Formulation of a Generalized Jacobian and Inverse Acceleration Analysis of a Linear Delta Manipulator Based on Reciprocal Screws Method

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Abstract

This paper presents the inverse velocity and acceleration analyses of a three DOF Delta linear parallel manipulators mainly focusing on screw algebra. First the generalized Jacobian of the Delta parallel manipulator is derived using reciprocal screws technique. Geometrical observation method is employed to determine the reciprocal screw systems. This generalized Jacobian will be used to formulate the acceleration of the platform. For the acceleration analysis a new Hessian matrix of serial kinematic chains (or limbs) is developed in an explicit and compact form using Lie brackets. This idea is then extended to cover parallel manipulators.

1. Introduction

A parallel manipulator is a mechanical system that uses several computer-controlled serial chains to support a single platform, or end-effector. Perhaps, the best known parallel manipulator is formed from six linear actuators that support a movable base for devices such as flight simulators. This device is called a Stewart platform or the Gough-Stewart platform in recognition of the engineers who first designed and used them [1].

Parallel manipulators have a kinematic structure that has some advantages over serial manipulators such as rigidity, precision, velocity and acceleration. Nowadays, these characteristics are highly demanded in the manufacturing of aeronautical components and for the automotive industry. Parallel manipulators have found a place in applications such as flight simulators or pointing devices, but they have not jumped into industrial production yet. one of the basic reason is the kinematic, stiffness, singularity and dynamics analysis of a parallel manipulators is not an easy task.

Velocity, accuracy stiffness and dynamics are the important performance factors that should essentially be considered in design of parallel manipulators, particularly in the many circumstances where high speed, high precision and high rigidity are basic requirements. The common manipulation required for velocity, accuracy, stiffness modelling is to formulate a specific linear map between two vector spaces at a given configuration. Huang [2] has recently developed a mathematical framework to unify all these three. He propose a general approach for Jacobian analysis of lower mobility manipulators by simultaneously taking into account both instantaneous permitted and restricted motions of the end-effector. This Jacobian is known as generalized Jacobian, unlike an overall Jacobian it incorporate restricted motions of the end-effector.

Jacobian analysis has been a focus of research area for many years as far as parallel manipulators are concerned. For this reason there are so many research works available on the subject. Jacobian analysis for different parallel manipulators has been conducted using different approaches. Among these approaches are direct differentiation method, the kinematic influence coefficient method and screw-based method. A Jacobian analysis of 6-DOF parallel manipulator was presented using a screw-based method [1] [3]. Screw-based method is more powerful and easily gives deep insight into the topological structure of robotic system and this approach is well outlined by Tian Huang [2]. A paper on inverse, direct, and intermediate Jacobians of a delta robot was published by Lopez using a direct differentiation method [4].

In relation to acceleration analysis, screw theory based approaches could potentially be the most powerful method for acceleration analysis. In order to overcome the difficulty of expressing the twist derivatives in a screw form, a novel term named the "accelerator" or "acceleration motor" was proposed and employed for the acceleration analysis of serial and parallel kinematic chains [5] . However, the terms associated with the second derivatives in the acceleration equations can only be written in a lengthy form of Lie brackets rather than in a compact form of the Hessian matrix [5] [6].

2. Delta robot

The delta robot is a parallel robot, i.e. it consists of multiple kinematic chains interconnecting the base with the end-effector. The key concept of the delta

robot is the use of parallelograms. These parallelograms restrict the movement of the end platform to pure translation. mostly the robot's base is mounted above the workspace and all the actuators are located on it. From the base, three middle jointed arms extend. The ends of these arms are connected to a small triangular platform. This type of configuration is widely applicable as a sorting and collating in various types of packaging and food industry [7]. Actuation of the input links will move the triangular platform along the X, Y or Z direction. Actuation can be done with linear or rotational actuators, with or without reductions. Nowadays linearly actuated delta robots are used in the manufacturing industry for high-speed machining centres.

3. Position Equation

A delta robot employs three parallelograms to completely restrain the orientation of the moving platform with only three purely translational degrees of freedom. The description of the kinematics chain is shown in figure 1. A reference coordinate system $O-xyz$ is attached to the centre of the base platform with X and Y axes lying on the fixed plane and Z axis pointing up vertically.
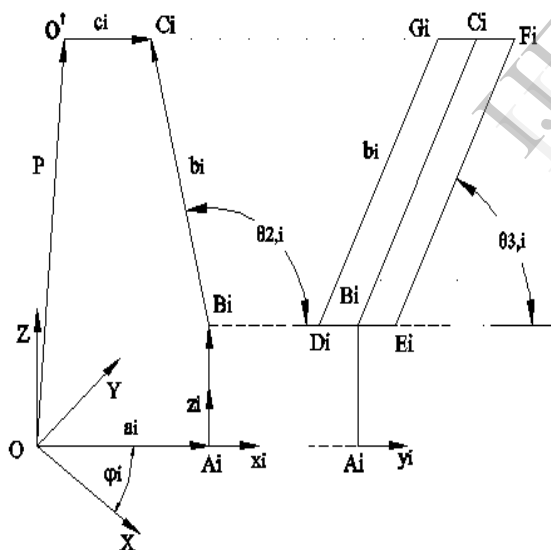
Figure 1. Description of the kinematic chain

Another coordinate system $A_i - x_i y_i z_i$ is attached to the fixed base at point A_i, such that x_i axis is in line with line $\overline{OA}$ and z_i axis is parallel to z axis. the constant angle φ_i is measured from the x axis to x_i. some variables associated with the ith limb are defined in Figure 1.

Therefore the loop-closure equation of each limb can be written as;

$$\overrightarrow{OA_i} + \overrightarrow{A_iB_i} + \overrightarrow{B_iC_i} = \overrightarrow{Oo'} + \overrightarrow{o'C_i} \qquad (1)$$

since the $\overrightarrow{A_iB_i}$ is the vector along the actuated joint, then

$$\overrightarrow{A_iB_i} = \overrightarrow{Oo'} + \overrightarrow{o'C_i} - \overrightarrow{OA_i} - \overrightarrow{B_iC_i} \qquad (2)$$

Where all the vectors in Eq. (2) are expressed in the reference coordinate system $O-xyz$.

