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Fixed Point Theorem in Fuzzy Metric Space

Manoj Kumar Tiwari,
Deptt. of Mathematics, Pt. Sundarlal Sharma (Open) University,
Chhattisgarh, Bilaspur.

***Abstract*:- In this research paper we have established a common fixed point theorem for compatible pair of self mappings in a fuzzy metric space. 2000 Mathematics Subject Classification: 54H25, 47H10.**

Key Words and phrases: Common fixed point, Fuzzy metric space, compatible maps.

1. INTRODUCTION

The concept of fuzzy sets was initiated by L.A. Zadeh [19] in 1965 and the concept of fuzzy metric space was introduced by Kramosil and Michalek [9]. Grabiec [5] proved the contraction principle in the setting of the fuzzy metric space which was further generalization of results by Subrahmanyam [17] for a pair of commuting mappings. George and Veeramani [4] have modified the notion of fuzzy metric spaces with the help of continuous t-norm, by generalizing the concept of probabilistic metric space to fuzzy situation. Also, Jungck and Rhoades [7] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points. Balasubramaniam *et.al.* [1] proved a fixed point theorem, which generalizes a result of Pant for fuzzy mappings in fuzzy metric space.Also, Jha *et.al.*[6] have proved a common fixed point theorem for four self maps in fuzzy metric space under the weak contractive conditions. Also, B. Singh and S. Jain [16] introduced the notion of semi-compatible mappings in fuzzy metric space and compared this notion with the notion of compatible map of type (α), compatible map of type (β) and obtained some fixed point theorems in complete fuzzy metric space in the sense of Grabiec [5]. As a generalization of fixed point results of Singh and Jain [16], Mishra *et. al.*[10] have proved a fixed point theorems in complete fuzzy metric space by replacing continuity condition with reciprocally continuity maps.

The aim of this paper is to obtain a common fixed point theorem for compatible pair of self mappings in fuzzy metric space.
Now,We have used the following notions:

DEFINITION 1.1([19]) Let X be any set. A fuzzy set A in X is a function with domain X and values in [0, 1].

DEFINITION 1.2([4]) A binary operation $\ast : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm if, ([0, 1], $\ast$) is an abelian topological monoid with unit 1 such that $a \ast b \leq c \ast d$ whenever $a \leq c$ and $b \leq d$, for all a, b, c, d in [0, 1].
For an example: $a * b = ab$, $a * b = \min \{a, b\}$.

DEFINITION 1.3([4]) The triplet (X, M, $\ast$) is called a fuzzy metric space if, X is an arbitrary set, $\ast$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, 1)$ satisfying the following conditions: for all x, y, z in X, and $s, t > 0$,
[i] $M(x, y, 0) = 0$, $M(x, y, t) > 0$;
[ii] $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
[iii] $M(x, y, t) = M(y, x, t)$,
[iv] $M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)$,
[v] $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.
In this case, M is called a fuzzy metric on X and the function $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.
Also, we consider the following condition in the fuzzy metric space (X, M, $\ast$):
[vi] $\lim_{t\to\infty} M(x, y, t) = 1$, for all $x, y \in X$.
It is important to note that every metric space (X, d) induces a fuzzy metric space (X,M, $\ast$) where $a \ast b = \min \{a, b\}$ and for all a, b $\in$ X, we have $M(x,y,t) = \frac{t}{t+d(x,y)}$, for all $t > 0$, and $M(x, y, 0) = 0$, so-called the fuzzy metric space induced by the metric d.

DEFINITION 1.4([4]) In a fuzzy metric space (X, M, $\ast$) a sequence $\{x_n\}$ is called a Cauchy sequence if, $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for every $t > 0$ and for each $p > 0$.
A fuzzy metric space (X, M, $\ast$) is complete if, every Cauchy sequence in X converges in X.

DEFINITION 1.5([4]) In a fuzzy metric space (X, M, $\ast$) A sequence $\{x_n\}$ is said to be convergent to x in X if, $\lim_{n\to\infty} M(x_n, x, t) = 1$, for each $t > 0$.
It is noted that since $\ast$ is continuous, it follows from the condition [iv] of Definition (1.3.) that the limit of a sequence in a fuzzy metric space is unique.

