# Fluctuating Flow of a Second Order Fluid between Two Co-Axial Circular Pipes

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***Abstract* - In this paper an analytical solution to the flow of a second order fluid is presented expressing the pressure gradient in the form of Fourier series. The effect of the amplitude coefficient of the mean-velocity for different values of frequency of excitation is shown in different graphs.**

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## I. INTRODUCTION

In everyday life, we encounter many different kinds of fluids. The study of flows of Newtonian and non-Newtonian fluids through pipes and tubes has became important not only because of their technological importance's but also in view of the interesting mathematical features presented by the equations governing the flow. Such studies have a considerable practical relevance because of their applications in petro-chemical industries, manufacturing of foods and paper and many other similar activities. Uchida (1) studied the pulsating flow Newtonian fluid due to the pressure gradient in the direction of the flow. Rajgopal etal.(2) Pontrelli (3) made important theoretical studies on these fluids, Rath and Jena (4) studied the flow of a viscous fluid generated in response to fluctuations in the axial velocity of the outer cylinder Biswal etal.(5) studied the above problem incase of visco-elastic liquid. Lui Ciqun and Huang Jungi (6) studied the axial flow of second order fluid and analyzed the flow characters of these fluids. Hayat etal.(7) studied the Fluctuating flow of a third grade fluid on porous plate in a rotating medium.Kaloni(8) analyzed the Fluctuating flow of an elastic viscous fluid past a porous flat plate.Hayat etal(9),Fetecau(10) studied the above problem on a porous plate.Ozer etal(11) studied the flow of a second grade fluid through a cylindrical permeable tube.Hayat etal(12) considered the MHD flow of the above fluid in a porous channel.Similar type of flows were investigated by Wang etal.(13),Tadhg etal.(14), Tiwary etal.(15), Hayat etal.(16) made analytical studies on transient rotating flow of a second grade fluid . Hayat etal. (17) studied the peristaltic flow of a second order fluid in the presence of an induced magnetic field.Jamil etal. (18), Hayat et al.(19) studied the flow of a second grade fluid in different mediums and got very interesting results.In this paper, we will study the fluctuating flow of a second order fluid in the annular region between two coaxial circular pipes and got the solution using Fourier series.

## II. BASIC EQUATIONS

We work through the cylindrical polar coordinates $(r, \theta, z)$. z-axis coincides with the common axis of the circular pipes. The radius of the outer pipe is 'a' and inner pipe is 'b'. Let $(0,0,w)$ be the unsteady rectilinear flow between the pipes. All physical quantities are independent of $\theta$ because of axial symmetry. The equation of the continuity reduces to

$$\frac{\partial w}{\partial z} = 0 \qquad \text{-------------- (1)}$$

Thus $w$ is independent of z and we can write $w = w(r, t)$.
The stress components for the problem under discussion are given below

$$\left.\begin{aligned}
\tau_{rr} &= -p + (2\mu_2 + \mu_3)\left(\frac{\partial w}{\partial r}\right)^2 \\
\tau_{\theta\theta} &= -p \\
\tau_{zz} &= -p + \mu_3\left(\frac{\partial w}{\partial r}\right)^2 \\
\tau_{r\theta} &= \tau_{\theta z} = 0 \\
\tau_{rz} &= \mu_1 \frac{\partial w}{\partial r} + \mu_2 \frac{\partial^2 w}{\partial t \partial r} \\
&= \left(\mu_1 + \mu_2 \frac{\partial}{\partial t}\right)\frac{\partial w}{\partial r}
\end{aligned}\right\} \qquad \text{------ (2)}$$

The equations of motion becomes

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} + 2(2\alpha + \beta)\left[\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial w}{\partial r}\right)\right] = 0 \text{ - (3)}$$

$$-\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} = 0 \text{ --------------- (4)}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \left(\upsilon + \alpha\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \text{------- (5)}$$

Where, $\rho$ = is the density

$\alpha$ = coefficient of visco-elasticity = $\dfrac{\mu_2}{\rho}$

$\beta$ = coefficient of cross-viscocity = $\dfrac{\mu_3}{\rho}$

And

$\upsilon$ = coefficient of kinematic-viscocity = $\dfrac{\mu_1}{\rho}$

On the basis of the equation (1), (3) & (5) we may assume

$$-\frac{1}{\rho}\frac{\partial \rho}{\partial z} = f(t) \qquad \text{--------------- (6)}$$

Equation (5) becomes

$$\frac{\partial w}{\partial t} = f(t) + \left(\upsilon + \alpha\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \text{---------- (7)}$$

Boundary conditions are

$$\left.\begin{aligned} r &= a, & w(r,t) &= 0 \\ r &= b, & w(r,t) &= 0 \end{aligned}\right\} \text{-- (8)}$$

## III. SOLUTION OF THE PROBLEM

The pressure-gradient (7) can be expressed in the form of a Fourier series as

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_{cn}\cos n\sigma t + a_{sn}\sin n\sigma t\right]$$

$$= a_0 + R_p\sum_{n=1}^{\infty} a_n e^{in\sigma t} \text{ ----------- (9)}$$

Where, $a_0$ = steady pressure gradient

$a_n = a_{cn} - ia_{sn}$, $R_P$ = real part of the expression

and $a_{cn}$ & $a_{sn}$ are constants which represents the amplitudes of the elemental vibrations of a pulsating pressure gradient superposed on $a_0$ ,where $a_0$ is steady pressure-gradient.

We assume the period of excitation as $\dfrac{2\pi}{\sigma}$ .