4. Screw and reciprocal screw

The concept of screw and reciprocal screws was first studied by Ball [8]. The instantaneous motion of a rigid body can be represented by a six-component vector called *twist*, $\$_t$. Similarly, all the forces acting on a rigid body can be described by another six-component vector called *wrench*, $\$_w$. In this regard, the first three components of a twist represent the angular velocity and the last three components represent the linear velocity of a point in the rigid body which is instantaneously coincident with the origin of the reference frame. On the other hand, the first three components of a wrench represent the resultant force and the last three components represent the resultant moment about the origin of the reference frame.

Although twist and wrench have different physical meanings, they have the same mathematical representation called screws. A unit screw $\hat{\$}$ is defined by a pair of vectors:

$$\hat{\$} = \begin{bmatrix} s \\ s_0 \times s + \lambda s \end{bmatrix} \qquad (3)$$
$$= S_1, S_2, S_3, S_4, S_5, S_6{}^T$$

where s is a unit vector along the direction of the screw axis, s_0 is the position vector of any point on the screw axis with respect to a reference frame, and λ is called the pitch [1]. A screw of intensity ρ can be written as $\$ = \rho\,\hat{\$}$. Two screws $\$_1$ and $\$_2$ are said to be reciprocal to each other if their orthogonal product equals to zero.

$$\langle \$_1 \quad \$_2 \rangle = 0 \qquad (4)$$

If one screw is twist and the other is wrench, the physical meaning for reciprocity between these two screws is that the instantaneous work for the wrench along the twist is zero.

A screw system is a span of k_i ($k_i \le 6$) linearly independent screws, and is often called a k_i system. For this screw system, any k_i linearly independent screws can be a basis of the system. The reciprocal screw system for the given one is a $6-k_i$ system in which any one screw is reciprocal to all the screws in the given system. Similarly, any $6-k_i$ linearly independent screws form a basis of the reciprocal system. Hence we only need to find

$6 - k_i$ convenient linearly independent screws to represent the reciprocal system.

As described above the twist space form a 6-dimensional vector space, which is composed of a pair of complementary subspaces called *The twist subspace of permissions* and *The twist subspace of restrictions*. The twist subspace of permissions is a k_i -dimensional subspace of the twist space spanned by the twists permitted by the system constraints with $\$_{ta}$ being its element. The twist subspace of restrictions is a $6 - k_i$ dimensional subspace of the twist space spanned by the twists restricted by the system constraints with $\$_{tc}$ being its element [4].

The wrenche space also form a 6-dimensional vector space, which is composed of a pair of complementary subspaces called *The* wrenche *subspace of actuations* and *The* wrenche *subspace of constraints*. The wrenche subspace of actuations is a k_i -dimensional subspace of the wrenche space spanned by the wrenches of actuations generated by k_i independent actuation forces and/or moments with $\$_{wa}$ being its element. The wrench subspace of constraints is a $6 - k_i$ dimensional subspace of the wrenche space spanned by the wrenches of constraints generated by $6 - k_i$ independent joint reaction forces and/or moments with $\$_{wc}$ being its element.

From the above definitions we can see that:

$$\langle \$_{ta} \quad \$_{wc} \rangle = 0 \text{ and}$$
$$\langle \$_{tc} \quad \$_{wa} \rangle = 0 \tag{5}$$

Hence if a screw system of twists represents the degrees of freedom (DOF) of a rigid body, the reciprocal system of wrenches is the constraint forces acting on it. The dual proposition also holds: When a screw system of wrenches act on a rigid body, then the reciprocal system of twists is the DOF of the body. Therefore, the reciprocity property makes it possible to obtain motion information from the corresponding constraint counterpart and vice versa.

5. Determination of the basis Elements of the Four Subspaces

This section employ geometrical observation method to determine the reciprocal screw systems of linearly actuated delta robot. we only needs to implement three simple observations to fully determine the basis of a reciprocal screws of a screw system [9]. Simply by inspecting the joint axes and applying these observations, one can easily obtain a basis of the reciprocal screw system. Relative motion between two rigid bodies can be achieved by mechanical joints.
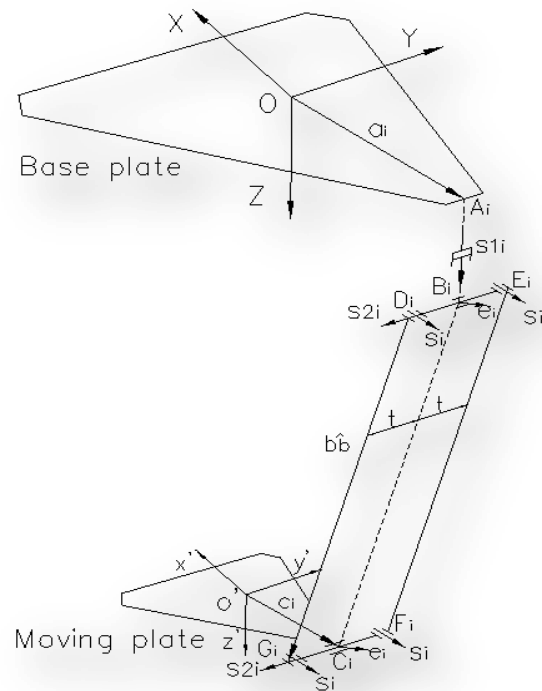
Figure.2 Schematic representation of a single limb of linearly actuated delta robot.

For example, rotation and translation can be obtained by revolute and prismatic joints respectively. In robot manipulator design, revolute and prismatic joints are widely used. In fact, most of the other joints can be conceived as the combination of several revolute and prismatic joints. For example, a universal joint can be substituted by two revolute joints whose axis intersects with each other perpendicularly and a spherical joint can be represented by three mutually perpendicular revolute joints with the axes of the joints passing through the center of the sphere.

At the same time, common constraints in statics are pure force or moment. For both the motion and constraint, the screw associated with rotation or a pure force is a zero pitch screw(λ =0) which we call a *line vector*, while the screw associated with translation or a pure moment is an infinite pitch screw which we call a *couple vector*.

5.1. The three basic observations

The following three basic observations will help us to find the basis of the reciprocal screws of a screw system. The proofs for the observations are discussed by Jianguo [9].

Observation 1: Two line vectors are reciprocal to each other if and only if they are coplanar.

Observation 2: Two couple vectors are always reciprocal to each other.

Observation 3: A line vector is reciprocal to a couple vector if and only if they are perpendicular to each other.

