Fractional Order Model on Giving up Smoking

A. George Maria Selvam[1],
[1] Sacred Heart College,
Tirupattur - 635 601, S.India

R. Janagaraj[2]
[2]Kongunadu College of Engineering andTechnology,
Thottiam-621 215, S.India.

D. Abraham Vianny[3]
[3]Knowledge Institute of Technology,
Kakapalayam-637 504, S.India

Abstract **- In this paper, we analyze a fractional order model on smoking in which the population is divided into three classes: potential smokers, smokers and quitters. Using fractional order differential dynamical theory, we study the effect of smokers on quitters. The equilibrium points are established and stability of the equilibrium points is discussed. Finally we examine the stability of both equilibriums and the results are illustrated by numerical simulations, and they exhibit rich dynamics of the fractional model.**

Keywords: Fractional Order, Differential equations, Smoking, Equilibrium points, stability.

1. INTRODUCTION

Tobacco epidemic threatens the lives of one billion men, women and children during this century. Tobacco use can kill in so many ways that it is a risk factor for six of the eight leading causes of death in the world. The cure for this devastating epidemic is dependent not on medicines or vaccines, but on the concerted actions of government and civil society. Research on health effects of tobacco has focused primarily on cigarette tobacco smoking. In 1950, Richard Doll published research findings in the British Medical Journal showing a close link between smoking and lung cancer. Four years later, in 1954, a study by some 40,000 doctors over 20 years, confirmed the suggestion, based on which the government issued advice that smoking and lung cancer rates were related. An estimated 5.4 million people die manually from lung cancer, heart disease and other illnesses due to tobacco. If Unchecked, the number may increase to more than 8 million a year by 2030.

2. REVIEW OF FRACTIONAL CALCULUS

Fractional order calculus deals with integrals and derivatives of arbitrary order. Fractional calculus is a natural extension of classical calculus. The origin of the theory of fractional calculus can be traced back to a letter dated September 30th, 1695 written by L'Hopital to Leibniz. Fractional calculus has emerged as one of the most important interdisciplinary subjects in Mathematics, Physics, Biology and Engineering.

Definition 1. [5] Riemann-Liouville definition is

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left[\frac{d}{dt}\right]^n \int_a^{\alpha} \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, (n-1 \le \alpha < n) $$

Definition 2.[5] The Caputo definition is

$$ {}_a^CD_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha-n)}\left[\frac{d}{dt}\right]^n \int_a^{\alpha} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, (n-1 \le \alpha < n) $$

The stability results for the fractional order linear system are given below.

Lemma 3. [6, 7] *The fractional - order autonomous system*

$$ D^{\alpha}x = Ax, \ \ x(0) = x_0 $$

where $0< \alpha < 1$ *,* $x \epsilon R^n$ *and* $A \epsilon R^{n\times n}$ *is*

(a) *asymptotically stable if and only if*

$$ |\arg(\lambda_i(A))| > \alpha\frac{\pi}{2}, (i = 1,2,\dots,n). $$

(b) *stable if and only if*

$$ |\arg(\lambda_i(A))| \ge \alpha\frac{\pi}{2}, (i = 1,2,\dots,n). $$

where $\arg(\lambda_i(A))$ *denotes the argument of the eigenvalue* λ_i *of* A.

3. MODEL DESCRIPTION OF FRACTIONAL ORDER

In 1997, Castillo-Garsow et al. [4] proposed a system described by PSQ model for giving up smoking. The total population $N(t)$ is divided into three classes: potential smokers (P), smokers (S), and people who have quit smoking permanently (Q), such that $N(t) = P(t) + S(t) + Q(t)$. The mathematical model of smoking is described by the following three nonlinear differential equations: [9]

$$ \begin{aligned} \frac{dP}{dt} &= \mu - \mu P(t) - \beta P(t)S(t) \\ \frac{dS}{dt} &= -(\mu + \gamma)S(t) + \beta P(t)S(t) + \alpha Q(t) \qquad (1) \\ \frac{dQ}{dt} &= -(\mu + \alpha)Q(t) + \gamma(1-\sigma) S(t) \end{aligned} $$

In (1), β is the contact rate between potential smokers and smokers, μ is the rate of natural death, α is the contact rate between smokers and quitters who revert back to smoking, γ is the rate of quitting smoking, $(1-\sigma)$ is the fraction of smokers who quit smoking (at a rate γ).