In view of the periodic pressure distribution, we can assume the solution for the velocity field as,

$$w(r,t) = w_0(r) + \sum_{n=1}^{\infty}\left[w_{cn}(r)\cos n\sigma t + w_{sn}(r)\sin n\sigma t\right]$$

$$= w_0(r) + R_P\sum_{n=1}^{\infty} w_n(r)e^{m\sigma t} \qquad \text{--------------- (10)}$$

And, $\quad w_n(r) = w_{cn}(r) - iw_{sn}(r)$ --------------- (11)

the function (11) satisfies the differential equation

$$w_n''(r) + \frac{1}{r}w_n'(r) + \frac{-in\sigma}{\upsilon + in\sigma}w_n(r) = \frac{-a_n}{\upsilon + in\sigma\alpha}$$
----------------- (12)

The boundary conditions (8) reduced to

$$\left.\begin{aligned} w_n(a) &= 0 \\ w_n(b) &= 0 \end{aligned}\right\} \quad n = 0,1,2 \text{ --------------- (13)}$$

The solution of (12) subject to the boundary conditions (13) is

$$w(r,t) = \frac{a_0}{4\upsilon}\left(a^2 - r^2\right) - \frac{a_0\left(a^2 - b^2\right)\left(\log a - \log r\right)}{4\upsilon\left(\log a - \log b\right)}$$
$$+ R_p\left[\sum_{n=1}^{\infty}\frac{a_n}{in\sigma}\left[1 + \left\{J_0\left(\bar{k}ri^{3/2}\right)\right.\right.\right.$$
$$\left.\begin{aligned}&\left\{k_0\left(\bar{k}ai^{1/2}\right) - k_0\left(\bar{k}bi^{1/2}\right)\right\} \\ &-k_0\left(\bar{k}ri^{1/2}\right)\left\{J_0\left(\bar{k}ai^{3/2}\right) - J_0\left(\bar{k}bi^{3/2}\right)\right\}\end{aligned}\right\} \Big/$$
$$\left.\left\{J_0\left(\bar{k}ai^{3/2}\right)k_0\left(\bar{k}bi^{1/2}\right) - J_0\left(\bar{k}bi^{3/2}\right)k_0\left(\bar{k}ai^{1/2}\right)\right\}e^{in\sigma t}\right]$$
--------------- (14)

In the above $J_0$ and $K_0$ are Bessel functions of zeroth order of first and second kind respectively where

$$\bar{k} = \frac{n\sigma}{\upsilon + in\sigma\alpha} \qquad \text{--------------- (15)}$$

and with $m_n^2 = \dfrac{in\sigma}{\upsilon + in\sigma\alpha}$ --------------- (16)

(14) reduces to

$$w(r,t) = \frac{a_0}{4\upsilon}\left(a^2 - r^2\right) - \frac{a_0\left(a^2 - b^2\right)(\log a - \log r)}{4\upsilon(\log a - \log b)}$$
$$+ R_p\left[\sum_{n=1}^{\infty}\frac{a_n}{in\sigma}\left[1 + \{J_0(im_nR)k_0(m_na) - k_0(m_nb)\right.\right.$$
$$\left. - k_0(m_nR)\{J_0(im_na) - J_0(im_nb)\}\}\}\right/$$
$$\left.\{J_0(im_na)k_0(m_nb) - J_0(im_nb)k_0(m_na)\}\right]e^{i\sigma t}\Big] \quad \text{---------------} (17)$$

To simplify the above equation, we introduce the following non-dimensional parameters,

$$k = a\sqrt{\sigma/\upsilon} = \text{Frequency parameter}$$

$$\tan^{-1}\frac{\eta\alpha\sigma}{\upsilon} = \in_n = \text{Non-dimensional viscoelastic parameter}$$

$k_n = k\sqrt{n}$ = Frequency parameter in the nth mode of excitation.

$$m_n = \frac{k_n}{a}(r_n + is_n)$$

Where
$$\left.\begin{aligned}
r_n &= \sqrt{\cos\in_n}\cos\left(\frac{\pi}{4} - \frac{\in_n}{2}\right) \\
s_n &= \sqrt{\cos\in_n}\sin\left(\frac{\pi}{4} - \frac{\in_n}{2}\right) \\
J_0(im_nr) &= f_1(\eta) + if_2(\eta) \\
J_0(im_na) &= f_1(1) + if_2(1) \\
J_0(im_nb) &= f_1(\rho) + if_2(\rho) \\
k_0(m_nr) &= t_1(\eta) + it_2(\eta) \\
k_0(m_na) &= t_1(1) + it_2(1) \\
k_0(m_nb) &= t_1(\rho) + it_2(\rho) \\
\text{where } \eta = \frac{r}{a} &\quad \text{and} \quad \rho = \frac{b}{a}
\end{aligned}\right\} \text{----} (18)$$

In the above expressions the suffix 'n' denote the quantity in the nth mode of excitation, which is dropped out in the case of the flow under a signal pulse. With the help of the above non-dimensional quantities, the velocity field can be written as

$$w(r,t) = \frac{a_0a^2\left(1-\eta^2\right)}{4\upsilon} - \frac{a_0a^2\left(1-\rho^2\right)}{4\upsilon}\log\left(\frac{n}{\rho}\right)$$
$$+ a^2\sum_{n=1}^{\infty}\frac{a_{cn}}{\upsilon k_n^2}\{(1-p)\sin n\sigma t - Q\cos n\sigma t\}$$
$$+ \frac{a_{sn}}{\upsilon k_n^2}\{(p-1)\cos n\sigma t - Q\sin n\sigma t\}\Big] \quad \text{----------} (19)$$

where

$$\left.\begin{aligned}
P &= At_1(\eta) - Bt_2(\eta) - A'f_1(\eta) + B'f_2(\eta) \\
Q &= Bt_1(\eta) - At_2(\eta) - B'f_1(\eta) - A'f_2(\eta)
\end{aligned}\right\} \quad \text{---------------} (20)$$