5.2. The basis Elements of the Four Subspaces

A single limb of linearly actuated Delta robot with 3-DOF translational motion capabilities is shown in figure 2. It consists of a base, a platform and three identical limbs made of prismatic-revolute-parallelogram-revolute(PRPaR) joints. In each limb, the axis of the prismatic joint is parallel to the z-axis of the base plate, the axes of two revolute joints are parallel to each other, and the axes of the revolute joints in the parallelogram joint are normal to the parallelogram$\square$ $DEFG_i$, Since the parallelogram joint can be visualized as a 1-DOF compound joint [2]. We have 4-screw system single limb i.e. $k_i = 4$. The joint axes are arranged such that, $s_{2,i} \perp s_i$, and $s_i \perp \hat{b}_i$.

Hence, the unit screws of permissions, $\hat{\$}_{ta,ja,i}(j_a = 1,2,3,4)$ in the ith limb($i = 1,2,3$) can be generated as follows:
Since a prismatic joint is a *couple vector*

$$\hat{\$}_{ta,1,i} = \begin{bmatrix} s_{1i} \\ 0 \end{bmatrix} \qquad (6)$$

Note that except the actuated joint the rest three joints are revolute joints *(i.e. line vector)* therefore the unit screws about a reference point o' due to the two vectors $s_{2,i}$ will give us

$$\hat{\$}_{ta,2,i} = \begin{bmatrix} (c_i - b\hat{b}_i) \times s_{2,i} \\ s_{2,i} \end{bmatrix} \qquad (7)$$

$$\text{and} \quad \hat{\$}_{ta,4,i} = \begin{bmatrix} c_i \times s_{2,i} \\ s_{2,i} \end{bmatrix}. \qquad (8)$$

$\hat{\$}_{ta,3,i}$ is a unit screw of permissions of a 1-DOF parallelogram joint. it will be the resultant unit screw of permissions associated with the revolute joints of the ith limb at D_i, E_i, F_i, and G_i

$$\hat{\$}_{ta,D_i} = \begin{pmatrix} c_i - ts_{2,i} - b\hat{b}_i \times s_i \\ s_i \end{pmatrix}, \qquad (9)$$

$$\hat{\$}_{ta,E_i} = \begin{pmatrix} c_i + ts_{2,i} - b\hat{b}_i \times s_i \\ s_i \end{pmatrix}, \qquad (10)$$

$$\hat{\$}_{ta,F_i} = \begin{pmatrix} c_i + ts_{2,i} \times s_i \\ s_i \end{pmatrix}, \qquad (11)$$

$$\hat{\$}_{ta,G_i} = \begin{pmatrix} c_i - ts_{2,i} \times s_i \\ s_i \end{pmatrix} \qquad (12)$$

where s_i is coincident with the axis of the revolute joint, and is also the normal vector of parallelogram $\square$ $DEFG_i$. Within the scope of velocity analysis, the instantaneous twist of the platform due only to the twist of the parallelogram joint can be given as

$$\$_{ta,3,i} = \dot{\theta}_{3,i}\hat{\$}_{ta,D_i} - \dot{\theta}_{3,i}\hat{\$}_{ta,G_i}$$

$$= \dot{\theta}_{3,i}\hat{\$}_{ta,E_i} - \dot{\theta}_{3,i}\hat{\$}_{ta,F_i} \qquad (13)$$

where $\dot{\theta}_{3,i}$ represents the angular velocity of the revolute joint. Substituting Eq.(9) through Eq.(12) into Eq. (13) gives

$$\$_{ta,3,i} = \dot{\theta}_{3,i}b\hat{\$}_{ta,3,i} = \dot{\theta}_{3,i}b\begin{pmatrix} e_i \\ 0 \end{pmatrix} \qquad (14)$$

Hence the unit screw becomes,

$$\hat{\$}_{ta,3,i} = \begin{pmatrix} e_i \\ 0 \end{pmatrix} \qquad (15)$$

where $e_i = s_i \times \hat{b}_i$. Therefore, $\hat{\$}_{ta,3,i}$ can be considered to be the unit screw of permissions associated with the 1-DOF parallelogram joint of limb i.

Next the two unit wrench of constraints, $\hat{\$}_{wc,k_c,i}$ ($k_c = 1,2$), which are orthogonal to all $\hat{\$}_{ta,j_a,i}$, can then be identified using the above observations:

Since the unit screw of permissions are composed of one couple vector and three line vectors we can take two linearly independent couple vectors which are perpendicular to s_{2i} . Note that by observation 2 any couple vector is a reciprocal of any other couple vector. So

$$\hat{\$}_{wc,1,i} = \begin{pmatrix} 0 \\ s_i \end{pmatrix}, \qquad (16)$$

$$\hat{\$}_{wc,2,i} = \begin{pmatrix} 0 \\ s_i \times s_{2i} \end{pmatrix} \qquad (17)$$

Locking sequentially the k_ath ($k_a = 1,2,...,4$) joint, the unit wrench $\hat{\$}_{wa,k_a,i}$, which is orthogonal to $\hat{\$}_{ta,j_a,i}$ ($j_a = 1,2,...,4$, $j_a \neq k_a$) and dual to $\hat{\$}_{ta,k_a,i}$, can be identified as by the three basic observations mentioned above, take a line vector along a vector $\hat{b}_i$, from observation 3 this line vector is not perpendicular to the couple vector s_{1i}

that means it will be a dual to s_{1i}. from observation 2 it will be a reciprocal to any coplanar line vector. From observation 3 this same line vector is perpendicular to e_i. Hence $\hat{\$}_{wa,1,i} = \begin{pmatrix} \hat{b}_i \\ c_i \times \hat{b}_i \end{pmatrix}$ will be a dual to $\hat{\$}_{ta,1,i}$ and orthogonal to $\hat{\$}_{ta,2,i}$, $\hat{\$}_{ta,3,i}$, and $\hat{\$}_{ta,4,i}$. Like wise

$$\hat{\$}_{wa,2,i} = \begin{pmatrix} e_{3,i} \\ c_i \times e_{3,i} \end{pmatrix}, \qquad (18)$$

$$\hat{\$}_{wa,3,i} = \begin{pmatrix} s_{2,i} \\ c_i \times s_{2,i} \end{pmatrix}, \qquad (19)$$

$$\hat{\$}_{wa,4,i} = \begin{pmatrix} e_{4,i} \\ c_i - b b_i \times e_{4,i} \end{pmatrix} \qquad (20)$$

where $e_{3,i}$ is a unit vector which is perpendicular to both $s_{1,i}$ and e_i passing through point C_i and $e_{4,i}$ is a unit vector which is perpendicular to both $s_{1,i}$ and e_i passing through point B_i. Hence $e_{3,i} = e_{4,i} = s_{1,i} \times e_i$.