Several authors formulated fractional order systems and analyzed the dynamical and qualitative behavior of the systems [1, 3, 8]. Following this trend, In this paper, we propose a system of fractional order smoking model. We assume the following fractional order Mathematical Model on Smoking.

$$\begin{aligned} D^{\alpha_1} P(t) &= \mu - \mu P(t) - \beta P(t)S(t) \\ D^{\alpha_2} S(t) &= -(\mu + \gamma)S(t) + \beta P(t)S(t) + \alpha Q(t) \qquad (2) \\ D^{\alpha_3} Q(t) &= -(\mu + \alpha)Q(t) + \gamma(1-\sigma) S(t) \end{aligned}$$

where the parameter $\mu, \beta, \gamma, \alpha$ and σ are all positive and $\alpha_1, \alpha_2, \alpha_3$ are all fractional derivative orders.

4. STABILITY ANALYSIS OF EQUILIBRIUM POINTS

To evaluate the equilibrium point, let us consider

$$D^{\alpha_1} P(t) = 0;\ D^{\alpha_2} S(t) = 0;\ D^{\alpha_3} Q(t) = 0.$$

The fractional order system has two equilibrium points; the smoking-free equilibrium $E_0 = (1,0,0)$ and the smoking-present equilibrium $E_1 = (P^*, S^*, Q^*)$, where

$$P^* = \frac{\mu}{\beta} + \frac{\gamma(\mu + \alpha\sigma)}{\beta(\mu+\alpha)}, S^* = \frac{\mu(\mu+\alpha)}{\mu(\mu+\alpha) + \gamma(\mu+\alpha\sigma)} - \frac{\mu}{\beta}, Q^* = \frac{\mu\gamma(\sigma-1)}{\beta(\mu+\alpha)} - \frac{\mu\gamma(\sigma-1)}{\mu(\mu+\alpha) + \gamma(\mu+\alpha\sigma)}.$$

Here $R_0 = \frac{\beta(\mu+\alpha)}{\mu(\mu+\alpha)+\gamma(\mu+\alpha\sigma)}$. Following lemma [2] is needed for the analysis of the stability properties.

Lemma 4. *If $R_0 < 1$, the smoking-free equilibrium E_0 is asymptotically stable. If $R_0 = 1$, E_0 is Stable; $R_0 > 1$, E_0 is Unstable.*

Based on (2), to investigate the stability of each equilibrium point (P^*, S^*, Q^*), we provide the Jacobian matrix $J(P^*, S^*, Q^*)$.

$$J = \begin{bmatrix} -\mu - \beta S & -\beta P & 0 \\ \beta S & -(\mu+\gamma) + \beta P & \alpha \\ 0 & \gamma(1-\sigma) & -(\mu+\alpha) \end{bmatrix} . (3)$$

For E_0, we have

$$J(E_0) = \begin{bmatrix} -\mu & -\beta & 0 \\ 0 & -(\mu+\gamma) + \beta & \alpha \\ 0 & \gamma(1-\sigma) & -(\mu+\alpha) \end{bmatrix}$$

Trace $J(E_0) = \beta - [3\mu + \alpha + \gamma]$ and Det $J(E_0) = \mu^2[\beta - \alpha - \gamma - \mu] + \mu\alpha[\beta - \gamma\sigma]$. The eigen values of matrix $J(E_0)$ are $\lambda_1 = -\mu$ and $\lambda_{2,3} = \frac{\beta-\alpha-\gamma}{2} - \mu \pm \frac{1}{2}\sqrt{(\alpha+\beta)^2 + \gamma[2(\alpha-\beta)+\gamma] - 4\alpha\gamma\sigma}$.