and

$$A = \left[\{f_1(1) - f_1(\rho)\}\begin{Bmatrix} f_1(1)t_1(\rho) - f_2(1)t_2(\rho) \\ - f_1(\rho)t_1(1) + f_2(\rho)t_2(1)\end{Bmatrix}\right.$$
$$\left. + \{f_2(1) - f_2(\rho)\}\begin{Bmatrix} f_1(1)t_2(\rho) + f_2(1)t_1(\rho) \\ -t_1(1)f_2(\delta) - t_2(1)f_1(\rho)\end{Bmatrix}\right] \Big/$$
$$\Big[\{f_1(1)t_1(\rho) - f_2(1)t_2(\rho) - f_1(\rho)t_1(1) + t_2(\rho)t_2(1))\}^2$$
$$+ \{f_1(1)t_2(\rho) + f_2(1)t_1(\rho) - t_1(1)t_2(\rho) + t_2(1)f_1(\rho)\}^2\Big] \quad \text{---------------} (21)$$

$$B = \left[\{f_2(1) - f_2(\rho)\}\begin{Bmatrix} f_1(1)t_1(\rho) - f_2(1)t_2(\rho) \\ - f_1(\rho)t_1(1) + f_2(\rho)t_2(1)\end{Bmatrix}\right.$$
$$\left. - \{f_1(1) - f_1(\rho)\}\begin{Bmatrix} f_1(1)t_2(\rho) + f_2(1)t_1(\rho) \\ -t_1(1)f_2(\rho) - t_2(1)f_1(\rho)\end{Bmatrix}\right] \Big/$$
$$\Big[\{f_1(1)t_1(\rho) - f_2(1)t_2(\rho) - f_1(\rho) - t_1(1) + f_2(\rho)t_2(1)\}^2$$
$$\{f_1(1)t_2(\rho) + f_2(1)t_1(\rho) - t_1(1)f_2(\rho)t_2(1)f_1(\rho)\}^2\Big]$$

$$\text{---------------} (22)$$

$$A' = \left[\{t_1(1) - t_1(\rho)\}\begin{Bmatrix} f_1(1)t_1(\rho) - f_2(1)t_2(\rho) \\ - f_1(\rho)t_1(1) + f_2(\rho)t_2(1)\end{Bmatrix}\right.$$
$$\left. + \{t_2(1) - t_2(\rho)\}\begin{Bmatrix} f_1(1)t_2(\rho) + f_2(1)t_1(\rho) \\ -t_1(1)f_2(\rho) - t_2(1)f_1(\rho)\end{Bmatrix}\right] \Big/$$
$$\Big[\{f_1(1)t_1(\rho) - f_2(1)t_2(\rho) - f_1(\rho)t_1(1) + f_2(\rho)t_2(1)\}^2 +$$
$$\{f_1(1)t_2(\rho) + f_2(1)t_1(\rho) - t_1(1)f_2(\rho) - t_2(1)f_1(\rho)\}^2\Big] \quad \text{---------------} (23)$$

$$B' = \left[ \{t_2(1) - t_2(\rho)\} \begin{Bmatrix} f_1(1)t_1(\rho) - f_2(1)t_2(\rho) \\ - f_1(\rho)t_1(1) + f_2(\rho)t_2(1) \end{Bmatrix} - \{t_1(1) - t_1(\rho)\} \begin{Bmatrix} f_1(1)t_2(\rho) + f_2(1)t_1(\rho) \\ - t_1(1) f_2(\rho) - t_2(1) f_1(\rho) \end{Bmatrix} \right] \Big/ \left[ \{f_1(1)t_1(\rho) - f_2(1)t_2(\rho) - f_1(\rho)t_1(1) + f_2(\rho)t_2(1)\}^2 + \{f_1(1)t_2(\rho) + f_2(1)t_1(\rho) - t_1(1)f_2(\rho) - t_2(1) + f_2(\rho)\}^2 \right]$$

--------------- (24)

The mean velocity over one period across the cross-section is denoted by $\overline{w}$ and is defined by

$$\overline{w} = \frac{\sigma}{2\pi}\int_0^{2\pi/\sigma} dt \cdot \frac{1}{\pi(a^2 - b^2)}\int_a^b w(r,t)\cdot 2\pi r dr$$

$$= \frac{a_0}{8\upsilon(a^2 - b^2)}\left[(a^2 - b^2)(a^2 + b^2) - \frac{(a^2 - b^2)^2}{\log a - \log b}\right]$$

$$= \frac{a_0}{8\upsilon}\left[(a^2 + b^2) - \frac{(a^2 - b^2)}{\log a - \log b}\right]$$

$$= \frac{a_0 a^2}{8\upsilon s}$$

--------------- (25)

where $s = (1 + \rho^2) + \dfrac{1 - \rho^2}{\cos\rho}$

The mean pressure gradient 'G' over one period is given by,

$$G = \frac{\sigma}{2\pi}\int_0^{2\pi/\sigma} f(t)dt = a_0$$

---------- (26)

The mean velocity in the pulsating motion under the influence of a periodic pressure gradient (9) is identified with that in the steady-state flow under the same value of pressure gradient as that in the pulsating flow and is not affected by the presence of the visco-elastic parameter $\alpha$. The non-dimensional expression for the velocity now reduces to

$$\overset{*}{w}(\eta_c, t) = \frac{1}{s}\left[\begin{array}{l} 2(1-\eta^2) - 2(1-\rho^2)\log\left(\dfrac{\eta}{\rho}\right) + \\ 8\sum_{n=1}^{\infty} \dfrac{a_{cn}}{a_0 k_n^2}\{(1-\rho)\sin n\sigma t - Q\cos n\sigma t\} \\ + 8\sum_{n=1}^{\infty} \dfrac{a_{sn}}{a_0 k_n^2}\begin{Bmatrix} (\rho - 1)\cos n\sigma t \\ -Q\sin n\sigma t \end{Bmatrix} \end{array}\right]$$

--------------- (27)

The expression for the non-dimensional pressure gradient is as follows

$$-\frac{1}{e}\left(\frac{\partial \overset{*}{p}}{\partial z}\right) = -\frac{2a}{\frac{1}{2}p\overline{w}^2}\cdot\frac{\partial p}{\partial z}$$

$$= \frac{64}{R_e S}\left[\left\{1 + \sum_{n=1}^{\infty}\frac{a_{cn}}{a_0}\cos n\sigma t + \sum_{n=1}^{\infty}\frac{a_{sn}}{a_0}\sin n\sigma t\right\}\right]$$

Where $R_e = \dfrac{2a\overline{w}}{\upsilon}$ = Reynold's number

the starred quantities denote the corresponding non-dimensional expressions.