Finally, with the k_cth ($k_c = 1,2$) constraints provided by $\hat{\$}_{wc,k_c,i}$ being sequentially released, the unit screw of restrictions, $\hat{\$}_{tc,j_c,i}$ ($j_c = 1,2$), which is orthogonal to $\hat{\$}_{wa,k_a,i}$ ($k_a = 1,2,\dots,4$) and $\hat{\$}_{wc,k_c,i}$ ($k_c = 1,2$, $k_c \neq j_c$) except for $\hat{\$}_{wc,j_c,i}$, can be identified as

$$\hat{\$}_{tc,1,i} = \begin{pmatrix} c_i \times s_i \\ s_i \end{pmatrix}, \quad \hat{\$}_{tc,2,i} = \begin{pmatrix} c_i \times b_i \\ b_i \end{pmatrix}$$

6. Jacobian Analyses

Jacobian analysis relates the linear map between actuator rates and velocity twist of the end-effector.. The Jacobian matrix ($\mathbf{J}$) relates the two by:

$$\mathbf{J}\$_t = \dot{\mathbf{q}} \qquad (21)$$

where $\$_t$ is the twist of the platform and $\dot{\mathbf{q}}$ is the actuated joints rate. According Huang [2] for parallel manipulators the twist, $\$_t$, of the platform to can be expressed as a linear combination of the basis elements of the twist space associated with the ith limb, for $i = 1,2,3$

$$\$_t = \$_{ta} + \$_{tc} = \$_{ta,i} + \$_{tc,i}$$

$$= \sum_{j_a=1}^{k_i} \delta \rho_{a,j_a,i} \hat{\$}_{ta,j_a,i} + \sum_{j_c=1}^{6-k_i} \delta \rho_{c,j_c,i} \hat{\$}_{tc,j_c,i} \qquad (22)$$

where $\hat{\$}_{ta,j_a,i}$ and $\hat{\$}_{tc,j_c,i}$ are unit screw of j_ath elements of the twist subspace of permissions and j_cth elements of the twist subspace of restrictions respectively. $\delta \rho_{a,j_a,i}$ and $\delta \rho_{c,j_c,i}$ are their respective intensity within the ith limb.

since $\hat{\$}_{wa,1,i}$ is the unit wrench of actuations associated with the actuated joint, labelled $j_a = 1$, in the ith ($i = 1,2,3$) limb. Note that $\hat{\$}_{wa,1,i}$ is dual to $\hat{\$}_{ta,1,i}$ but orthogonal to $\hat{\$}_{ta,j_a,i}$ ($j_a = 1,2,\cdots,4$, $j_a \neq 1$) and $\hat{\$}_{tc,j_c,i}$ ($j_c = 1,2$). Similarly, let $\hat{\$}_{wc,k_c,i}$ be the k_cth unit wrench of constraints in the ith limb. Also, note that $\hat{\$}_{wc,k_c,i}$ is orthogonal to $\hat{\$}_{ta,j_a,i}$ ($j_a = 1,2,\dots,4$) and $\hat{\$}_{tc,j_c,i}$ ($j_c = 1,2$, $j_c \neq k_c$) and dual to $\hat{\$}_{tc,k_c,i}$. Thus, Taking inner products on both sides of equation (22) with $\hat{\$}_{wa,1,i}$ and $\hat{\$}_{wc,kc,i}$, respectively gives

$$\hat{\$}_{wa,1,i}^{\mathrm{T}}\$_t = \delta \rho_{a,1,i} \hat{\$}_{wa,1,i}^{\mathrm{T}} \hat{\$}_{ta,1,i} , i = 1,2,3 \qquad (23)$$

$$\hat{\$}_{wc,k_c,i}^{\mathrm{T}}\$_t = \delta \rho_{c,k_c,i} \hat{\$}_{wc,k_c,i}^{\mathrm{T}} \hat{\$}_{tc,k_c,i}, k_c = 1,2 , \qquad (24)$$

Therefore Eq.(21) and Eq.(22) can be written as

$$\mathbf{J}_x \$_t = \mathbf{J}_\rho \delta \boldsymbol{\rho} \qquad (25)$$

$$\mathbf{J}_x = \begin{bmatrix} \mathbf{J}_{xa} \\ \mathbf{J}_{xc} \end{bmatrix}, \ \mathbf{J}_\rho = \begin{bmatrix} \mathbf{J}_{\rho a} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{\rho c} \end{bmatrix}$$

The overall Jacobian matrix will be

$$\mathbf{J} = \mathbf{J}_\rho^{-1} \mathbf{J}_x \qquad (26)$$

Hence $\mathbf{J}\$_t = \dot{\mathbf{q}}$

Where, $\dot{q} = (\dot{q}_a^{\mathrm{T}} \ \mathbf{0})^{\mathrm{T}} \quad \dot{\mathbf{q}}_a = \begin{bmatrix} \dot{q}_{a1,1} & \dot{q}_{a1,2} & \dot{q}_{a1,3} \end{bmatrix}^{\mathrm{T}}$

Therefore substituting all the screws components of the generalized Jacobian for a linearly delta robot will be;

$$\mathbf{J}_{xa} = \begin{bmatrix} \hat{b}_1^{\mathrm{T}} & c_1 \times \hat{b}_1^{\ T} \\ \hat{b}_2^{\mathrm{T}} & c_2 \times \hat{b}_2^{\ T} \\ \hat{b}_3^{\mathrm{T}} & c_3 \times \hat{b}_3^{\ T} \end{bmatrix}, \mathbf{J}_{\rho a} = \begin{bmatrix} \hat{b}_1^{\mathrm{T}} \cdot s_{1,1} & & 0 \\ & \hat{b}_2^{\mathrm{T}} \cdot s_{1,2} & \\ 0 & & \hat{b}_3^{\mathrm{T}} \cdot s_{1,3} \end{bmatrix}$$

$$\mathbf{J}_{xc} = \begin{bmatrix} 0_{3\times1} & 0_{3\times1} & 0_{3\times1} & 0_{3\times1} & 0_{3\times1} & 0_{3\times1} \\ s_1 & s_1 \times s_{21} & s_2 & s_2 \times s_{22} & s_3 & s_3 \times s_{23} \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{J}_{\rho c} = \begin{bmatrix} 1 & & & & & 0 \\ & s_1 \times s_{21} & & & & \\ & & 1 & & & \\ & & & s_2 \times s_{22} & & \\ & & & & 1 & \\ 0 & & & & & s_3 \times s_{23} \end{bmatrix}$$

From Eq. (26) we can see that the generalized Jacobian matrix of a linearly actuated 3-DOF delta parallel manipulator will be a 9x6 matrix.