while for E_1 we have

$$J(E_1) = \begin{bmatrix} -A & -\mu - B & 0 \\ A - \mu & B - \gamma & \alpha \\ 0 & \gamma(1-\sigma) & -(\mu+\alpha) \end{bmatrix}$$

where $A = \frac{\mu\beta(\mu+\alpha)}{\mu(\mu+\alpha)+\gamma(\mu+\alpha\sigma)}$ and $B = \frac{\gamma(\mu+\alpha\sigma)}{(\mu+\alpha)}$. The characteristic polynomial $J_{(E_1)}(\lambda)$ for $J(E_1)$ is

$$J_{(E_1)}(\lambda) = \lambda^3 + s_1\lambda^2 + s_2\lambda + s_3$$

where

$$s_1 = [\mu + \alpha + \gamma] - [B - A],$$
$$s_2 = (\mu+\alpha)[A - B + \gamma] - \alpha\gamma(1-\sigma) + A(\mu+\gamma) - \mu(\mu+B),$$
$$s_3 = (\mu+\alpha)[A(\mu+\gamma) - \mu(\mu+B)] - A\alpha\gamma(1-\sigma).$$

If $s_1 s_2 > s_3$, the Routh Hurwitz criterion implies that all roots of $J_{(E_1)}(\lambda)$ have negative real parts, E_1 is stable. This conditions are in contrast to the existence condition of E_1. It means that E_1 is unstable.

5. NUMERICAL SOLUTIONS AND EXAMPLES

Numerical solution of the fractional- order smoking model is given as follows [7]:

$$P(t_k) = \left(\mu - \mu P(t_{k-1}) - \beta P(t_{k-1})S(t_{k-1})\right)h^{\alpha_1} - \sum_{j=v}^{k} c_j^{(\alpha_1)} P(t_{k-j})$$

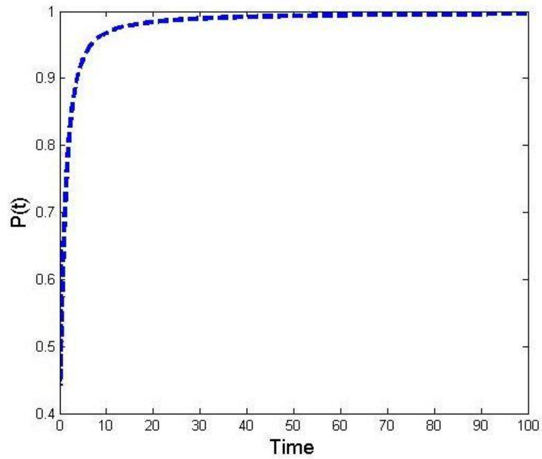
$$S(t_k) = \left(-(\mu+\gamma)S(t_{k-1}) + \beta P(t_{k-1})S(t_{k-1}) + \alpha Q(t_{k-1})\right)h^{\alpha_2} - \sum_{j=v}^{k} c_j^{(\alpha_2)} S(t_{k-j})$$

$$Q(t_k) = \left(-(\mu+\alpha)Q(t_{k-1}) + \gamma(1-\sigma) S(t_{k-1})\right)h^{\alpha_3} - \sum_{j=v}^{k} c_j^{(\alpha_3)} Q(t_{k-j})$$

where T_{sim} is the simulation time, $k = 1,2,3,\ldots,N$, for $N = [T_{sim}/h]$, and $(P(0), S(0), Q(0))$ is the start point (initial conditions). In this section we develop several numerical simulations of the fractional order model (2).

