Fixed Point Theorem in Fuzzy Metric Space

Manoj Kumar Tiwari,
Deptt. of Mathematics, Pt. Sundarlal Sharma (Open) University,
Chhattisgarh, Bilaspur.

Abstract: In this research paper we have established a common fixed point theorem for compatible pair of self mappings in a fuzzy metric space. 2000 Mathematics Subject Classification: 54H25, 47H10.

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1. INTRODUCTION

The concept of fuzzy sets was initiated by L.A. Zadeh [19] in 1965 and the concept of fuzzy metric space was introduced by Kramosil and Michalek [9]. Grabiec [5] proved the contraction principle in the setting of the fuzzy metric space which was further generalization of results by Subrahmanyam [17] for a pair of commuting mappings. George and Veeramani [4] have modified the notion of fuzzy metric spaces with the help of continuous t-norm, by generalizing the concept of probabilistic metric space to fuzzy situation. Also, Jungck and Rhoades [7] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points. Balasubramaniam et.al. [1] proved a fixed point theorem, which generalizes a result of Pant for fuzzy mappings in fuzzy metric space. Also, Jha et al.[6] have proved a common fixed point theorem for four self maps in fuzzy metric space under the weak contractive conditions. Also, B. Singh and S. Jain [16] introduced the notion of semi-compatible mappings in fuzzy metric space and compared this notion with the notion of compatible map of type (α), compatible map of type (β) and obtained some fixed point theorems in complete fuzzy metric space in the sense of Grabiec [5]. As a generalization of fixed point results of Singh and Jain [16], Mishra et al.[10] have proved a fixed point theorems in complete fuzzy metric space by replacing continuity condition with reciprocally continuity maps.

The aim of this paper is to obtain a common fixed point theorem for compatible pair of self mappings in fuzzy metric space. Now, We have used the following notions:

DEFINITION 1.1([19]) Let X be any set. A fuzzy set A in X is a function with domain X and values in [0, 1].
DEFINITION 1.2([4]) A binary operation * : [0, 1] × [0, 1] → [0, 1] is called a continuous t-norm if, ([0, 1], *) is an abelian topological monoid with unit 1 such that a * b ≤ c * d whenever a ≤ c and b ≤ d, for all a, b, c, d in [0, 1]. For an example: a * b = ab, a * b = min {a, b}.
DEFINITION 1.3([4]) The triplet (X, M, *) is called a fuzzy metric space if, X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on X×X×[0, 1) satisfying the following conditions: for all x, y, z in X, and s, t > 0,
[i] M(x, y, 0) = 0, M(x, y, t) > 0;
[ii] M(x, y, t) = 1 for all t > 0 if and only if x = y,
[iii] M(x, y, t) = M(y, x, t),
[iv] M(x, y, t) * M(y, z, s) ≤ M(x, z, t + s),
[v] M(x, y, ·) : [0, ∞) → [0, 1] is left continuous.
In this case, M is called a fuzzy metric on X and the function M(x, y, t) denotes the degree of nearness between x and y with respect to t.

Also, we consider the following condition in the fuzzy metric space (X, M, *):
[vi] lim t→+∞M(x, y, t) = 1, for all x, y ∈ X.

It is important to note that every metric space (X, d) induces a fuzzy metric space (X,M, *) where a * b = min {a, b} and for all a, b ∈ X, we have M(x, y, t) = \frac{t}{d(x,y)}, for all t > 0, and M(x, y, 0) = 0, so-called the fuzzy metric space induced by the metric d.

DEFINITION 1.4([4]) In a fuzzy metric space (X, M, *) a sequence {xn} is called a Cauchy sequence if, limn→∞ M(xn+p, xn, t) = 1 for every t > 0 and for each p > 0.

A fuzzy metric space (X, M, *) is complete if, every Cauchy sequence in X converges in X.

DEFINITION 1.5([4]) In a fuzzy metric space (X, M, *) A sequence {xn} is said to be convergent to x in X if, limn→∞ M(xn, x, t) = 1, for each t > 0.

It is noted that since * is continuous, it follows from the condition [iv] of Definition (1.3.) that the limit of a sequence in a fuzzy metric space is unique.
Therefore, we get $A u = S u$.

We first consider the case when $(A, S)$ are compatible maps.

PROPOSITION 1.12: Let $A$ and $B$ be compatible, self-mappings of a fuzzy metric space $X$.