7. Acceleration Analysis

7.1. Acceleration analysis of a limb

The formulae for expressing the angular acceleration, and the acceleration of an arbitrary point of the end effector of a serial chain have been developed by Rico and Duffy [6] [8]. Those formulae contained the direction and moment parts of the screws representing the kinematic pairs that join the end effector with the body, or link, employed as reference system. Here, the coordinate system must have the arbitrary point chosen as the origin. That same paper shows that if the angular velocity of the end effector, with respect to the reference system, vanishes, then the acceleration of an arbitrary point in the end effector can be expressed in a simpler form. In fact, the acceleration state of the rigid body can be written in terms of the screws, representing the kinematic pairs which join the end effector to the reference frame, together with their Lie products.

As outlined in a recent work by Liu and Huang [5] acceleration analysis of a parallel manipulator can be best performed by the concept of a Lie bracket. Taking the variation of Eq. (22) and expressing the derivatives of screws in the form of Lie brackets yields

$$\mathbf{A} = \mathbf{J}_i \delta^2 \rho_i + \$_{Li}, \quad i = 1,2,3 \tag{27}$$

Where $A = \left(\begin{array}{cc} \delta^2 r - \delta\alpha \times \delta r^{\mathrm{T}} & \delta^2\alpha^{\mathrm{T}} \end{array} \right)^{\mathrm{T}}$,

$$\delta^2 \rho_i = \left(\begin{array}{cc} \delta^2 \rho_{a,i}{}^{\mathrm{T}} & \delta^2 \rho_{c,i}{}^{\mathrm{T}} \end{array} \right)^{\mathrm{T}}$$

$$\delta^2 \boldsymbol{\rho}_{a,i} = \begin{array}{cccc} \delta^2 \rho_{a,1,i} & \delta^2 \rho_{a,2,i} & \cdots & \delta^2 \rho_{a,n_i,i} \end{array}^{\mathrm{T}}$$

$$\delta^2 \boldsymbol{\rho}_{c,i} = \begin{array}{cccc} \delta^2 \rho_{c,1,i} & \delta^2 \rho_{c,2,i} & \cdots & \delta^2 \rho_{c,6-n_i,i} \end{array}^{\mathrm{T}}$$

$$\$_{Li} = \left[\delta\rho_{a,1,i} \hat{\$}_{ta,1,i} \quad \delta\rho_{a,2,i} \hat{\$}_{ta,2,i} + \cdots + \delta\rho_{c,6-n_i,i} \hat{\$}_{tc,6-n_i,i} \right]$$
$$+ \left[\delta\rho_{a,2,i} \hat{\$}_{ta,2,i} \quad \delta\rho_{a,3,i} \hat{\$}_{ta,3,i} + \cdots + \delta\rho_{c,6-n_i,i} \hat{\$}_{tc,6-n_i,i} \right]$$
$$+ \cdots + \left[\delta\rho_{c,5-n_i,i} \hat{\$}_{tc,5-n_i,i} \quad \delta\rho_{c,6-n_i,i} \hat{\$}_{tc,6-n_i,i} \right]$$

$\delta^2 r$, $\delta^2 \alpha$, $\delta^2 \rho_{a,j_a,i}$, and $\delta^2 \rho_{c,j_c,i}$ denote, respectively, the variation of δr , $\delta\alpha$, $\delta\rho_{a,j_a,i}$, and $\delta\rho_{c,j_c,i}$. The bracket $* \; *$ in $\$_{Li}$ denotes the Lie product. From the properties of the Lie product, $\$_{Li}$ can also be written as

$$\$_{Li} = \$_{a,i} + \$_{ac,i} + \$_{c,i} \tag{28}$$

where

$$\$_{a,i} = \delta\rho_{a,1,i}\delta\rho_{a,2,i} \left[\hat{\$}_{ta,1,i} \quad \hat{\$}_{ta,2,i} \right] + \cdots$$
$$+ \delta\rho_{a,1,i}\delta\rho_{a,k_i,i} \left[\hat{\$}_{ta,1,i} \quad \hat{\$}_{ta,k_i,i} \right]$$
$$+ \delta\rho_{a,2,i}\delta\rho_{a,3,i} \left[\hat{\$}_{ta,2,i} \quad \hat{\$}_{ta,3,i} \right] + \cdots$$
$$+ \delta\rho_{a,2,i}\delta\rho_{a,k_i,i} \left[\hat{\$}_{ta,2,i} \quad \hat{\$}_{ta,k_i,i} \right] + \cdots$$
$$+ \delta\rho_{a,k_i-1,i}\delta\rho_{a,k_i,i} \left[\hat{\$}_{ta,k_i-1,i} \quad \hat{\$}_{ta,k_i,i} \right]$$

$$\$_{ac,i} = \delta\rho_{a,1,i}\delta\rho_{c,1,i} \left[\hat{\$}_{ta,1,i} \quad \hat{\$}_{tc,1,i} \right]$$
$$+ \cdots + \delta\rho_{a,1,i}\delta\rho_{c,6-k_i,i} \left[\hat{\$}_{ta,1,i} \quad \hat{\$}_{tc,6-k_i,i} \right]$$
$$+ \delta\rho_{a,2,i}\delta\rho_{c,1,i} \left[\hat{\$}_{ta,2,i} \quad \hat{\$}_{tc,1,i} \right] + \cdots$$
$$+ \delta\rho_{a,2,i}\delta\rho_{c,6-k_i,i} \left[\hat{\$}_{ta,2,i} \quad \hat{\$}_{tc,6-k_i,i} \right] + \cdots$$
$$+ \delta\rho_{a,k_i,i}\delta\rho_{c,1,i} \left[\hat{\$}_{ta,k_i,i} \quad \hat{\$}_{tc,1,i} \right] + \cdots$$
$$+ \delta\rho_{a,k_i,i}\delta\rho_{c,6-k_i,i} \left[\hat{\$}_{ta,k_i,i} \quad \hat{\$}_{tc,6-k_i,i} \right]$$

$$\$_{c,i} = \delta\rho_{c,1,i}\delta\rho_{c,2,i} \left[\hat{\$}_{tc,1,i} \quad \hat{\$}_{tc,2,i} \right] + \cdots$$
$$+ \delta\rho_{c,1,i}\delta\rho_{c,6-k_i,i} \left[\hat{\$}_{tc,1,i} \quad \hat{\$}_{tc,6-k_i,i} \right]$$
$$+ \delta\rho_{c,2,i}\delta\rho_{c,3,i} \left[\hat{\$}_{tc,2,i} \quad \hat{\$}_{tc,3,i} \right] + \cdots$$
$$+ \delta\rho_{c,2,i}\delta\rho_{c,6-k_i,i} \left[\hat{\$}_{tc,2,i} \quad \hat{\$}_{tc,6-k_i,i} \right] + \cdots$$
$$+ \delta\rho_{c,5-k_i,i}\delta\rho_{c,6-k_i,i} \left[\hat{\$}_{tc,5-k_i,i} \quad \hat{\$}_{tc,6-k_i,i} \right]$$