Example 1. *Let us consider the parameter values $\alpha = 0.5, \beta = 0.05, \gamma = 0.5, \mu = 0.9$ and $\sigma = 0.4$ with the initial conditions $P(0) = 0.65, S(0) = 0.30, Q(0) = 0.05$. Also we take the fractional derivatives $\alpha_1 = \alpha_2 = \alpha_3 = 0.85$. The eigen values are $\lambda_1 = -0.90, \lambda_2 = -0.9869$ and $\lambda_3 = -1.7631$. The time plot of potential smokers(nonsmokers) $P(t)$, smokers $S(t)$ and quit smoking $Q(t)$ diagram illustrate the result, see Figure-1. Here reproduction number $R_0 = \frac{\beta(\mu+\alpha)}{\mu(\mu+\alpha)+\gamma(\mu+\alpha\sigma)} = \frac{0.07}{1.81} = 0.0386 < 1$ and for the fractional order system(2), $|\arg(\lambda_{1,2,3})| = 3.1416 > 1.3357 = \alpha\frac{\pi}{2}$. Hence Lemma (4) and Lemma (3), the smoking-free equilibrium E_0 of the system (2) isasymptotically stable and the characteristic equation of the linearized system(2) at the smoking-free equilibrium E_0 is ([7])*

$$\lambda^{255} + 2.65\lambda^{170} + 3.315\lambda^{85} + 1.566 = 0$$



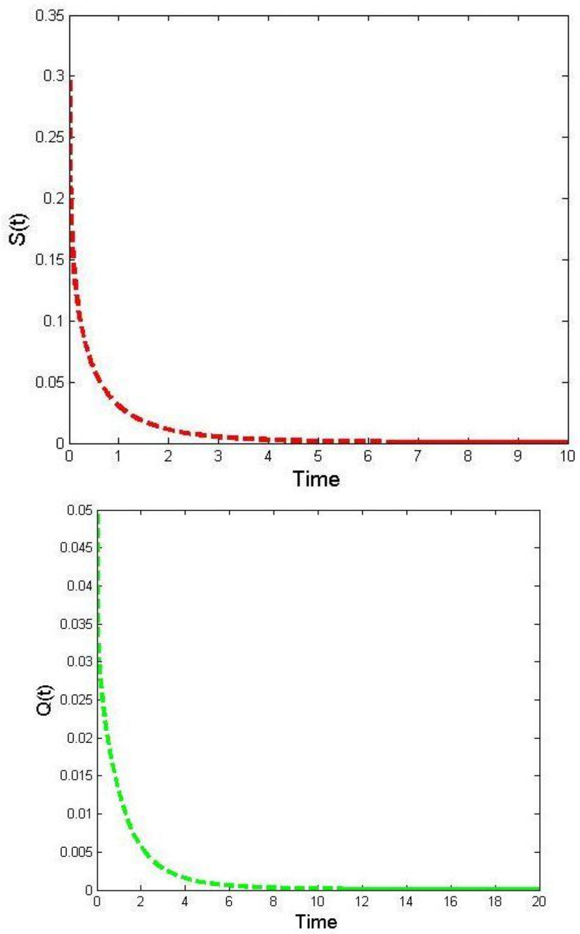


Figure.1 Time series of smoking-free equilibrium E_0 with Stability of $R_0 < 1$.

Example 2. Let us consider the parameter values $\alpha = 0.25, \beta = 0.15, \gamma = 0.02, \mu = 0.04$ *and* $\sigma = 0.3$ *with the initial conditions* $P(0) = 0.65, S(0) = 0.30, Q(0) = 0.05$ *. Also we take the fractional derivatives* $\alpha_1 = \alpha_2 = \alpha_3 = 0.85$, *it is the smoking-present equilibrium with the approximate solutions* $\big(P(t), S(t), Q(t)\big) = (0.319, 0.567, 0.027)$. *The eigen values are* $\lambda_1 = -0.0567, \lambda_2 = -0.0693$ *and* $\lambda_3 = -0.3012$. *The time plot of potential smokers(nonsmokers)* $P(t)$, *smokers* $S(t)$ *and quit smoking* $Q(t)$ *diagram illustrate the result, see Figure-2. Here reproduction number* $R_0 = \frac{\beta\ (\mu+\alpha)}{\mu(\mu+\alpha)+\gamma(\mu+\alpha\sigma)} = \frac{0.0435}{0.0139} = 3.129 > 1$ *and for the fractional order autonomous system(2),* $|\arg(\lambda_{1,2,3})| = 3.1416 > 1.3357 = \alpha\frac{\pi}{2}$. *Hence Lemma (4) and Lemma (3), the smoking-present equilibrium* E_1 *of the system (2) is stable and the characteristic equation of the linearized system(2) at the smoking-present equilibrium* E_1 *is ([7])*