## IV. SECTIONAL MEAN –VELOCITY

The expression for the instantaneous mass flow across a section of tubes is derived from the sectional mean-velocitie. But the sectional-mean velocity "$w_{Mv}$" is given as

$$w_{Mv} = \frac{1}{\pi(a^2 - b^2)}\int_b^a [w(r,t)\cdot 2\pi r]dr$$

$$= \frac{2}{a^2 - b^2}\left[\begin{array}{l} \dfrac{a_0}{16\upsilon}(a^2 - b^2)^2 - \dfrac{a_0(a^2 - b^2)^2\log a}{8\upsilon(\log a - \log b)} \\ -- \dfrac{a_0(a^2 - b^2)^2}{16\upsilon(\log a - \log b)} \end{array}\right]$$

$$+\frac{a_0\left(a^2-b^2\right)-\left(a^2\log a-b^2\log b\right)}{8\upsilon(\log a-\log b)}$$

$$+\sum_{n=1}^{\infty}\frac{a_{cn}}{\sigma}\left[\left\{\frac{a^2-b^2}{2}-\frac{a^2}{2}\{S_n(\rho)-\rho C_n(\rho)\}\right\}\sin n\sigma t\right.$$

$$\left.-\frac{a^2}{2}\{T_n(\rho)-\rho D_n(\rho)\}\cos n\sigma t\right]$$

$$+\sum_{n=1}^{\infty}\frac{a_{cn}}{n\sigma}\left[\frac{a^2}{2}\left\{S_n(\rho)-\rho C_n(\rho)-\frac{a^2-b^2}{2}\right\}\cos n\sigma t\right.$$

$$\left.-\frac{a^2}{2}\{T_n(\rho)-\rho D_n(\rho)\}\sin n\sigma t\right] \quad \text{---------- (28)}$$

Where

$$\left.\begin{aligned}
&\int_b^a \rho r dr=\frac{a^2}{2}[S_n(\rho)-\rho C_n(\rho)]\\
&\int_b^a Q r dr=\frac{a^2}{2}[T_n(\rho)-\rho D_n(\rho)]\\
&S_n(\rho)=At_2'(1)+Bt_1'(1)-A'f_2'(1)-B'f_1'(1)\\
&C_n(\rho)=At_2'(\rho)+Bt_1'(\rho)-A'f_2'(\rho)-B'f_1'(\rho)\\
&T_n(\rho)=Bt_2'(1)-At_1'(1)-B'f_2'(1)-A'f_1'(1)\\
&D_n(\rho)=Bt_2'(\rho)-At_1'(\rho)-B'f_2'(\rho)+A'f_1'(\rho)
\end{aligned}\right\} \quad \text{--------------- (29)}$$

The sectional mean velocity in dimensionless form is given by

$$\overset{*}{W}_{MV^n}=\frac{W_{MV^n}}{W}=\frac{1}{s}\left[\frac{\left(1+\rho^2\right)\log\rho+(1-\rho)^2}{\log\rho}+8\sum_{n=1}^{\infty}\frac{a_{cn}}{a_0}\frac{1}{k_n^2}\right.$$

$$\left[\left\{1-\left(\frac{S_n(\rho)-\rho C_n(\rho)}{1-\rho^2}\right)\sin n\sigma t-\left\{\frac{T_n(\rho)-\rho D_n(\rho)}{1-\rho^2}\right\}\cos n\sigma t\right\}\right]$$

$$\left.+8\sum_{n=1}^{\infty}\frac{a_{sn}}{a_0}\frac{1}{k_n^2}\left[\left\{\frac{S_n(\rho)-\rho C_n(\rho)}{1-\rho^2}\right\}\cos n\sigma t-\left\{\frac{T_n(\rho)-\rho D_n(\rho)}{1-\rho^2}\right\}\{\sin n\sigma t\}\right]\right]$$

--------------- (30)

We define the amplitude coefficient and phase lag in the nth mode of the sectional mean-velocity from the wave of the pressure-gradient by the following expressions respectively.

$$A_{MV^n}=\frac{8}{K_n^2 S}\left[\left\{1-\left\{\frac{S_n(\rho)-\rho C_n(\rho)}{\left(1-\rho^2\right)}\right\}^2\right\}+\left\{\frac{T_n(\rho)-\rho D_n(\rho)}{1-\rho^2}\right\}^2\right]^{\frac{1}{2}} \quad \text{--- (31)}$$

and

$$\theta_{MV^n}=\tan^{-1}\left[\frac{T_n(\rho)-\rho D_n(\rho)}{(1-\rho)^2-\{S_n(\rho)-\rho D_n(\rho)\}}\right] \quad \text{------------- (32)}$$

with help of the equations (31) and (32) we get the non-dimensional form of sectional mean-velocity as follows

$$W_{MV^n}=1+\sum_{n=1}^{\infty}A_{MV^n}\left[\frac{a_{cn}}{a_0}\sin\left(n\sigma t-\theta_{MV^n}\right)-\frac{a_{sn}}{a_0}\cos\left(n\sigma t-\theta_{MV^n}\right)\right] \quad \text{- (33)}$$