Then equation (27) can be written as

$$A = \mathbf{J}_i \delta^2 \rho_i + \delta^{\mathrm{T}} \rho_i \mathbf{H}_i \delta\rho_i, \quad i = 1,2,3 \tag{29}$$

Where $\mathbf{H}_i = \begin{bmatrix} \mathbf{H}_{a,i} & \mathbf{H}_{ac,i} \\ \mathbf{0} & \mathbf{H}_{c,i} \end{bmatrix}$,

$$\mathbf{H}_{a,i} = \begin{bmatrix} 0 & [\hat{\$}_{ta,1,i} \ \hat{\$}_{ta,2,i}] & [\hat{\$}_{ta,1,i} \ \hat{\$}_{ta,3,i}] & \cdots & [\hat{\$}_{ta,1,i} \ \hat{\$}_{ta,k_i,i}] \\ 0 & 0 & [\hat{\$}_{ta,2,i} \ \hat{\$}_{ta,3,i}] & \cdots & [\hat{\$}_{ta,2,i} \ \hat{\$}_{ta,k_i,i}] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [\hat{\$}_{ta,k_i-1,i} \ \hat{\$}_{ta,k_i,i}] \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\mathbf{H}_{c,i} = \begin{bmatrix} 0 & [\hat{\$}_{tc,1,i} \ \hat{\$}_{tc,2,i}] & [\hat{\$}_{tc,1,i} \ \hat{\$}_{tc,3,i}] & \cdots & [\hat{\$}_{tc,1,i} \ \hat{\$}_{tc,6-k_i,i}] \\ 0 & 0 & [\hat{\$}_{tc,2,i} \ \hat{\$}_{tc,3,i}] & \cdots & [\hat{\$}_{tc,2,i} \ \hat{\$}_{tc,6-k_i,i}] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [\hat{\$}_{tc,5-k_i,i} \ \hat{\$}_{tc,6-k_i,i}] \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\mathbf{H}_{ac,i} = \begin{bmatrix} [\hat{\$}_{ta,1,i} \ \hat{\$}_{ta,2,i}] & \cdots & [\hat{\$}_{ta,1,i} \ \hat{\$}_{tc,6-k_i,i}] \\ \vdots & \cdots & \vdots \\ [\hat{\$}_{ta,k_i,i} \ \hat{\$}_{tc,1,i}] & \cdots & [\hat{\$}_{ta,k_i,i} \ \hat{\$}_{tc,6-k_i,i}] \end{bmatrix}$$

$\mathbf{H}_i$ is known as the Hessian matrix of the ith limb. It is a three-dimensional matrix having six layers, each containing a 6×6 upper triangular matrix where [* *] represent the Lie product between two elements of the Lie algebra e(3). The constituent parts of $\mathbf{H}_i$, $\mathbf{H}_{a,i} \in R^{6\times k_i \times k_i}$, $\mathbf{H}_{ac,i} \in R^{6\times k_i \times \ 6-k_i}$ and $\mathbf{H}_{c,i} \in R^{6\times \ 6-k_i \ \times \ 6-k_i}$, are also three-dimensional matrices having six layers.

In acceleration analysis where only ideal motions of the platform are considered, replacements can be made in Eq. (29) such that

$$\delta\boldsymbol{\rho}_i = \begin{bmatrix} \delta\boldsymbol{\rho}_{a,i}^{\mathrm{T}} & \delta\boldsymbol{\rho}_{c,i}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \rightarrow \dot{\boldsymbol{\theta}}_i = \begin{bmatrix} \dot{\boldsymbol{\theta}}_{a,i}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

$$\delta^2\boldsymbol{\rho}_i = \begin{pmatrix} \delta^2\boldsymbol{\rho}_{a,i}^{\ \mathrm{T}} & \delta^2\boldsymbol{\rho}_{c,i}^{\ \mathrm{T}} \end{pmatrix}^{\mathrm{T}} \rightarrow \ddot{\boldsymbol{\theta}}_i = \begin{bmatrix} \ddot{\boldsymbol{\theta}}_{a,i}^{\mathrm{T}} & \mathbf{0} \end{bmatrix}^{\mathrm{T}}$$

$$A = \begin{pmatrix} \delta^2\mathbf{r} - \delta\boldsymbol{\alpha}\times\delta\mathbf{r}^{\ \mathrm{T}} & \delta^2\boldsymbol{\alpha}^{\ \mathrm{T}} \end{pmatrix}^{\mathrm{T}} \rightarrow A = \begin{pmatrix} \dot{\mathbf{v}} - \boldsymbol{\omega}\times\mathbf{v}^{\ \mathrm{T}} & \dot{\boldsymbol{\omega}}^{\ \mathrm{T}} \end{pmatrix}^{\mathrm{T}}$$

Thus

$$A = \mathbf{J}_i\ddot{\boldsymbol{\theta}}_i + \dot{\boldsymbol{\theta}}_i^{\mathrm{T}}\mathbf{H}_i\dot{\boldsymbol{\theta}}_i \quad (i=1,2,3) \qquad (30)$$

and A becomes the accelerator of an $k_i - \text{DOF}$ limb with $\dot{\mathbf{v}}$ and $\dot{\boldsymbol{\omega}}$ being the linear acceleration of the reference point and the angular acceleration of the platform respectively.