$$\lambda^{255} + 0.4272\lambda^{170} + 0.0419\lambda^{85} + 0.0012 = 0.$$

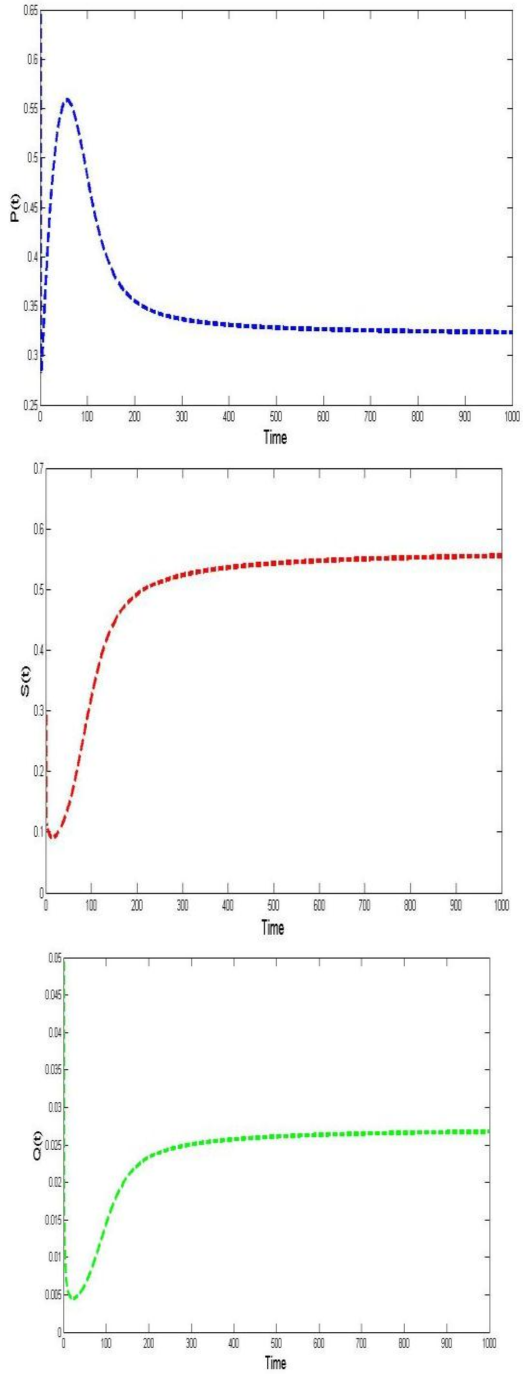


Figure.2 Time series of smoking-present equilibrium E_1 with Stability of $R_0 > 1$.