DEFINITION 1.6([1]) Two self mappings A and B of a fuzzy metric space (X, M, ∗) are said to be compatible if, $\lim_{n\to\infty}$ M(ABx_n, BAx_n, t) = 1 whenever {x_n} is a sequence such that $\lim_{n\to\infty}$ Ax_n = $\lim_{n\to\infty}$ Bx_n = p, for some p in X.
LEMMA 1.11([14]) Let (X, M, ∗) be a fuzzy metric space. If there exists k ∈ (0, 1) such that M(x, y, kt) ≥ M(x, y, t) then $x = y$.
PROPOSITION 1.12: Let A and B be compatible, self mappings of a fuzzy metric space X,
(1) If Ay = By then ABy = Bay.
(2) If Ax_n , Bx_n → y, for some y is X then
(a) BAx_n → Ay if A is continuous.
(b) If A and B are continuous at y then Ay = By and ABy = BAy.
PROOF: (1) Let Ay = By and {x_n} be a sequence in X such that x_n = y for all n. Then Ax_n , Bx_n → Ay. Now by the compatibility of A and B, we have
M(ABy, BAy, t) = M(ABx_n, BAx_n, t) = 1 which yields ABy = BAy.
(2) If Ax_n , Bx_n → y, for some y is X then
(a) By the continuity of A, ABx_n → Ay and by compatibility of A, B
M(ABx_n, BAx_n, t) = 1 as n→∞, which yields BAx_n → Ay.
(b) If A and B are continuous then from (a) we have BAx_n → Ay. But by the continuity of B, BAx_n → By. Thus by uniqueness of the limit Ay = By. Hence ABy = BAy from (1).

2. MAIN RESULTS

THEOREM 2.1. Let (X, M, ∗) be a complete fuzzy metric space with additional condition [vi]and with a ∗ a ≥ a for all a ∈ [0, 1]. Let A, B, S and T be mappings from X into itself such that :
[i] A(X) ⊆ T(X), B(X) ⊆ S(X)
[ii] One of the A, B, S or T is continuous,
[iii] (A, S) and (B, T) are compatible pairs of mappings,
[iv] M(Ax, By, t) ≥ ϕ(min{M(Sx, Ty, t), M(Ax, Ty, αt), M(Sx, By, (2 − α)t), }) for all x, y ∈ X, α∈ (0, 2) and t > 0. where ϕ : [0, 1] → [0, 1] is a continuous function such that ϕ(t) > t for some 0 < t < 1.Then A, B, S and T have a unique common fixed point in X.

PROOF: Let x_0 ∈ X be an arbitrary point. Then, since A(X) ⊆ T(X), B(X) ⊆ S(X), there exists x_1, x_2 ∈ X such that Ax_0 = Tx_1 and Bx_1 = Sx_2. Inductively, we construct the sequences {y_n} and {x_n} in X such that y_{2n} = Ax_{2n} = Tx_{2n+1} and y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}, for n = 0, 1, 2, ...
Now, we put α = 1 − q with q ∈ (0, 1) in [iv], we have
M(y_{2n}, y_{2n+1}, t) = M(Ax_{2n}, Bx_{2n+1}, t)
≥ ϕ(min{M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, Tx_{2n+1}, (1 − q)t), M(Sx_{2n}, Bx_{2n+1}, (1 + q)t)}).
That is,
M(y_{2n}, y_{2n+1}, t) ≥ ϕ (min{ M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n-1}, y_{2n+1}, (1 + q)t)})
≥ ϕ (min{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n-1}, y_{2n}, t), M(y_{2n-1}, y_{2n+1}, qt)})
≥ M(y_{2n-1}, y_{2n}, t) ∗ M(y_{2n}, y_{2n+1}, t) ∗ M(y_{2n}, y_{2n+1}, qt).
Since t-norm ∗ is continuous, letting q → 1, we have
M(y_{2n}, y_{2n+1}, t) ≥ ϕ (min{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t)}
≥ ϕ (min{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t)}).
It follows that, M(y_{2n}, y_{2n+1}, t) > M(y_{2n-1}, y_{2n}, t), since ϕ (t) > t for each 0 < t < 1.
Similarly, M(y_{2n+1}, y_{2n+2}, t) > M(y_{2n}, y_{2n+1}, t). Therefore, in general, we have
M(y_n, y_{n+1}, t) ≥ ϕ (M(y_{n-1}, y_n, t)) > M(y_{n-1}, y_n, t).
Therefore, {M(y_n, y_{n+1}, t)} is an increasing sequence of positive real numbers in [0, 1] and tends to a limit, say λ ≤ 1. We claim that λ = 1. If λ < 1, then M(y_n, y_{n+1}, t) ≥ ϕ (M(y_{n-1}, y_n, t)).
So, on letting n →∞, we get $\lim_{n\to\infty}$ M(y_n, y_{n+1}, t) ≥ ϕ($\lim_{n\to\infty}$ M(y_n, y_{n+1}, t))
that is, λ ≥ r(λ) > λ, a contradiction. Thus, we have λ = 1.
Now, for any positive integer p, we have
M(y_n, y_{n+p}, t) ≥ M(y_n, y_{n+1}, t) ∗M(y_{n+1}, y_{n+2}, t/p) ∗ ... ∗M(y_{n+p-1}, y_{n+p}, t/p).
Letting n →∞, we get $\lim_{n\to\infty}$ M(y_n, y_{n+p}, t) ≥ 1 ∗1 ∗ …... ∗ 1 = 1.
Thus, we have $\lim_{n\to\infty}$ M(y_n, y_{n+p}, t) = 1. Hence, {y_n} is a Cauchy sequence in X. Since X is complete metric space, so the sequence {y_n} converges to a point u (say) in X and consequently, the subsequences {Ax_{2n}}, {Sx_{2n}}, {Tx_{2n+1}} and {Bx_{2n+1}} also converges to u.
We first consider the case when (A, S) and (B, T) are compatible mappings. Since A and S are compatible maps, so we have $\lim_{n\to\infty}$ M(ASx_n, SAx_n, t) = 1 whenever {x_n} is a sequence such that $\lim_{n\to\infty}$ Ax_n = $\lim_{n\to\infty}$ Sx_n = u, for some u in X. Therefore, we get Au = Su. And also B and T are compatible maps, so we have $\lim_{n\to\infty}$ M(BTx_n, TBx_n, t) = 1 whenever {x_n} is a sequence such that $\lim_{n\to\infty}$ Bx_n = $\lim_{n\to\infty}$ Tx_n = u, for some u in X.
We claim that Au = u. For this, suppose that Au ≠ u.