7.2. Acceleration analysis of a parallel manipulator

Following the same approach, the acceleration equation of a parallel manipulator is obtained by taking inner products on both sides of Eq. (30) with $\hat{\$}_{wa,1,i}$ and $\hat{\$}_{wc,k_c,i}$ respectively [3], to give

$$\mathbf{J}A = \ddot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^{\mathrm{T}}\mathbf{H}\dot{\boldsymbol{q}} \qquad (31)$$

Where $\ddot{\boldsymbol{q}} = \begin{bmatrix} \ddot{\boldsymbol{q}}_a^{\mathrm{T}} & \mathbf{0} \end{bmatrix}^{\mathrm{T}}$, $\ddot{\boldsymbol{q}}_a = \begin{bmatrix} \ddot{q}_{a,1,1} & \ddot{q}_{a,1,2} & \ddot{q}_{a,1,3} \end{bmatrix}^{\mathrm{T}}$, $\ddot{q}_{a,1,i}$ is the acceleration of the actuated joint numbered 1 in the ith limb. $\mathbf{H} \in R^{N\times N\times N}$ ($N = f + \sum_{i=1}^{f} 6-k_i$) is known as the Hessian matrix of an f-DOF parallel manipulator. It is a three dimensional matrix composed of $\mathbf{H}_a \in R^{f\times N\times N}$ and $\mathbf{H}_c \in R^{\sum_{i=1}^{f} 6-k_i \ \times N\times N}$. The expression of $U_{a(c)}^{\text{th}}$ layer of $\mathbf{H}_{a(c)}$ is given as follow: the U_a^{th} layer of $\mathbf{H}_a$ ($U_a = 1,2,...f$) and the U_c^{th} layer of $\mathbf{H}_c$ ($U_c = 1,2,\cdots,6-k_i,$ $\cdots,\sum_{i=1}^{f}6-k_i$)

$$\mathbf{H}_{n,i} = \psi_1 *$$

$$\begin{bmatrix} 0 & m_1[\hat{\$}_{ta,1,i} \ \hat{\$}_{ta,2,i}] & m_1[\hat{\$}_{ta,1,i} \ \hat{\$}_{ta,3,i}] & \cdots & m_1[\hat{\$}_{ta,1,i} \ \hat{\$}_{ta,6-ki,i}] \\ 0 & 0 & m_1[\hat{\$}_{ta,2,i} \ \hat{\$}_{ta,3,i}] & \cdots & m_1[\hat{\$}_{ta,2,i} \ \hat{\$}_{ta,6-ki,i}] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & m_1[\hat{\$}_{ta,5-ki,i} \ \hat{\$}_{ta,6-ki,i}] \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\psi_2$$

where $\begin{cases} n=a, \begin{cases} m_1 = \hat{\$}_{wa,1,i} \\ m_2 = \hat{\$}_{ta,1,i} \end{cases} \\ n=c, \begin{cases} m_1 = \hat{\$}_{wc,k_c,i} \\ m_2 = \hat{\$}_{tc,k_c,i} \end{cases} \end{cases}$ $\psi_1 = \dfrac{\mathbf{J}_i^{LP\ \mathrm{T}}}{m_1^T m_2}$ $\psi_2 = \mathbf{J}_i^{LP}$

where, $\mathbf{J}_i^{LP} = \mathbf{J}_i^{-1}\mathbf{J}^+ = \begin{bmatrix} \mathbf{J}_{a,i}^{L} \\ \mathbf{J}_{c,i}^{L} \end{bmatrix}\mathbf{J}^+$,

$$\mathbf{J}_{a,i}^{L} = \begin{bmatrix} \hat{\$}^T_{wa,1,i} \Big/ \hat{\$}^T_{wa,1,i}\hat{\$}_{ta,1,i} \\ \vdots \\ \hat{\$}^T_{wa,k_i,i} \Big/ \hat{\$}^T_{wa,k_i,i}\hat{\$}_{ta,k_i,i} \end{bmatrix},$$

$$\mathbf{J}_{c,i}^{L} = \begin{bmatrix} \hat{\$}^T_{wc,1,i} \Big/ \hat{\$}^T_{wc,1,i}\hat{\$}_{tc,1,i} \\ \vdots \\ \hat{\$}^T_{wc,6-k_i,i} \Big/ \hat{\$}^T_{wc,6-ni,i}\hat{\$}_{tc,6-k_i,i} \end{bmatrix}$$

Where, $\mathbf{J}^+$ is the pseudo-inverse of $\mathbf{J}$ and the superscript $\mathbf{L}$ simply identifies that the matrix applies to a single limb.

Hence the inverse acceleration equation of a parallel manipulator will be derived from Eq. (31)

$$\ddot{q} = \mathbf{J}A - \$_t^{\mathbf{T}} \mathbf{J}^{\mathbf{T}} \mathbf{H} \mathbf{J} \$_t \qquad (32)$$

where $\ddot{q} = \begin{bmatrix} \ddot{q}_{a,i}^{\mathrm{T}} & \mathbf{0} \end{bmatrix}^{\mathrm{T}}$ $\ddot{q} = \begin{bmatrix} \ddot{q}_{a1,1} & \ddot{q}_{a1,2} & \ddot{q}_{a1,3} \end{bmatrix}^{\mathrm{T}}$

The Hessian matrix, H, of the manipulator can be found by substituting the expressions just found for $\hat{\$}_{ta,j_a,i}$ ($j_a = 1,2,\ldots,4$), $\hat{\$}_{tc,1,i}$ and $\hat{\$}_{tc,2,i}$ and Jacobians $\mathbf{J}_i^{LP}$ and J into the expressions given above and then into Eq.(32) to give layers of H as follows

Where $\mathbf{H}_{a,i} = \dfrac{\mathbf{J}_i^{LP^{\mathrm{T}}} \mathbf{L}_{a,i} \mathbf{J}_i^{LP}}{\hat{\mathbf{b}}_i^{\mathrm{T}} s_{1,i}}$, $i = 1,2,3$

$$\mathbf{L}_{a,i} = \begin{bmatrix} \mathbf{M}_{a1,i} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{a2,i} \end{bmatrix}_{6 \times 6},$$

$$\mathbf{M}_{a1,i} = \begin{bmatrix} 0 & \hat{b}_i^{\mathrm{T}} \ s_{1,i} \times s_{2,i} & 0 & \hat{b}_i^{\mathrm{T}} \ s_{1,i} \times s_{2,i} \\ 0 & 0 & 0 & b\left(\hat{b}_i^{\mathrm{T}} b_i\right) \end{bmatrix}_{2 \times 4}$$

$$\mathbf{M}_{a2,i} = \begin{bmatrix} \hat{b}_i^{\mathrm{T}} \ e_i \times s_i & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{4 \times 2}$$