## V. *RESISTANCE COEFFICIENTS*

The shearing stress on the wall is given by,

$$S_F=\tau=-\mu_1\left(\frac{dW}{dR}\right) \quad \text{--------- (34)}$$

where $\mu_1$ is the coefficient of viscosity. We denote the non-dimensional frictional force at the outer wall by $\overset{*}{S}_{Fa^n}$ and that at the inner wall by $S_{Fb^n}$ and get their corresponding expressions as

$$\overset{*}{S}_{Fa^n}=\frac{S_{Fa^n}}{\frac{1}{2}\rho\overline{W}^2}$$

$$=\frac{16}{R_e}\left[\frac{1+\left(1-\rho^2\right)/2\log\rho}{S}+\sum_{n=1}^{\infty}\frac{a_{cn}}{a_0 k_n^2}\{S_n(\rho)\cos n\sigma t-T_n(\rho)\sin n\sigma t\}\right]$$

$$+\sum_{n=1}^{\infty}\frac{a_{sn}}{a_0 k_n^{\,2}}\{T_n(\rho)\cos n\sigma t+S_n(\rho)\sin n\sigma t\}\Bigg] \text{-- (35)}$$

and,
$$\overset{*}{S}_{Fb^n}=\frac{S_{Fb^n}}{\frac{1}{2}\rho W^2}$$

$$=-\frac{16}{R_e}\left[\begin{array}{l}\dfrac{\rho+(1-\rho^2)/2\rho\log\rho}{S}\\ +\displaystyle\sum_{n=1}^{\infty}\frac{a_{cn}}{a_0 k_n^{\,2}}\left\{\begin{array}{l}C_n(\rho)\cos n\sigma t\\ -D_n(\rho)\sin n\sigma t\end{array}\right\}\end{array}\right.$$

$$+\sum_{n=1}^{\infty}\frac{a_{sn}}{a_n k_n^{\,2}}\{D_n(\rho)\cos n\sigma t+C_n(\rho)\sin n\sigma t\}\Bigg] \text{ (36)}$$

We define the amplitude coefficient and phase lag in the nth mode of the resistance coefficient behind the wave of the imposed pressure gradient by the expressions

$$\left.\begin{array}{l}A_{SF_a^n}=\dfrac{1}{K_n^2 S}\sqrt{S_n^{\,2}(\rho)+T_n^{\,2}(\rho)}\\ A_{SF_b^n}=\dfrac{1}{K_n^2 S}\sqrt{C_n^{\,2}(\rho)+D_n^{\,2}(\rho)}\end{array}\right\} \text{----------- (37)}$$

$$\left.\begin{array}{l}\theta_{SF_a^n}=\tan^{-1}\dfrac{S_n(\rho)}{T_n(\rho)}\\ \theta_{SF_b^n}=\tan^{-1}\dfrac{C_n(\rho)}{D_n(\rho)}\end{array}\right\} \text{----------- (38)}$$

Here the suffixes 'a' and 'b' denote the corresponding values on the outer and inner wall respectively.

With these substitutions the equations (35) and (36) respectively reduce to

$$\overset{*}{S}_{Fa^n}=\frac{16}{R_e}\left[\begin{array}{l}\dfrac{1+(1-\rho^2)/2\log\rho}{S}-\\ \displaystyle\sum_{n=1}^{\infty}ASF_{a^n}\left\{\begin{array}{l}\dfrac{a_{cn}}{a_0}\sin\left(n\sigma t-\theta_{SF_a{}^n}\right)\\ -\dfrac{a_{sn}}{a_0}\cos\left(n\sigma t-\theta_{SF_a{}^n}\right)\end{array}\right\}\end{array}\right] \text{---}$$

---------------------(39)

and

$$\overset{*}{S}_{Fb^n}=-\frac{16}{R_e}\left[\begin{array}{l}\dfrac{\rho+(1-\rho^2)/2\rho\log\rho}{S}\\ -\displaystyle\sum_{n=1}^{\infty}A_{SF_b^n}\left\{\begin{array}{l}\dfrac{a_{cn}}{a_0}\sin\left(n\sigma t-\theta_{SF_b{}^n}\right)\\ -\dfrac{a_{sn}}{a_0}\cos\left(n\sigma t-\theta_{SF_b{}^n}\right)\end{array}\right\}\end{array}\right]$$

--------------- (40)

Mean rate of work done is given by

$$W_e=\pi\left(a^2-b^2\right)W_{MV}\left(-\frac{\partial p}{\partial z}\right) \text{----------- (41)}$$

The total mean-rate of work done is

$$\overline{W}_e=\frac{\sigma}{2\pi}\int_0^{\frac{2\pi}{\sigma}}W_e\,dt \text{ -------------- (42)}$$

But

$$W_{MV^n}=\overline{W}\left[1+\sum_{n=1}^{\infty}A_{MV^n}\left\{\begin{array}{l}\dfrac{a_{cn}}{a_0}\sin\left(n\sigma t-\theta_{MV^n}\right)\\ +\dfrac{a_{sn}}{a_0}\cos\left(n\sigma t-\theta_{MV^n}\right)\end{array}\right\}\right]$$

--------------- (43)

and the pressure gradient is given by

$$-\frac{\partial p}{\partial z}=\frac{8\mu_1\overline{W}}{a^2}\left[1+\sum_{n=1}^{\infty}\frac{a_{cn}}{a_0}\cos n\sigma t+\sum_{n=1}^{\infty}\frac{a_{sn}}{a_0}\sin n\sigma t\right]$$

