Fractional Shift Invariant System in the Linear Canonical Transform Domain

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Abstract

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Abstract:

Shift invariant systems are especially important in signal processing, image processing. Fourier transform plays an important role in dealing with shift invariant system. Gudadhe had extended this concept to fractional domain and apply it to fractional Fourier transform.

Here we extend the concept of shift invariant system in the Linear Canonical transform domain.

1. Introduction

Linear Canonical transform (LCT) which is the generalized of Fractional Fourier transform (FrFT) of time and frequency plane is defined by,

$$\left[LCT[f(t)]\right](u) = F_A(u) = \langle f(t), K_A(t,u)\rangle$$

where the kernel is,

$$K_A\text{ (u, t)}=\sqrt{\frac{1}{2\pi ib}}\cdot e^{i\pi\frac{d}{b}u^2}\cdot e^{-i\pi\frac{1}{b}ut}\cdot e^{i\pi\frac{a}{b}t^2}, \qquad b\neq 0$$

(1)

$$=\sqrt{d}\cdot e^{i\pi cd\,u^2}\cdot f(du), \qquad b=0 \qquad (1)$$

and $a, b, c, d$are real numbers with $ad - bc = 1$. The kernel can also be viewed as a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with determinant $ad - bc = 1$.

This transform is linear but not shift invariant. Gudadhe and Thakare [3] had defined unitary canonical shift operator $\left(L_\rho^A f\right)(s) = f(s - a\rho)e^{\frac{ia}{2b}\rho^2(1-ad)}\cdot e^{-i\frac{1}{b}s\rho(1-ad)}$

which shifts the support of signal $f(s)$ in the time-frequency plane by the amount ρ.

Linearity and time invariance are the basic properties of system. Many physical processes can be modeled by linear time invariant systems which can be analyzed in details. Moreover the complete characterization of linear shift invariant system can be developed in terms of its response to a unit impulse.

As mentioned in [4], one of the most important properties of Fourier transform with regard to its use in dealing with linear time invariant system, is its effect on convolution operator.

It is derived that if $f(t)$, $g(t)$ are input and output of linear time invariant system with impulse response $h(t)$ then $g(t) = f(t) \star h(t)$, where $\star$ denotes convolution operator given by

$$g(t) = \int_{-\infty}^{\infty} f(t)h(T - t)dT \quad (2)$$

Then the convolution theorem of Fourier transform gives

$$G(w) = H(w)F(w) \quad (3)$$

where $F(w), G(w)$ are Fourier transform of $f(t)$, $g(t)$ resp. Moreover $H(w)$ is Fourier transform of impulse response $\delta(t-\rho)$ i.e. $H(w) = e^{-iw\rho}$.

Thus (3) gives $G(w) = e^{-iw\rho} F(w)$

This result is consistent with shifting property of Fourier transform. That is if

$\{Ff(t)\}(w) = F(w)$ then

$\{Ff(t-\rho)\}(w) = e^{-iw\rho} F(w)$

In this paper a new linear canonical invariant system is formulated which will give similar type of consistency with the corresponding shifting property of Linear Canonical transform as shown above for conventional Fourier transform.

It is known that if LCT of $f(t)$is denoted by

$\{L_A f(t)\}(s)$ then

$$\{L_A f(t-\rho)\}(s) = \{L_A f(t)\}(s - a\rho)e^{\frac{ia}{2b}\rho^2(1-ad)} \cdot e^{-i\frac{1}{b}s\rho(1-ad)}$$

Simple calculation shows that right hand side of above equation can also be expressed as

$$\{L_A f(t-\rho)\}(s) = \sqrt{\frac{1}{2\pi ib}} \int_{-\infty}^{\infty} e^{\left[\frac{id}{2b}(s-a\rho)^2 - \frac{i}{b}(s-a\rho)t + \frac{ia}{2b}t^2\right]} f(t)dt \cdot e^{\frac{ia}{2b}\rho^2(1-ad)} \cdot e^{-i\frac{1}{b}s\rho(1-ad)}$$

$\{L_A f(t-\rho)\}(s) =$
$e^{\left[\frac{ia}{2b}\rho^2-\frac{i}{b}s\rho\right]}\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty} e^{\left[\frac{id}{2b}s^2-\frac{i}{b}st+\frac{ia}{2b}t^2\right]}e^{i\frac{a}{b}\rho t}f(t)dt\cdot e^{\frac{id}{2b}a^2\rho^2-\frac{id}{2b}2as\rho-\frac{id}{2b}a^2\rho^2+\frac{id}{b}as\rho}$

$$\{L_A f(t-\rho)\}(s) = e^{\left[\frac{ia}{2b}\rho^2-\frac{i}{b}s\rho\right]}\left\{L_A e^{i\frac{a}{b}\rho t}f(t)\right\}(s) \quad (4)$$

2. Convolution for Linear Canonical Transform

[2] gives new definition for canonical type convolution $\star$ in terms of usual convolution $*$.

For any function$f(x)$, let us define the functions $\bar{f}(x)$ by $\bar{f}(x) = f(x)e^{i\pi\frac{a}{b}x^2}$. For any two functions f and g, we define the convolution operation $\star$ by

$$h(x) = (f \star g)(x) = \sqrt{-i}\cdot\ e^{-i\pi\frac{d}{b}x^2}\cdot(\bar{f} * \bar{g})(x) \quad (5)$$

Let $h(x) = (f \star g)(x)$ and F_A, G_A, H_A denote the Linear Canonical transform of f, g, h respectively. Then

$$H_A(u) = F_A(u)G_A(u)e^{-i\pi\frac{d}{b}u^2} \quad (6)$$

This can be considered as generalization of (3) since the coefficient term is of unit modulus.

3. Linear Canonical shift invariant system

We define linear canonical shift invariant system as canonical convolution of two functions, one is $e^{i\frac{a}{b}\rho t}f(t)$ and other is $h(t)$, where $h(t)$ is impulse response at t

That is $g(t) = e^{i\frac{a}{b}\rho t}f(t) \star h(t)$

By using convolution theorem as given in (6) for canonical domain we have

$$\{L_A g(t)\}(s) = e^{-\frac{id}{2b}s^2}\left\{L_A e^{i\frac{a}{b}\rho t}f(t)\right\}(s)\{L_A\delta(t-\rho)\}(s)$$

$$= e^{-\frac{id}{2b}s^2}e^{\frac{i}{2b}(ds^2+a\rho^2)-\frac{i}{b}s\rho}\left\{L_A e^{i\frac{a}{b}\rho t}f(t)\right\}(s)$$

$$= e^{\frac{ia}{2b}\rho^2-\frac{i}{b}s\rho}\left\{L_A e^{i\frac{a}{b}\rho t}f(t)\right\}(s) \quad (7)$$

Note that this result coincides with the shifting property for Linear Canonical transform in its new form as given in (4).

Thus output of the linear canonical shift invariant system is given by canonical convolution of the functions $e^{i\frac{a}{b}\rho t}x(t)$ and $h(t)$, where $h(t)$ is the impulse response.

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4. Conclusion

A new definition of linear canonical shift invariant system is introduced. This definition generalized the idea of linear shift invariant system in the Linear Canonical transform domain.

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