$\mathbf{H}_{c1,i} = \mathbf{J}_i^{LP^{\mathrm{T}}} \mathbf{L}_{c1,i} \mathbf{J}_i^{LP}$, $i = 1,2,3$

$\mathbf{H}_{c2,i} = \dfrac{\mathbf{J}_i^{LP^{\mathrm{T}}} \mathbf{L}_{c2,i} \mathbf{J}_i^{LP}}{s_1 \times s_{21}^{\ T} \hat{b}_i}$ $i = 1,2,3$

$\mathbf{L}_{c1,i} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{c1,i} \end{bmatrix}$

$\mathbf{L}_{c2,i} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{c2,i} & \mathbf{0} \end{bmatrix}$

$\mathbf{M}_{c1,i} = \begin{bmatrix} 0 & \hat{b}_i^{\mathrm{T}} \ s_{2,i} \times \hat{b}_i & 0 & \hat{b}_i^{\mathrm{T}} \ s_{2,i} \times \hat{b}_i & 0 & 0 \end{bmatrix}^{\mathrm{T}}$

$\mathbf{M}_{c2,i} = \begin{bmatrix} 0 & \eta^T \eta & 0 & \eta^T \eta & 0 & 0 \end{bmatrix}^{\mathrm{T}}$

where, $\eta = s_i \times s_{2,i}$

$H_{a,i}$ ($H_{c1,i} \& H_{c2,i}$) represents the K_ath (K_cth) layer of H_a (H_c).

Numerical Example

Let's take the following numeric values for the dimensions of a linearly actuated delta robot, where c=0.2m, b=0.5m, t=0.05m, a=0.4m $\varphi_1 = 0, \varphi_2 = 2\pi/3, \quad \varphi_3 = 4\pi/3$, and the platform is assumed to have the following velocity and acceleration behaviour;

$a_x = a_{max} \cos(2\pi t/T)$ $a_y = a_{max} \sin(2\pi t/T)$, $a_{max} = 1 m/s$, $v_z = 1 m/s$, $a_z = 0$, $T = 5 s$,

$v_x = T a_{max} \sin(2\pi t/T)/2$
$v_y = T a_{max} \cos(2\pi t/T)/2$.

When the platform of the manipulator moves according to the preceding rules, in Fig. 3 and Fig. 4 the velocity and the acceleration of the actuated joints versus time can be evaluated using the proposed approach. These results are shown in Fig. 5 and Fig. 6.
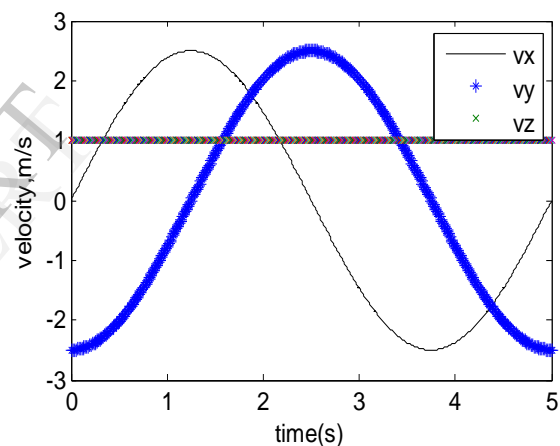

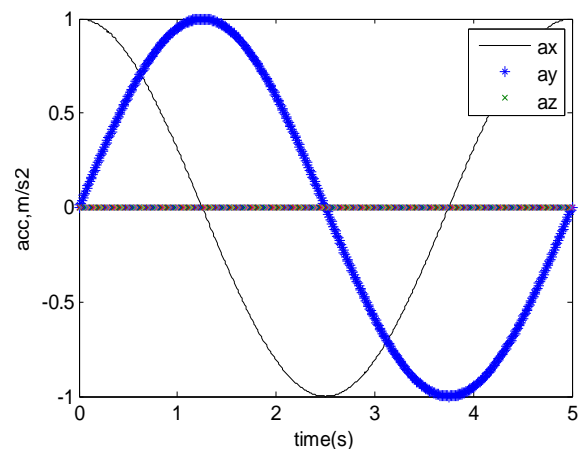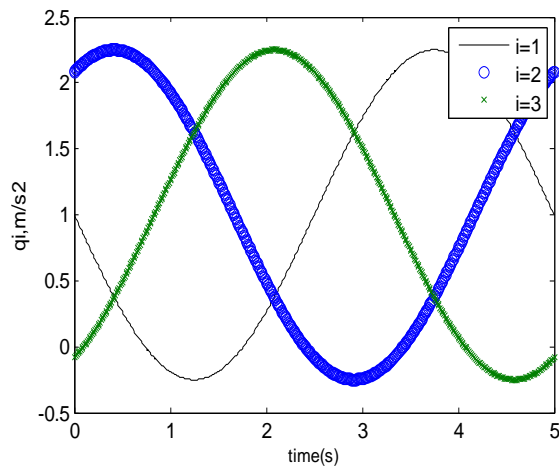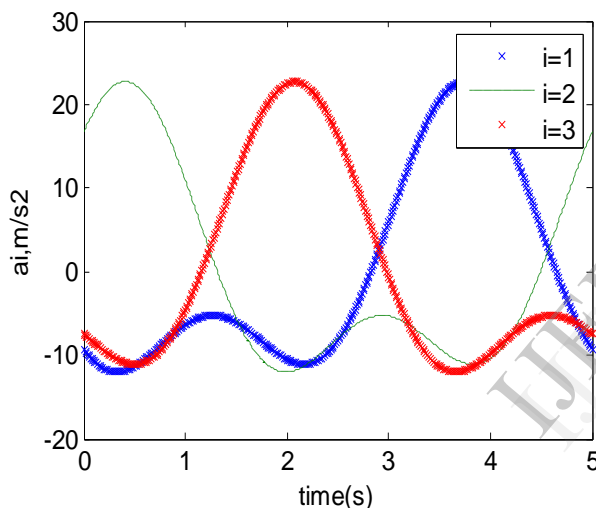**Figure 3. Velocity of the moving platform**

Figure 4. Acceleration of the moving platform

Figure 5. Velocity of the three actuated joints

Figure 6. Acceleration of the three actuated joints

8. Conclusions

This paper presents the inverse velocity and acceleration analyses of a linear delta parallel manipulators using a general and systematic approach based on screw theory. With this approach, the process of acceleration modelling is much simplified with help of the generalized Jacobian. It results in a new Hessian matrix for a linearly actuated delta parallel manipulator being developed in a general and compact form.

Acknowledgment

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