Then, setting x = u and y = x2n+1 in contractive condition [iv] with α = 1, we get

$$M(Au, Bx_{2n+1}, t) \geq \phi(\min\{ M(Su, Tx_{2n+1}, t), M(Au, Tx_{2n+1}, t), M(Su, Bx_{2n+1}, t)\}).$$

Letting n →∞, we get M(Au, u, t) ≥ r(M(Au, u, t)) > M(Au, u, t), which implies that u = Au.
Thus, we have u = Au = Su. Since A(X) ⊆ T(X), so there exists v in X such that u = Au = Tv.
Therefore, setting x = x2n and y = v in contractive condition [iv] with α = 1, we get

$$M(Ax_{2n}, Bv, t) \geq \phi(\min\{ M(Sx_{2n}, Tv, t), M(Ax_{2n}, Tv, t), M(Sx_{2n}, Bv, t)\}).$$

Letting n →∞, we get M(Au, Bv, t) ≥ ϕ(M(Au, Bv, t)) > M(Au, Bv, t), which implies that u = Bv.
Thus, we have u = Bv = Tv. Therefore, we get u = Au = Su = Bv = Tv.
Now, since u = Bv = Tv, so by the compatibility of (B, T), it follows that BTv = TBv and so we get Bu = BTv = TBv = Tu. Thus, from the contractive condition (iv) with α = 1, we have

$$M(Au, Bu, t) \geq \phi(\min\{ M(Su, Tu, t), M(Au, Tu, t), M(Su,Bu, t)\}),$$

that is, M(u, Bu, t) > M(u, Bu, t), which is a contradiction.

⇒ u = Bu. Similarly, using condition [iv] with α = 1, one can show that Au = u. Therefore, we have u = Au = Bu = Tu = Su. Hence, the point u is a common fixed point of A, B, S and T.

UNIQUENESS :

We easily verified the uniqueness of a common fixed point of the mappings A, B, S and T by using [iv]. In fact, if u0 be another fixed point for mappings A, B, S and T. Then, for α = 1, we have

$$M(u, u_0, t) = M(Au, Bu_0, t) \geq \phi(\min\{M(Su, Tu_0, t), M(Au, Tu_0, t), M(Su,Bu_0, t)\}),$$
$$\geq \phi(M(u, u_0, t)) > M(u, u_0, t), \text{ and hence, we get } u = u_0.$$

This completes the proof of the theorem.

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