1. If $A y = B y$ then $A b y = B a y$.
2. If $A x n$ and $B x \to y$, for some $y$ is $X$ then
   (a) $B A x n \to A y$ if $A$ is continuous.
   (b) If $A$ and $B$ are continuous at $y$ then $A y = B y$ and $A B y = B A y$.

PROOF: (1) Let $A y = B y$ and $\{x n\}$ be a sequence in $X$ such that $x n = y$ for all $n$. Then $A x n$, $B x n \to A y$. Now by the compatibility of $A$ and $B$, we have

$$M(B A x n, B A y, t) = M(A B x n, B A y, t) = 1$$

which yields $A B y = B A y$.

(2) If $A x n$, $B x n \to y$, for some $y$ is $X$ then

(a) By the continuity of $A$, $B A x n \to A y$ and by compatibility of $A$, $B$ we have

$$M(B A x n, B A y, t) = 1$$

as $n \to \infty$, which yields $B A x n \to A y$.

(b) If $A$ and $B$ are continuous then from (a) we have $B A x n \to A y$. But by the continuity of $B$, $B A y \to B y$. Thus by uniqueness of the limit $A y = B y$. Hence $A B y = B A y$ from (1).

2. MAIN RESULTS

THEOREM 2.1: Let $(X, M, *)$ be a complete fuzzy metric space with additional condition [vi] and with $a * a \geq a$ for all $a \in [0, 1]$. Let $A$, $B$, $S$ and $T$ be mappings from $X$ into itself such that:

[i] $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$

[ii] One of the $A$, $B$, $S$ or $T$ is continuous.

[iii] $(A, S)$ and $(B, T)$ are compatible pairs of mappings.

(iv) $M(A x, B y, t) \geq \phi(\min(M(S x, T y, t), M(A x, T y, \alpha t), M(S x, B y, (2 - \alpha) t), \ldots))$ for all $x, y \in X$, $\alpha \in (0, 2)$ and $t > 0$. where $\phi : [0, 1] \to [0, 1]$ is a continuous function such that $\phi(t) > t$ for some $0 < t < 1$. Then $A$, $B$, $S$ and $T$ have a unique common fixed point in $X$.

PROOF: Let $x 0 \in X$ be an arbitrary point. Then, since $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, there exists $x 1, x 2 \in X$ such that $A x 0 = T x 1$ and $B x 1 = S x 2$. Inductively, we construct the sequences $\{y n\}$ and $\{x n\}$ in $X$ such that $y 2 n = A x 2 n = T x 2 n + 1$ and $y 2 n + 1 = B x 2 n + 1 = S x 2 n + 2$, for $n = 0, 1, 2, \ldots$

Now, we put $\alpha = 1 - q$ with $q \in (0, 1)$ in [iv], we have

$$M(y 2 n, y 2 n + 1, t) = M(A x 2 n, B x 2 n + 1, t) \geq \phi(\min(M(S x 2 n, T x 2 n + 1, t), M(A x 2 n, T x 2 n + 1, (1 - q) t), M(S x 2 n, B x 2 n + 1, (1 + q) t))).$$

That is,

$$M(y 2 n, y 2 n + 1, t) \geq \phi(\min(M(y 2 n - 1, y 2 n, t), M(y 2 n, y 2 n + 1, t), M(y 2 n - 1, y 2 n + 1, (1 + q) t))) \geq \phi(\min(M(y 2 n - 1, y 2 n, t), M(y 2 n, y 2 n + 1, t), M(y 2 n - 1, y 2 n + 1, q t))).$$

Since $\alpha$-norm * is continuous, letting $q \to 1$, we have

$$M(y 2 n, y 2 n + 1, t) \geq \phi(M(y 2 n - 1, y 2 n, t), M(y 2 n, y 2 n + 1, t), M(y 2 n - 1, y 2 n + 1, t)) \geq \phi(M(y 2 n - 1, y 2 n, t), M(y 2 n, y 2 n + 1, t))).$$