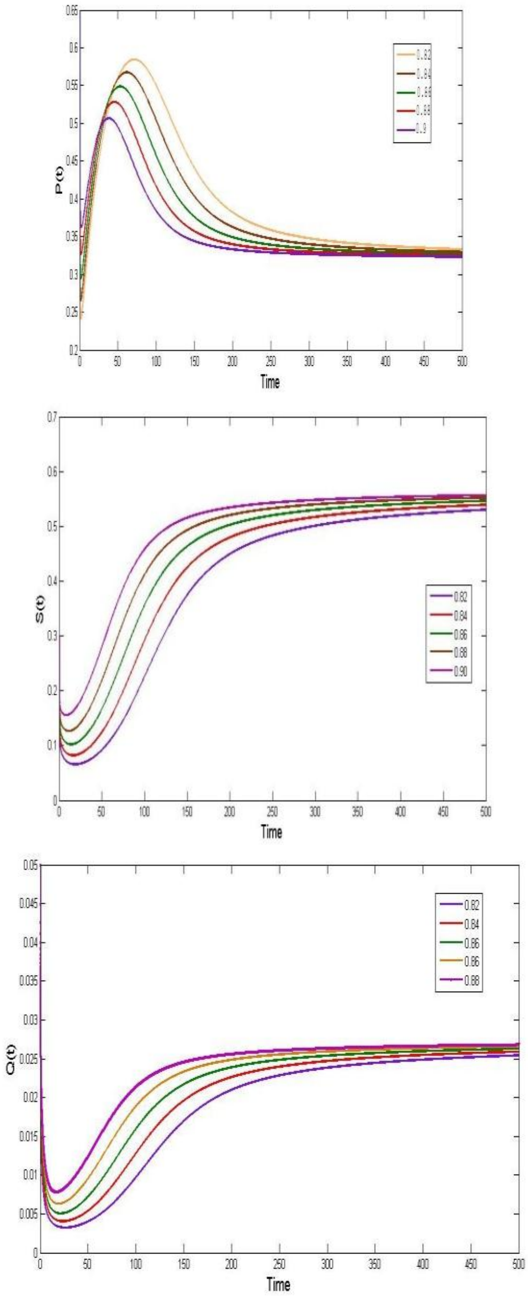


Figure.3 Time series of smoking-present equilibrium E_1 and different fractional derivatives ($\alpha' s$) with Stability.

6. CONCLUSION

In this paper, we presented the Fractional Order Model on giving up smoking. Dynamic properties are discussed by computing equilibrium points and their stability properties. Finally numerical examples are presented for various fractional orders.

REFERENCES

[1] A. A. M. Arafa, S. Z. Rida, and M. Khalil, *A fractional-order model of HIV infection with drug therapyeffect*, Journal of the Egyptian Mathematical Society, 2013.

[2] Binay Kumar Mishra, Santosh Kumar Srivastava, Bimal Kumar Mishra, *A quarantine model on the spreading behavior of worms in wireless sensor network*, Transaction on IoT and CloudComputing 2(1) 1-12 2014.

[3] Carla M. A. Pinto, J. A. Tenreiro Machado, *Fractional Dynamics of Computer Virus Propagation*,Hindawi Publishing Corporation, Mathematical Problems in Engineering, Volume 2014, Article ID476502, 7 pages.

[4] C. Castillo-Garsow, G. Jordan-Salivia, and A. Rodriguez Herrera, *Mathematical models for the dynamics of tobacco use, recovery, and relapse*, Technical Report Series BU-1505-M, Cornell University,Ithaca, NY, USA, 1997.

[5] Leticia Adriana RamrezHernndez, Mayra Guadalupe Garca Reyna and Juan Martnez Ortiz, *Populationdynamics with fractional equations (Predator-Prey)*, Vol. 23 (NE-2) ClculofraccionarioNoviembre 2013.

[6] MehdiDalir and Majid Bashour, *Applications of Fractional Calculus*, Applied Mathematical Sciences, Vol. 4, 2010, no. 21, 1021 - 1032.

[7] IvoPetras, *Fractional order Nonlinear Systems - Modeling, Analysis and Simulation*, Higher Education Press, Springer International Edition, April 2010.

[8] J. A. T.Machado, *Analysis and design of fractional-order digital control systems*, Systems AnalysisModelling Simulation, vol. 27, no. 2-3, pp. 107122, 1997.

[9] Zainab Alkhudhari, Sarah Al-Sheikh, and Salma Al-Tuwairqi, *Global Dynamics of a Mathematical Modelon Smoking*, Hindawi Publishing Corporation, ISRN Applied Mathematics, Volume 2014, Article ID847075,7pages,ttp://dx.doi.org/10.1155/2014/847075.