--------------- (44)

$$W_e=\pi\left(a^2-b^2\right)W_{MV^n}\left(-\frac{\partial p}{\partial z}\right)$$

$$=8\pi\left(1-\rho^2\right)\frac{W^2\mu_1}{S}\times$$

$$\left[\begin{array}{l}S+\\ 8\displaystyle\sum_{n=1}^{\infty}\frac{a_{cn}}{K_n^2}\frac{1}{a_0}\left\{1-\frac{S_n(\rho)-\rho c_n(\rho)}{1-\rho^2}\right\}\sin n\sigma t\\ +\left\{\dfrac{T_n(\rho)-\rho D_n(\rho)}{1-\rho^2}\right\}\cos n\sigma t\end{array}\right]$$

$$+8\sum_{n=1}^{\infty}\frac{a_{sn}}{a_n}\cdot\frac{1}{k_n^2}\left[\begin{array}{l}\left\{\frac{S_n(\rho)-\rho c_n(\rho)}{1-\rho^2}-1\right\}\cos n\sigma t\\-\left\{\frac{T_n(\rho)-\rho D_n(\rho)}{1-\rho^2}\right\}\sin n\sigma t\end{array}\right]$$

$$\left[1+\sum_{n=1}^{\infty}\frac{a_{cn}}{a_0}\cos n\sigma t+\sum_{n=1}^{\infty}\frac{a_{sn}}{a_0}\sin n\sigma t\right] \text{------- (45)}$$

On simplification, we get

$$W_e=8\mu_1\frac{W^2}{S}(1-\rho^2)\left\{S+8\sum_{n=1}^{\infty}\frac{1}{K_n^2}\frac{a_{cn}}{a_0}\left[\begin{array}{l}\left\{1-\frac{S_n(\rho)-\rho C_n(\rho)}{1-\rho^2}\right\}\sin n\sigma t\\-\left\{\frac{T_n(\rho)-\rho D_n(\rho)}{1-\rho^2}\right\}\cos n\sigma t\end{array}\right]\right.$$

$$-8\sum_{n=1}^{\infty}\frac{1}{K_n^2}\frac{a_{sn}}{a_0}\left[\begin{array}{l}\left\{1-\frac{S_n(\rho)-\rho C_n(\rho)}{1-\rho^2}\right\}\cos n\sigma t\\+\left\{\frac{T_n(\rho)-\rho D_n(\rho)}{1-\rho^2}\right\}\sin n\sigma t\end{array}\right]$$

$$+\sum_{n=1}^{\infty}\frac{a_{cn}}{a_0}\cos n\sigma t\left\{S+8\sum_{n=1}^{\infty}\frac{1}{K_n^2}\frac{a_{cn}}{a_0}\left[\begin{array}{l}\left\{1-\frac{S_n(\rho)-\rho C_n(\rho)}{1-\rho^2}\right\}\sin n\sigma t\\-\left\{\frac{T_n(\rho)-\rho D_n(\rho)}{1-\rho^2}\right\}\cos n\sigma t\end{array}\right]\right.$$

$$\left.-8\sum_{n=1}^{\infty}\frac{1}{K_n^2}\frac{a_{sn}}{a_0}\left[\begin{array}{l}\left\{1-\frac{S_n(\rho)-\rho C_n(\rho)}{1-\rho^2}\right\}\cos n\sigma t\\+\left\{\frac{T_n(\rho)-\rho D_n(\rho)}{1-\rho^2}\right\}\sin n\sigma t\end{array}\right]\right\}$$

$$+\sum_{n=1}^{\infty}\frac{a_{sn}}{a_0}\sin n\sigma t\left\{S+8\sum_{n=1}^{\infty}\frac{1}{K_n^2}\frac{a_{cn}}{a_0}\left[\begin{array}{l}\left\{1-\frac{S_n(\rho)-\rho C_n(\rho)}{1-\rho^2}\right\}\sin n\sigma t\\-\left\{\frac{T_n(\rho)-\rho D_n(\rho)}{1-\rho^2}\right\}\cos n\sigma t\end{array}\right]\right.$$

$$\left.-8\sum_{n=1}^{\infty}\frac{1}{K_n^2}\frac{a_{sn}}{a_0}\left[\begin{array}{l}\left\{1-\frac{S_n(\rho)-\rho C_n(\rho)}{1-\rho^2}\right\}\cos n\sigma t\\+\left\{\frac{T_n(\rho)-\rho D_n(\rho)}{1-\rho^2}\right\}\sin n\sigma t\end{array}\right]\right\} \text{- (46)}$$

and $$\overline{W_e}=\frac{\sigma}{2\pi}\int_0^{\frac{2\pi}{\sigma}}W_e\,dt$$

$$=\frac{8\mu_1W^2(1-\rho^2)}{S}\left\{\begin{array}{l}S-\\4\left[\sum_{n=1}^{\infty}\left\{\frac{T_n(\rho)-\rho D_n(\rho)}{(1-\rho^2)K_n^2}\right\}\left\{\begin{array}{l}\left(\frac{a_{cn}}{a_0}\right)^2\\+\left(\frac{a_{sn}}{a_0}\right)^2\end{array}\right\}\right]\end{array}\right\}$$

------------- (47)

The total mean rate of change of kinetic energy across the cross section is

$$\overline{W_k}=\frac{\sigma}{2\pi}\int_0^{\frac{2\pi}{\sigma}}W_k\,dt=0 \quad \text{---------- (48)}$$

where $W_k$ is the rate of increase of kinetic energy of the fluid in a unit length pipe which is given by

$$W_k=\frac{1}{2}\rho\int_b^a\left(\frac{\partial w}{\partial t}\right)^2 2\pi R\,dR \quad \text{------------ (49)}$$

The total rate of change of dissipation of energy due to internal friction is given by

$$\varphi=\tau_{RZ}\theta_{RZ}=W_i$$

Where $W_i$ stands for the total rate of change of dissipation of energy. The total mean-rate of change of dissipation of energy due to internal friction is given by

$$\overline{W_i}=\left(\frac{\sigma}{2\pi}\right)\int_0^{\frac{2\pi}{\sigma}}dt\int_b^a\varphi 2\pi R\,dR$$

$$=\frac{8\mu_1\overline{W}^2(1-\rho^2)}{S}\left\{S-4\left[\sum_{n=1}^{\infty}\left\{\frac{T_n(\rho)-\rho D_n(\rho)}{(1-\rho^2)K_n^2}\right\}\left\{\begin{array}{l}\left(\frac{a_{cn}}{a_0}\right)^2\\+\left(\frac{a_{sn}}{a_0}\right)^2\end{array}\right\}\right]\right\}$$

--------- (50)

Work done = The total mean rate r

Thus we get the mean-rate of work done = The total mean rate of change of dissipation of energy and this fact leads to the same conclusions as in Uchida(1) that the pressure gradient does work equal to the energy loss due to dissipation of energy after a full cycle of the motion. Also the kinetic energy changes instantaneously but there is not loss in it after a complete cycle. Thus we see the energy loss is caused by the dissipation and is increased by the existence of the components in the fluctuating motion.