It follows that, $M(y 2 n, y 2 n + 1, t) > M(y 2 n - 1, y 2 n, t)$, since $\phi(t) > t$ for each $0 < t < 1$. Similarly, $M(y 2 n + 1, y 2 n + 2, t) > M(y 2 n, y 2 n + 1, t)$. Therefore, in general, we have

$$M(y n, y n + 1, t) \geq \phi(M(y n - 1, y n, t), M(y n - 1, y n, t)).$$

Therefore, $\{M(y n, y n + 1, t)\}$ is an increasing sequence of positive real numbers in $[0, 1]$ and tends to a limit, say $\lambda \leq 1$. We claim that $\lambda = 1$. If $\lambda < 1$, then $M(y n, y n + 1, t) \geq \phi(M(y n - 1, y n, t)).$

So, on letting $n \to \infty$, we get $\lim n \to \infty M(y n, y n + 1, t) \geq \phi(\lim n \to \infty M(y n, y n + 1, t))$ that is, $\lambda \geq \phi(\lambda) > \lambda$, a contradiction. Thus, we have $\lambda = 1$.

Now, for any positive integer $p$, we have

$$M(y n, y n + p, t) \geq M(y n, y n + 1, t) * M(y n + 1, y n + 2, t / p) * \ldots * M(y n + p - 1, y n + p, t / p) \to \lim n \to \infty M(y n, y n + p, t) \geq 1."
Then, setting \( x = u \) and \( y = x_{2n+1} \) in contractive condition [iv] with \( \alpha = 1 \), we get
\[
M(Au, Bx_{2n+1}, t) \geq \phi(\min\{ M(Su, Tx_{2n+1}, t), M(Au, Tx_{2n+1}, t), M(Su, Bx_{2n+1}, t) \}).
\]
Letting \( n \to \infty \), we get \( M(Au, u, t) \geq \tau(M(Au, u, t)) > M(Au, u, t) \), which implies that \( u = Au \). Thus, we have \( u = Au = Su \). Since \( A(X) \subseteq T(X) \), there exists \( v \) in \( X \) such that \( u = Au = TV \).

Therefore, setting \( x = x_{2n} \) and \( y = v \) in contractive condition [iv] with \( \alpha = 1 \), we get
\[
M(Ax_{2n}, Bv, t) \geq \phi(\min\{ M(Sx_{2n}, Tv, t), M(Ax_{2n}, Tv, t), M(Sx_{2n}, Bv, t) \}).
\]
Letting \( n \to \infty \), we get \( M(Au, Bu, t) \geq \phi(\min\{ M(Au, Bu, t), M(Au, Bv, t) \}) > M(Au, Bv, t) \), which implies that \( u = Bv \).

Thus, we have \( u = Bv = TV \). Therefore, we get \( u = Au = Su = Bv = Tv \).

Thus, \( u = Bv = Tv \). Therefore, we get \( u = Au = Su = Bv = Tv \).

Leaving \( x = x_{2n} \) and \( y = v \) in contractive condition [iv] with \( \alpha = 1 \), we get
\[
M(Ax_{2n}, Bv, t) \geq \phi(\min\{ M(Sx_{2n}, Tv, t), M(Ax_{2n}, Tv, t), M(Sx_{2n}, Bv, t) \}).
\]
Letting \( n \to \infty \), we get \( M(Au, Bu, t) \geq \phi(\min\{ M(Su, Bu, t), M(Au, Bu, t) \}) \), which is a contradiction.

Thus, from the contractive condition [iv] with \( \alpha = 1 \), we have
\[
M(Au, Bu, t) \geq \phi(\min\{ M(Su, Tu, t), M(Au, Tu, t), M(Su, Bu, t) \}),
\]
that is, \( M(u, Bu, t) > M(u, Bu, t) \), which is a contradiction.

\( \Rightarrow u = Bu \). Similarly, using condition [iv] with \( \alpha = 1 \), one can show that \( Au = u \). Therefore, we have \( u = Au = Bu = Tu = Su \). Hence, the point \( u \) is a common fixed point of \( A, B, S \) and \( T \).

**UNIQUENESS:**

We easily verified the uniqueness of a common fixed point of the mappings \( A, B, S \) and \( T \) by using [iv]. In fact, if \( u \neq u' \) be another fixed point for mappings \( A, B, S \) and \( T \). Then, for \( \alpha = 1 \), we have
\[
M(u, u', t) = M(Au, Bu', t) \geq \phi(\min\{ M(Su, Tu', t), M(Au, Tu', t), M(Su, Bu', t) \}) \geq \phi(M(u, u', t)) > M(u, u', t),
\]
and hence, we get \( u = u' \).

This completes the proof of the theorem.

**REFERENCES**


