqrt{2}(x^2+1)\sqrt{(x^2+1)}}{\sqrt{2}(x^2+1)\sqrt{(x^2+1)}4x\sqrt{x}}}$$

$$= \frac{4x\sqrt{x}\left[(x+1)^3 + 4\sqrt{2}(x^2+1)\sqrt{(x^2+1)}\right]}{(x+1)^3\left[4x\sqrt{x} + \sqrt{2}(x^2+1)^{\frac{3}{2}}\right]}$$

$$= \frac{4x^{\frac{3}{2}}(x+1)^3 + 4\sqrt{2}(x^2+1)^{\frac{3}{2}}}{(x+1)^3\left[4x^{\frac{3}{2}} + \sqrt{2}(x^2+1)^{\frac{3}{2}}\right]}$$

By adopting the same procedure again differentiate the above function w. r. t. x we get

$$K'_{SH-SG}(x) = \left[\frac{D''_{SH}(x)}{D''_{SG}(x)}\right]'$$

$$(x+1)^3\left[4x^{\frac{3}{2}} + \sqrt{2}(x^2+1)^{\frac{3}{2}}\right]\left[\left[4x^{\frac{3}{2}}(3(x+1)^2 + 12\sqrt{2}(x^2+1)^{\frac{1}{2}}.x\right]\right.$$

$$\left.+\left[(x+1)^3 + 4\sqrt{2}(x^2+1)^{\frac{3}{2}}.6x^{\frac{1}{2}}\right]\right] - 4x^{\frac{3}{2}}\left[(x+1)^3 + 4\sqrt{2}(x^2+1)^{\frac{3}{2}}\right]$$

$$= \frac{\left[(x+1)^3\left[6x^{\frac{1}{2}} + 3\sqrt{2}(x^2+1)^{\frac{1}{2}}.x\right] + \left[4x^{\frac{3}{2}} + \sqrt{2}(x^2+1)^{\frac{3}{2}}\right]3(x+1)^2\right]}{\left[(x+1)^3\left[4x^{\frac{3}{2}} + \sqrt{2}(x^2+1)^{\frac{3}{2}}\right]\right]^2}$$

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$$= \frac{\begin{array}{c}192\sqrt{2}x^4(x^2+1)^{\frac{1}{2}}(x+1)^3+6\sqrt{2}x^{\frac{1}{2}}(x^2+1)^{\frac{3}{2}}(x+1)^6+48x^{\frac{1}{2}}(x^2+1)^3(x+1)^3\\ -12\sqrt{2}x^{\frac{5}{2}}(x^2+1)^{\frac{1}{2}}(x+1)^6-192\sqrt{2}x^3(x^2+1)^{\frac{3}{2}}(x+1)^2-96x^{\frac{3}{2}}(x^2+1)^3(x+1)\end{array}}{\left[(x+1)^3\left[4x^{\frac{3}{2}}+\sqrt{2}(x^2+1)^{\frac{3}{2}}\right]\right]^2}$$

$$= \frac{6\sqrt{2}x^{\frac{1}{2}}(x^2+1)^{\frac{1}{2}}(x+1)^2\left[32x^{\frac{5}{2}}(x-1)+(x+1)^4\left\{(x^2+1)-2x^{\frac{3}{2}}\right\}+4\sqrt{2}(x^2+1)^{\frac{5}{2}}(1-x)\right]}{\left[(x+1)^3\left[4x^{\frac{3}{2}}+\sqrt{2}(x^2+1)^{\frac{3}{2}}\right]\right]^2}$$

We again see that the value of the above function varies as follows

> 0 for $x < 1$ and

< 0 for $x > 1$

So, we conclude that the function increases in x ∈ (0, 1) and decreases in x ∈ (1, ∞).

CONCLUSION

In the present chapter we have proposed some new probabilistic mean divergence measures containing one parameter. Here, we have studied their limiting cases and properties also. Bounds on these proposed measures have also defined by using csiszar's f-divergence. Then we discuss some theorems related to the proposed measures. The concept of weightage can also be used these proposed measures. All these proposed measures can be used in various decision making problems and better results can be found due to the flexibility of parameters. Comparison of one measure can also be done with some other measures and may get some beneficial and fruitful results. These newly developed measures can also be made in fuzzy or generalized fuzzy set theory. Then application can be done in some decision making situations.

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