We define the coefficient of excess work as the extra energy dissipated due to the pulsation of amplitude which is equal to $a_0=\sqrt{a_{cn}^2+a_{sn}^2}$ .

Then we have in the n-th mode of vibration, the coefficient of excess work is given by

$$(C.E.W)_n = -4\left[\frac{T_n(\rho)-\rho D_n(\rho)}{(1-\rho^2)K_n^2}\right]$$

## VI. DISCUSSION OF THE RESULTS

In this paper we have studied the flow of a second order fluid in the annual region between two coaxial circular pipes. The pressure gradient expressed in the form of Fourier series, The following conclusions are made.

Fig-1 and Fig-2 shows the effect of the amplitude coefficient of the mean- velocity $A_{MV}$ for different values of K which is the frequency of excitation for $\rho = 0.2$ and 0.4. we see that $A_{MV}$ does not rise above the value zero until K=1.6.

For small values of $\rho$ i.e. $\rho = 0.2$ and 0.4, $A_{MV}$ -records larger values in the case of a Newtonian fluid $(i.e \in = 0)$, the maximum value occurring for values of K between 3 and 4, with $\in = \pm 60^0$ the mean velocity amplitude coefficient has negligible values whatever be the values of $\rho$ . This is also seen in Table-1 and Table- 2.

In the Fig.3 and Fig. 4 we see that for low frequency there is not much difference in the amplitude coefficient $A_{MV}$ when there is change in the values of $\in$ though as a rule the Newtonian value $(\in = 0^0)$ are smaller than the corresponding values of the non-Newtonian case $(\in \neq 0^0)$. It is to be noted that unlike the for going intermediate frequency case, $A_{MV}$ have their largest-values with extremely slow pulsation and drop to almost to zero value when K has a value slightly greater than 1. It records a slight rise for larger value of K. the value of $A_{MV}$ for slow pulsation in the case of low frequency is higher, the larger the radii ratio.

The effect of k on $\varphi_{MV}$, $A_{SFa}$, $A_{SFb}$ and C.E.W is shown in Table 1 &Table 2 for $\rho = 0.2$ and 0.4.

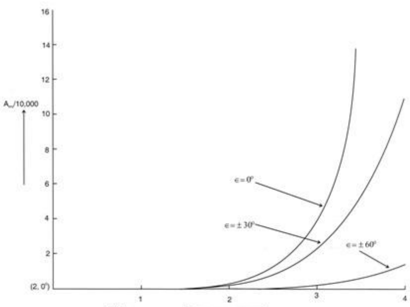

Fig. 1
ρ = 0·2
Intermediate Frequency Variation
for different values of K

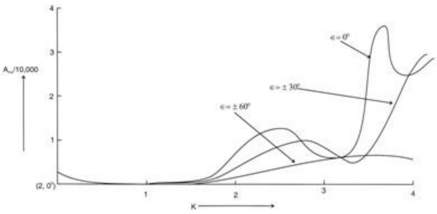

Fig. 2
ρ = 0·4
Intermediate Frequency Variation
for different values of K

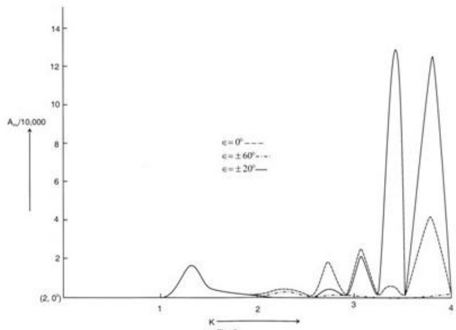

Fig. 3
ρ = 0·6
Intermediate Frequency Variation of $A_{MV}$
for different values of K

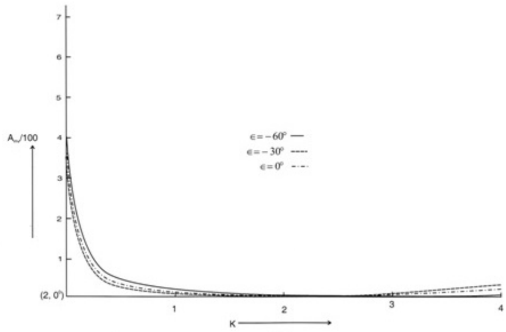

Fig. 4
ρ = 0·2
Low Frequency Variation of $A_{MV}$
for different values of K

**Table-1**

**Low frequency**

$\rho = 0.2,\ \epsilon = \pm 60^0$

| $K$ | $A_{MV}$ | $\theta_{MV}$ | $A_{SFa}$ | $A_{SFb}$ | C.E.W. | $\theta_{SFa}$ | $\theta_{SFb}$ |
|---|---|---|---|---|---|---|---|
| 4.0 | 8890.1200 | $6.4106^0$ | 1066.6572 | 0.4601 | -219.2985 | $-155.2349^0$ | $25.0065^0$ |
| 4.6 | 24733.0319 | $8.7633^0$ | 2967.8318 | 0.3650 | -829.5737 | $-14.2319^0$ | $36.8098^0$ |
| 5 | 44219.4912 | $10.6366^0$ | 5306.2294 | 0.3274 | -1790.5468 | $-308.3636^0$ | $49.4772^0$ |
| 5.6 | 70564.1819 | $14.1516^0$ | 10867.5929 | 0.3028 | -4815.4902 | $-231.8887^0$ | $83.8542^0$ |
| 6 | 129668.0719 | $17.0291^0$ | 15560.0668 | 0.3022 | -8195.2480 | $-192.6129^0$ | $128.3792^0$ |
| 6.5 | 834651.081 | $-5.6676^0$ | 2665250.50 | 0.9577 | 485039.62 | $218.7553^0$ | $62.866^0$ |
| 7 | 1075262.0034 | $57.4990^0$ | 129031.4300 | 0.2823 | -168940.9010 | $-57.047^0$ | $-22.7816^0$ |
| 7.5 | 711828.2514 | $2.9138^0$ | 85419.7189 | 1.6012 | 8020.5996 | $-45.7490^0$ | $93.5712^0$ |
| 8 | 1166812.5034 | $104.3661^0$ | 140017.6566 | 0.8348 | -226839.8700 | $-31.4296^0$ | $-26.106^0$ |
| 8.5 | 1751295.5034 | $-72.8293^0$ | 210156.2816 | 4.0784 | -305277.6210 | $45.0394^0$ | $43.3966^0$ |
| 9 | 1289960.7592 | $80.2790^0$ | 154795.3700 | 0.3006 | -283733.6800 | $-7.3500^0$ | $-4.6213^0$ |
| 9.5 | 1516472.7534 | $-950.2696^0$ | 181978.297 | 7.4942 | -288219.3100 | $34.4306^0$ | $33.8766^0$ |
| 10 | 1096105.7500 | $65.3183^0$ | 131532.7191 | 0.1593 | -242768.1908 | $-2.8639^0$ | $-1.0420^0$ |
| 10.5 | 1307616.7500 | $-118.0116^0$ | 156916.6253 | 13.1629 | -260876.5929 | $27.7955^0$ | $27.5812^0$ |
| 11 | 923764.1260 | $236.1645^0$ | 110851.7033 | 0.0953 | -204809.0019 | $1.1981^0$ | $0.0202^0$ |
| 11.5 | 1141626.7500 | $135.9920^0$ | 137000.0316 | 24.0425 | 233318.9021 | $24.1204^0$ | $24.0650^0$ |
| 12 | 783667.2500 | $54.1015^0$ | 94040.0780 | 0.0637 | 173778.7508 | 0.5002 | 0.2634 |

**Table-2**

**Low frequency**

$\rho = 0.4,\ \epsilon = \pm 60^0$

| $K$ | $A_{MV}$ | $\theta_{MV}$ | $A_{SFa}$ | $A_{SFb}$ | C.E.W. | $\theta_{SFa}$ | $\theta_{SFb}$ |
|---|---|---|---|---|---|---|---|
| 4.0 | 4230.8990 | $46.1421^0$ | 443.7784 | 0.9722 | -322.8464 | $28.7480^0$ | $-70.9608^0$ |
| 4.6 | 2170.8237 | $20.7376^0$ | 228.3349 | 1.6884 | -89.8927 | $15.3326^0$ | $-157.020^0$ |
| 5 | 15678.2558 | $74.5876^0$ | 1646.4692 | 0.6441 | -1905.5864 | $8.2044^0$ | $-2.1606^0$ |
| 5.6 | 33046.5235 | $-48.0554^0$ | 3622.4145 | 383.5247 | -2583.5537 | $0.9582^0$ | $68.9375^0$ |
| 6 | 50025.4454 | $-19.5959^0$ | 5529.6504 | 708.9641 | -1969.7429 | $-1.9896^0$ | $173.8215^0$ |
| 6.5 | 15351.368 | $116.0598^0$ | 6652354.01 | 349.2758 | 7867.0117 | $-3.9244^0$ | $-110.3475^0$ |
| 7 | 8333.3945 | $116.0598^0$ | 875.1004 | 0.2348 | -1013.5915 | $0.0507^0$ | $-0.0991^0$ |
| 7.5 | 2709.6630 | $230.4220^0$ | 305.1672 | 156.8091 | -109.2305 | $95.7303^0$ | $-6.0746^0$ |
| 8 | 6382.0781 | $-20.1201^0$ | 607.1931 | 0.1872 | -7776.2529 | $0.0287^0$ | $-0.1412^0$ |
| 8.5 | 4041.0810 | $-252.8441^0$ | 454.5305 | 76.6686 | -485.4173 | $8.4099^0$ | $-1.2210^0$ |
| 9 | 5040.7998 | $0.1421^0$ | 529.3452 | 0.1531 | -613.1131 | $0.0045^0$ | $-0.1527^0$ |
| 9.5 | 3916.8901 | $-242.7218^0$ | 430.6397 | 48.5973 | -474.2753 | $5.21150^0$ | $0.2384^0$ |
| 10 | 4081.2880 | $19.5608^0$ | 428.5861 | 0.1275 | -496.4075 | $-0.0110^0$ | $-0.1537^0$ |
| 10.5 | 3491.4106 | $-62.2166^0$ | 380.4548 | 34.6955 | -423.5269 | $4.0732^0$ | $0.7768^0$ |
| 11 | 3371.6855 | $169.8753^0$ | 354.0699 | 0.1077 | -410.0983 | $-0.0199^0$ | $-0.1512^0$ |
| 11.5 | 3075.7612 | $-305.1079^0$ | 333.5599 | 26.5280 | -373.5227 | $3.1338^0$ | $0.9853^0$ |
| 12 | 2832.2636 | $43.3908^0$ | 207.4245 | 0.0920 | -340.4885 | $0.0243^0$ | $-0.1477^0$ |

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